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Coordinatewise decomposition of group-valued Borel functions

by

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Abstract. Answering a question of Kłopotowski, Nadkarni, Sarbadhikari, and Srivastava, we characterize the Borel sets $S \subseteq X \times Y$ with the property that every Borel function $f : S \to \mathbb{C}$ is of the form $f(x, y) = u(x) + v(y)$, where $u : X \to \mathbb{C}$ and $v : Y \to \mathbb{C}$ are Borel.

Given a set X, let $X^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} X^n$. The *length* of $s \in X^{<\mathbb{N}}$ is the unique natural number $|s|$ such that $s \in X^{|s|}$. The *restriction* of $\alpha \in X^{\mathbb{N}}$ to n is the sequence $\alpha|n \in X^n$ whose ith coordinate agrees with that of α, for all $i < n$. We say that s is an *initial segment* of $\alpha \in X^{\mathbb{N}}$, or $s \subseteq \alpha$, if there exists $n \in \mathbb{N}$ such that $s = \alpha|n$. The *basic clopen set associated with* $s \in X^{<\mathbb{N}}$ is given by $\mathcal{N}_s = \{\alpha \in X^{\mathbb{N}} : s \subseteq \alpha\}$. The *diagonal* on X is defined by $\Delta(X) = \{(x, x) : x \in X\}$. Given a set $S \subseteq X \times Y$, let $S^{-1} = \{(y, x) \in Y \times X : (x, y) \in S\}$.

A *graph* on X is an irreflexive, symmetric set $\mathcal{G} \subseteq X \times X$. A *$\mathcal{G}$-path* from x to y is a sequence $\langle x_i\rangle_{i\le n} \in X^{<\mathbb{N}}$ such that $x = x_0$, $y = x_n$, and $\forall i < n\ ((x_i, x_{i+1}) \in \mathcal{G})$. Such a path is a *$\mathcal{G}$-cycle* if $n \ge 3$, $x = y$, and $\langle x_i\rangle_{i<n}$ is injective. We say that $\mathcal{G}$ is *acyclic* if there are no $\mathcal{G}$-cycles. The *graph metric* associated with such a graph is given by

$$d_{\mathcal{G}}(x, y) = \begin{cases} n & \text{if there is an injective } \mathcal{G}\text{-path } \langle x_i\rangle_{i\le n} \text{ from } x \text{ to } y, \\ \infty & \text{if there is no } \mathcal{G}\text{-path from } x \text{ to } y. \end{cases}$$

Suppose that E is an equivalence relation on X. The *E-class* of $x \in X$ is given by $[x]_E = \{y \in X : xEy\}$. The *$E$-saturation* of a set $B \subseteq X$ is given by $[B]_E = \{x \in X : \exists y \in B\ (xEy)\}$. We say that B is *E-invariant* if $B = [B]_E$, and we say that B is an *E-complete section* if $X = [B]_E$. A *reduction* of an equivalence relation E on X to an equivalence relation F on Y is a function $\pi : X \to Y$ such that $\forall x_1, x_2 \in X\ (x_1 E x_2 \Leftrightarrow \pi(x_1) F \pi(x_2))$.

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An *embedding* is an injective reduction. A *separating family* for E is a family of sets $B_0, B_1, \ldots \subseteq X$ such that

$$\forall x, y \in X \ (xEy \Leftrightarrow \forall n \in \mathbb{N} \ (x \in B_n \Leftrightarrow y \in B_n)).$$

We say that an equivalence relation E on a Polish space is *smooth* if there is a Borel reduction of E to $\Delta(2^{\mathbb{N}})$, or equivalently, if E admits a Borel separating family. A *transversal* of E is a set $B \subseteq X$ such that $\forall x \in X \ (|B \cap [x]_E| = 1)$.

Suppose that $S \subseteq X \times Y$, Γ is a group, and $f : S \to \Gamma$. A *coordinatewise decomposition* of f is a pair (u, v), where $u : X \to \Gamma$, $v : Y \to \Gamma$, and

$$\forall (x, y) \in S \ (f(x, y) = u(x)v(y)).$$

While our main goal here is to study coordinatewise decompositions in the descriptive set-theoretic context, we will first study the existence of coordinatewise decompositions without imposing any definability restrictions.

For the sake of notational convenience, we will assume that $X \cap Y = \emptyset$. The *graph associated with* S is the graph on the set $Z_S = X \cup Y$ given by $\mathcal{G}_S = S \cup S^{-1}$. The following fact was proven essentially by Cowsik, Kłopotowski and Nadkarni [1]:

PROPOSITION 1. *Suppose that X, Y are disjoint, $S \subseteq X \times Y$, and Γ is a non-trivial group. Then the following are equivalent*:

(1) *Every function $f : S \to \Gamma$ admits a coordinatewise decomposition.*
(2) *$\mathcal{G}_S$ is acyclic.*

Proof. To see $\neg(2) \Rightarrow \neg(1)$, suppose that there is a $\mathcal{G}_S$-cycle of the form $\langle x_0, y_0, x_1, y_1, \ldots, x_{n+1} \rangle$. Fix $\gamma_0 \in \Gamma \setminus \{1_\Gamma\}$ and define $f : S \to \Gamma$ by

$$f(x, y) = \begin{cases} \gamma_0 & \text{if } (x, y) = (x_0, y_0), \\ 1_\Gamma & \text{otherwise.} \end{cases}$$

If (u, v) is a coordinatewise decomposition of f, then

$$\begin{aligned} \gamma_0 &= f(x_0, y_0) f(x_1, y_0)^{-1} \cdots f(x_n, y_n) f(x_{n+1}, y_n)^{-1} \\ &= (u(x_0)v(y_0))(u(x_1)v(y_0))^{-1} \cdots (u(x_n)v(y_n))(u(x_{n+1})v(y_n))^{-1} \\ &= u(x_0)u(x_1)^{-1} \cdots u(x_n)u(x_{n+1})^{-1} = u(x_0)u(x_{n+1})^{-1} = 1_\Gamma, \end{aligned}$$

which contradicts our choice of γ_0.

To see $(2) \Rightarrow (1)$, let E_S be the equivalence relation whose classes are the connected components of $\mathcal{G}_S$, fix a transversal $B \subseteq Z_S$ of E_S, and define

$$B_n = \{z \in Z_S : \exists w \in B \ (d_{\mathcal{G}_S}(w, z) = n)\}.$$

For each $z \in B_{n+1}$, let $g_n(z)$ denote the unique $\mathcal{G}_S$-neighbor of z in B_n, and define $u : X \to \Gamma$, $v : Y \to \Gamma$ recursively by setting $u(x) = v(y) = 1_\Gamma$ for $x, y \in B_0$, and

$$u(x) = f(x, g_n(x))v(g_n(x))^{-1}, \quad v(y) = u(g_n(y))^{-1} f(g_n(y), y)$$

for $x, y \in B_{n+1}$. To see that (u, v) is a coordinatewise decomposition of f, suppose that $(x, y) \in S$, and fix $n \in \mathbb{N}$ such that $g_n(x) = y$ or $g_n(y) = x$. If $g_n(x) = y$, then $u(x) = f(x, y)v(y)^{-1}$, thus $f(x, y) = u(x)v(y)$. If $g_n(y) = x$, then $v(y) = u(x)^{-1}f(x, y)$, thus $f(x, y) = u(x)v(y)$. ∎

As a corollary of the proof of Proposition 1, we obtain a sufficient condition for the existence of Borel coordinatewise decompositions:

COROLLARY 2. *Suppose that X, Y are disjoint Polish spaces, $S \subseteq X \times Y$ is Borel, Γ is a standard Borel group, $\mathcal{G}_S$ is acyclic, and E_S has a Borel transversal. Then every Borel function $f : S \to \Gamma$ admits a Borel coordinatewise decomposition.*

Proof. It is sufficient to check that the functions u and v constructed in the proof of Proposition 1 are Borel. Letting $B_n \subseteq Z_S$ ands $g_n : Z_S \to Z_S$ be as above, it follows from the fact that $\mathcal{G}_S$ is acyclic that

$$\begin{aligned} z \in B_{n+1} &\Leftrightarrow z \notin \bigcup_{i \leq n} B_i \text{ and } \exists w \in B_n\ ((z, w) \in \mathcal{G}_S) \\ &\Leftrightarrow z \notin \bigcup_{i \leq n} B_i \text{ and } \exists! w \in B_n\ ((z, w) \in \mathcal{G}_S), \end{aligned}$$

and results of Suslin and Luzin (see, for example, Theorems 14.11 and 18.11 of Kechris [5] or Theorem 4.4.3 and Corollary 4.12.2 of Srivastava [8]) then imply that each of these sets is Borel. As

$$\operatorname{graph}(g_n) = \mathcal{G}_S \cap (B_{n+1} \times B_n),$$

it follows that g_n is Borel as well (see, for example, Theorem 14.12 of [5] or Theorem 4.5.2 of [8]), and this easily implies that u and v are Borel. ∎

Our main theorem is that the sufficient condition given above is also necessary to guarantee the existence of Borel coordinatewise decompositions:

THEOREM 3. *Suppose that X, Y are disjoint Polish spaces, $S \subseteq X \times Y$ is Borel, and Γ is a non-trivial standard Borel group. Then the following are equivalent:*

(1) *Every Borel function $f : S \to \Gamma$ admits a Borel coordinatewise decomposition.*
(2) *$\mathcal{G}_S$ is acyclic and E_S admits a Borel transversal.*

Proof. As (2)⇒(1) follows from Corollary 2, we need only show (1)⇒(2). Towards this end, suppose that (1) holds. As the map f described in the proof of ¬(2)⇒¬(1) of Proposition 1 is clearly Borel, it follows that $\mathcal{G}_S$ is acyclic, thus E_S is Borel (by Theorems 14.11 and 18.11 of [5] or Theorem 4.4.3 and Corollary 4.12.2 of [8]).

Fix a non-trivial countable subgroup $\Delta \leq \Gamma$, endow Δ with the discrete topology, and endow $\Delta^{\mathbb{N}}$ with the corresponding product topology. Define

E_0^Δ on $\Delta^{\mathbb{N}}$ by

$$\alpha E_0^\Delta \beta \Leftrightarrow \exists n \in \mathbb{N}\ \forall m > n\ (\alpha(m) = \beta(m)),$$

and define $F_0^\Delta \subseteq E_0^\Delta$ on $\Delta^{\mathbb{N}}$ by

$$\alpha F_0^\Delta \beta \Leftrightarrow \exists n \in \mathbb{N}\ \forall m > n\ (\alpha(0) \cdots \alpha(m) = \beta(0) \cdots \beta(m)).$$

Let Δ act freely on $\Delta^{\mathbb{N}}$ by left multiplication on the 0th coordinate, i.e.,

$$\delta \cdot \alpha = \langle \delta\alpha(0), \alpha(1), \alpha(2), \ldots \rangle.$$

Lemma 4. *The action of Δ on $\Delta^{\mathbb{N}}$ induces a free action of Δ on $\Delta^{\mathbb{N}}/F_0^\Delta$.*

Proof. It is enough to observe that

$$\forall \delta \in \Delta\, \forall \alpha, \beta \in \Delta^{\mathbb{N}}\ (\alpha F_0^\Delta \beta \Rightarrow \delta \cdot \alpha F_0^\Delta \delta \cdot \beta),$$

which is a trivial consequence of the definition of F_0^Δ. ∎

Suppose now that $F \subseteq E$ are Borel equivalence relations on a Polish space X. We say that E is *relatively ergodic* over F if there is no F-invariant Borel set $B \subseteq X$ such that both B and $X \setminus B$ are E-complete sections.

Lemma 5. *E_0^Δ is relatively ergodic over F_0^Δ.*

Proof. Suppose, towards a contradiction, that $B \subseteq \Delta^{\mathbb{N}}$ is an F_0^Δ-invariant Borel set such that both B and $\Delta^{\mathbb{N}} \setminus B$ are E_0^Δ-complete sections. As B is an E_0^Δ-complete section, it follows that B is non-meager, thus there exists $s \in \Delta^{<\mathbb{N}}$ such that B is comeager in $\mathcal{N}_s$. Define $C \subseteq \Delta^{\mathbb{N}}$ by

$$C = \Delta^{\mathbb{N}} \setminus [\mathcal{N}_s \setminus B]_{E_0^\Delta},$$

and observe that C is an E_0^Δ-invariant comeager Borel set and $\mathcal{N}_s \cap C \subseteq B \cap C$. It only remains to show that $C \subseteq B$, which implies that $\Delta^{\mathbb{N}} \setminus B$ is meager, contradicting the fact that $\Delta^{\mathbb{N}} \setminus B$ is an E_0^Δ-complete section. Towards this end, given $\alpha \in C$, set $n = |s|$ and define $\delta \in \Delta$ by

$$\delta = (s(0) \cdots s(n-1))^{-1}(\alpha(0) \cdots \alpha(n)).$$

Then $\alpha F_0^\Delta \langle s(0), \ldots, s(n-1), \delta, \alpha(n+1), \alpha(n+2), \ldots \rangle$, thus $\alpha \in B$. ∎

Recall that E_0 is the equivalence relation on $2^{\mathbb{N}}$ given by

$$\alpha E_0 \beta \Leftrightarrow \exists n \in \mathbb{N}\ \forall m > n\ (\alpha(m) = \beta(m)).$$

Lemma 6. *There is a Borel embedding $\pi_1 : \Delta^{\mathbb{N}} \to 2^{\mathbb{N}}$ of E_0^Δ into E_0.*

Proof. Fix an enumeration (k_n, δ_n) of $\mathbb{N} \times \Delta$. Define $\pi_1 : \Delta^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$[\pi_1(\alpha)](n) = \begin{cases} 1 & \text{if } \alpha(k_n) = \delta_n, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that π_1 is the desired embedding. ∎

Now suppose, towards a contradiction, that E_S has no Borel transversal.

Lemma 7. *There is a Borel embedding $\pi_2 : 2^{\mathbb{N}} \to Z_S$ of E_0 into $E_S|X$.*

Proof. Suppose, towards a contradiction, that there is no Borel embedding of E_0 into $E_S|X$. As E_S is Borel, so too is $E_S|X$. It follows from Theorem 1.1 of Harrington–Kechris–Louveau [3] that $E_S|X$ is smooth. Fix a Borel separating family $B_0, B_1, \ldots$ for $E_S|X$, and observe that the sets

$$A_n = B_n \cup \{y \in Y : \exists x \in B_n\ ((x,y) \in S)\}$$

form a $\mathbf{\Sigma}^1_1$ separating family for $E_S|(X \cup \mathrm{proj}_Y[S])$, where $\mathrm{proj}_Y : X \times Y \to Y$ denotes the projection function. It easily follows that E_S has a $\sigma(\mathbf{\Sigma}^1_1)$ separating family, thus Theorem 1.1 of [3] implies that E_S is smooth. As $\mathcal{G}_S$ is acyclic, it follows from Hjorth [4] (see also Miller [7]) that E_S admits a Borel transversal, which contradicts our assumption that it does not. ∎

For $x_1 E_S x_2$, we say that z is $\mathcal{G}_S$-*between* x_1 and x_2 if z lies along the unique injective $\mathcal{G}_S$-path from x_1 to x_2. Define $B \subseteq Z_S$ by

$$B = \{z \in Z_S : \exists x_1, x_2 \in \mathrm{rng}(\pi_2 \circ \pi_1)\ (z \text{ is } \mathcal{G}_S\text{-between } x_1 \text{ and } x_2)\}.$$

As $\mathcal{G}_S$ is acyclic and $\mathrm{rng}(\pi_2 \circ \pi_1)$ intersects every E_S-class in a countable set, it follows that B is Borel. As $E_S \cap (B \times \mathrm{rng}(\pi_2 \circ \pi_1))$ has countable sections, the Luzin–Novikov uniformization theorem (see, for example, Theorem 18.10 of [5] or Theorem 5.8.11 of [8]) ensures that it has a Borel uniformization $\pi_3 : B \to \mathrm{rng}(\pi_2 \circ \pi_1)$. We can clearly assume that $\pi_3|\mathrm{rng}(\pi_2 \circ \pi_1) = \mathrm{id}$. Define $\pi : B \to \varDelta^{\mathbb{N}}$ by

$$\pi = (\pi_2 \circ \pi_1)^{-1} \circ \pi_3,$$

and finally, define $f : S \to \varDelta$ by

$$f(x,y) = \begin{cases} 1_\varGamma & \text{if } x \notin B \text{ or } y \notin B, \\ \delta & \text{if } x, y \in B \text{ and } \delta \cdot \pi(y) F_0^\varDelta \pi(x). \end{cases}$$

Now suppose, towards a contradiction, that there is a Borel coordinatewise decomposition (u,v) of f.

LEMMA 8. *Suppose that* $x, x' \in B \cap X$ *and* $x E_S x'$. *Then*:

(1) $u(x)u(x')^{-1} \in \varDelta$.
(2) $u(x)u(x')^{-1} \cdot \pi(x') F_0^\varDelta \pi(x)$.

Proof. Let $\langle x_0, y_0, \ldots, x_n, y_n, x_{n+1}\rangle$ be the unique injective $\mathcal{G}_S$-path from x to x'. To see (1), observe that, for all $i \le n$,

$$u(x_i)u(x_{i+1})^{-1} = (u(x_i)v(y_i))(u(x_{i+1})v(y_i))^{-1} = f(x_i,y_i)f(x_{i+1},y_i)^{-1},$$

thus $u(x_i)u(x_{i+1})^{-1} \in \varDelta$. Noting that

$$u(x_0)u(x_{n+1})^{-1} = u(x_0)u(x_1)^{-1}u(x_1)u(x_2)^{-1} \cdots u(x_n)u(x_{n+1})^{-1},$$

it follows that $u(x)u(x')^{-1} \in \varDelta$.

To see (2), observe that for all $i \le n$,

$$\begin{aligned} u(x_i)u(x_{i+1})^{-1} \cdot [\pi(x_{i+1})]_{F_0^\Delta} &= f(x_i, y_i)f(x_{i+1}, y_i)^{-1} \cdot [\pi(x_{i+1})]_{F_0^\Delta} \\ &= f(x_i, y_i) \cdot [\pi(y_i)]_{F_0^\Delta} = [\pi(x_i)]_{F_0^\Delta}. \end{aligned}$$

Setting $C_i = [\pi(x_i)]_{F_0^\Delta}$, it follows that

$$\begin{aligned} u(x_0)u(x_{n+1})^{-1} \cdot C_{n+1} &= u(x_0)u(x_1)^{-1} \cdots u(x_n)u(x_{n+1})^{-1} \cdot C_{n+1} \\ &= u(x_0)u(x_1)^{-1} \cdots u(x_{n-1})u(x_n)^{-1} \cdot C_n \\ &\vdots \\ &= C_0, \end{aligned}$$

thus $u(x_0)u(x_{n+1})^{-1} \cdot [\pi(x_{n+1})]_{F_0^\Delta} = [\pi(x_0)]_{F_0^\Delta}$. ∎

Define $w : \Delta^{\mathbb{N}} \to \Gamma$ by $w = u \circ \pi_2 \circ \pi_1$, fix a countable Borel separating family $\Gamma_0, \Gamma_1, \ldots \subseteq \Gamma$ for Γ, and define $n : \Delta^{\mathbb{N}} \to \Gamma$ by

$$n(\alpha) = \min\{n \in \mathbb{N} : \exists \delta_1, \delta_2 \in \Delta\ (\delta_1 w(\alpha) \in \Gamma_n \text{ and } \delta_2 w(\alpha) \notin \Gamma_n)\}.$$

Lemma 8 ensures that if $\alpha E_0^\Delta \beta$, then $w(\alpha)w(\beta)^{-1} \in \Delta$, thus

$$\Delta w(\alpha) = \Delta w(\alpha)w(\beta)^{-1}w(\beta) = \Delta w(\beta),$$

and it follows that $n(\alpha) = n(\beta)$. As $\pi_3|\mathrm{rng}(\pi_2 \circ \pi_1) = \mathrm{id}$, Lemma 8 also ensures that $w(\alpha)w(\beta)^{-1} \cdot \beta F_0^\Delta \alpha$. It follows that if $\alpha = \delta \cdot \beta$, then $w(\alpha)w(\beta)^{-1} = \delta$, thus $w(\alpha) = \delta w(\beta)$. Defining $A \subseteq \Delta^{\mathbb{N}}$ by

$$A = \{\alpha \in \Delta^{\mathbb{N}} : w(\alpha) \in \Gamma_{n(\alpha)}\},$$

it follows that A is an F_0^Δ-invariant Borel set and both A and $\Delta^{\mathbb{N}} \setminus A$ are E_0^Δ-complete sections, which contradicts Lemma 5. ∎

Kłopotowski, Nadkarni, Sarbadhikari and Srivastava [6] have studied coordinatewise decomposition using another equivalence relation L which, modulo straightforward identifications, is the equivalence relation whose classes are the connected components of the dual graph $\breve{\mathcal{G}}_S$ on S, consisting of all pairs $((x_1, y_1), (x_2, y_2))$ of distinct elements of S such that either $x_1 = x_2$ or $y_1 = y_2$. The equivalence classes of L are the *linked components* of S, and the linked components of S are said to be *uniquely linked* if $\mathcal{G}_S$ is acyclic.

Conjecture 9 (Kłopotowski–Nadkarni–Sarbadhikari–Srivastava [6]). *Suppose that X, Y are disjoint Polish spaces and $S \subseteq X \times Y$ is Borel. Then the following are equivalent*:

(1) *Every Borel function $f : S \to \mathbb{C}$ has a Borel coordinatewise decomposition.*
(2) *The linked components of S are uniquely linked and L is smooth.*

In light of Theorem 3 and the above remarks, the following observation implies that Conjecture 9 is indeed correct:

PROPOSITION 10. *Suppose that X, Y are disjoint Polish spaces, $S \subseteq X \times Y$ is Borel, and $\mathcal{G}_S$ is acyclic. Then the following are equivalent*:

(1) *E_S admits a Borel transversal.*
(2) *L is smooth.*

Proof. To see (1)$\Rightarrow$(2), suppose that E_S admits a Borel transversal $B \subseteq Z_S$. Let $\pi_1 : Z_S \to Z_S$ be the function which sends z to the unique element of $B \cap [z]_{E_S}$, and let $\pi_2 = \mathrm{proj}_X |S$. Then π_1 is a Borel reduction of E_S to $\Delta(Z_S)$ and π_2 is a Borel reduction of L to E_S, thus $\pi_1 \circ \pi_2$ is a Borel reduction of L to $\Delta(Z_S)$, so L is smooth.

To see (2)$\Rightarrow$(1), suppose that L is smooth, and fix a Borel reduction $\pi_1 : S \to 2^{\mathbb{N}}$ of L to $\Delta(2^{\mathbb{N}})$. Put $Z = \mathrm{proj}_X[S] \cup \mathrm{proj}_Y[S]$. By the Jankov–von Neumann uniformization theorem (see, for example, Theorem 18.1 of [5] or Theorem 5.5.2 of [8]), there is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable reduction $\pi_2 : Z \to S$ of $E_S|Z$ to L, thus $\pi_1 \circ \pi_2$ is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable reduction of $E_S|Z$ to $\Delta(2^{\mathbb{N}})$. It easily follows that there is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable reduction of E_S to $\Delta(2^{\mathbb{N}})$, thus Theorem 1.1 of [3] implies that E_S is smooth. As $\mathcal{G}_S$ is acyclic, it follows from [4] (see also [7]) that E_S admits a Borel transversal. ∎

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