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Ordinals in topological groups

by

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Abstract. We show that if an uncountable regular cardinal τ and $\tau + 1$ embed in a topological group G as closed subspaces then G is not normal. We also prove that an uncountable regular cardinal cannot be embedded in a torsion free Abelian group that is hereditarily normal. These results are corollaries to our main results about ordinals in topological groups. To state the main results, let τ be an uncountable regular cardinal and G a T_1 topological group. We prove, among others, the following statements: (1) If τ and $\tau+1$ embed closedly in G then $\tau\times(\tau+1)$ embeds closedly in G; (2) If τ embeds in G, G is Abelian, and the order of every non-neutral element of G is greater than $2^N - 1$ then $\prod_{i\in N}\tau$ embeds in G; (3) The previous statement holds if τ is replaced by $\tau + 1$; (4) If G is Abelian, algebraically generated by $\tau + 1 \subset G$, and the order of every element does not exceed $2^N - 1$ then $\prod_{i\in N}(\tau + 1)$ is not embeddable in G.

1. Introduction. The paper is devoted to the following problem.

Problem. *Let G be a topological group and $X, Y \subset G$. What conditions on X and/or Y and/or G guarantee that $X \times X$ and/or $X \times Y$ embed* (*closedly*) *in G?*

It is known, in particular, that X^n is homeomorphic to a closed subspace of the Abelian free topological group over X for every natural number n [S-T]. We restrict ourselves to cases when X and Y in this problem are spaces of ordinals. A corollary to one of our main results implies that if ω_1 and $\omega_1 + 1$ embed closedly in G then so does $\omega_1 \times (\omega_1 + 1)$, thus making G not normal. Another result of the paper implies that ω_1 is not embeddable in a torsion free Abelian group that is hereditarily normal. This is because, once such a group contains ω_1 it contains $\omega_1 \times \omega_1$ as well. In short, investigating the above problem is one approach to study separation and covering properties in topological groups. We refer the reader to surveys [COM] and [TKA] that contain some results and references related to the topic in ques-

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tion. The main results of the paper are those cited in the abstract and are given by statements 2.6, 3.3, 3.4, 4.7, and 4.8.

To shorten our statements, let us agree that whenever we claim that $X = \{x_\alpha : \alpha < \tau\}$ is homeomorphic to an ordinal τ we assume that $x_\alpha \leftrightarrow \alpha$ is a homeomorphism inducing an order on X.

We will use the $\star$ sign for the binary operation in a given topological group G. The neutral element of G is denoted by e_G. The order of $g \in G$ is denoted by $o(g)$. If the order of every element of G does not exceed N we write $o(G) \leq N$. The equality $o(G) = N$ means that $o(G) \leq N$ and the order of at least one element equals N.

Let G be a topological group and $\langle x_\alpha : \alpha < \tau \rangle$ a sequence in G, where τ is an infinite regular cardinal. We say that $\langle x_\alpha : \alpha < \tau \rangle$ converges to $x \in G$ if for any open neighborhood U of x there exists an ordinal $\alpha_U < \tau$ such that $x_\alpha \in U$ whenever $\alpha > \alpha_U$. We will use the following fact quite often.

FACT. *Let G be a topological group and τ an infinite regular cardinal. If $\langle x_\alpha : \alpha < \tau \rangle$ and $\langle y_\alpha : \alpha < \tau \rangle$ converge in G to x and y, respectively, then $\langle x_\alpha \star y_\alpha : \alpha < \tau \rangle$ converges to $x \star y$ and $\langle x_\alpha^{-1} : \alpha < \tau \rangle$ to x^{-1}.*

An ordinal τ, when treated as a topological space, is endowed with the topology of linear order. The symbol α^n denotes $\alpha \star \cdots \star \alpha$, where α is an element of a group G. The topological second power of the space τ will be denoted by $\tau \times \tau$. The topological Nth power of τ will be denoted by $\prod_{i \in N} \tau$.

In notation and terminology we will follow [ENG]. All spaces are assumed to satisfy T_1. It is a classical theorem of Kolmogorov and Pontryagin that a T_1 topological group is Tikhonov (see, for example, [PON]).

2. $\tau \times (\tau + 1)$ in topological groups. In this section we will prove that if a topological group contains closed copies of τ and $\tau + 1$, where τ is an uncountable regular cardinal, then G contains a closed copy of $\tau \times (\tau + 1)$. It follows in particular that if ω_1 and $\omega_1 + 1$ embed in a topological group G as closed subspaces then G is not normal.

The strategy is straightforward. Given $X = \{x_\alpha : \alpha < \tau\}$ and $Y = \{y_\alpha : \alpha < \tau + 1\}$ in G, we carefully select a closed unbounded subset T of τ such that $\{x_\alpha : \alpha \in T\} \star \{y_\alpha : \alpha \in T \cup \{\tau\}\}$ is as desired.

LEMMA 2.1. *Let G be a topological group and $X = \{x_\alpha : \alpha < \tau\}, Y = \{y_\alpha : \alpha < \tau + 1\} \subset G$ homeomorphic to ordinals τ and $\tau + 1$, respectively, where τ is an infinite regular cardinal. Then for any $\lambda < \tau$ there exists $\lambda^* < \tau$ such that $x_\lambda \star y_\alpha \neq x_\beta \star y_\gamma$ for any $\alpha, \beta, \gamma \geq \lambda^*$.*

Proof. Assume the contrary. Then we can select non-decreasing sequences $\langle \beta_i \rangle_{i<\tau}$ of elements of τ and $\langle \alpha_i \rangle_{i<\tau}$, $\langle \gamma_i \rangle_{i<\tau}$ of elements of $\tau + 1$ such that

(1) $x_\lambda \star y_{\alpha_i} = x_{\beta_i} \star y_{\gamma_i}$;

(2) $\beta_i > \lambda$;
(3) $\langle \alpha_i \rangle_{i<\tau}$ and $\langle \gamma_i \rangle_{i<\tau}$ converge to τ.

By (3), $\langle y_{\alpha_i} : i < \tau \rangle$ and $\langle y_{\gamma_i} : i < \tau \rangle$ converge to y_τ. Therefore, the sequence $\langle x_\lambda \star y_{\alpha_i} : i < \tau \rangle$ converges to $x_\lambda \star y_\tau$. By (1), the sequence $\langle x_{\beta_i} \star y_{\gamma_i} : i < \tau \rangle$ also converges to $x_\lambda \star y_\tau$. Therefore, $\langle (x_{\beta_i} \star y_{\gamma_i}) \star y_{\gamma_i}^{-1} : i < \tau \rangle$ converges to $(x_\lambda \star y_\tau) \star y_\tau^{-1} = x_\lambda$. That is, $\langle x_{\beta_i} : i < \tau \rangle$ converges to x_λ, contradicting (2) and the fact that $x_\alpha \leftrightarrow \alpha$ is a homeomorphism of X with τ. ∎

LEMMA 2.2. *Let G be a topological group and $X = \{x_\alpha : \alpha < \tau\}, Y = \{y_\alpha : \alpha < \tau + 1\} \subset G$ homeomorphic to ordinals τ and $\tau + 1$, respectively, where τ is an uncountable regular cardinal. Then for any $\lambda < \tau$ there exists $\lambda^* < \tau$ such that $x_\alpha \star y_\lambda \neq x_\beta \star y_\gamma$ for any $\alpha, \beta, \gamma \geq \lambda^*$.*

Proof. Assume the contrary. Then we can select non-decreasing sequences $\langle \alpha_n \rangle_{n<\omega}$ and $\langle \beta_n \rangle_{n<\omega}$ of elements of τ and $\langle \gamma_n \rangle_{n<\omega}$ of elements of $\tau + 1$ such that

(1) $x_{\alpha_n} \star y_\lambda = x_{\beta_n} \star y_{\gamma_n}$;
(2) $\lim_{n\to\infty} x_{\alpha_n} = \lim_{n\to\infty} x_{\beta_n} = p$ for some $p \in G$;
(3) $\gamma_n > \lambda$.

Put $q = \lim_{n\to\infty} y_{\gamma_n}$. The limit exists by countable compactness of Y. Recall that $y_\alpha \leftrightarrow \alpha$ is a homeomorphism of Y with $\tau+1$. Applying this fact and (3), we have $q > y_\lambda$.

By continuity of $\star$, $\lim_{n\to\infty} x_{\alpha_n} \star y_\lambda = p \star y_\lambda$ and $\lim_{n\to\infty} x_{\beta_n} \star y_{\gamma_n} = p \star q$. By (1), $p \star y_\lambda = p \star q$. Therefore, $y_\lambda = q$, contradicting $q > y_\lambda$. ∎

LEMMA 2.3. *Let G be a topological group and $X = \{x_\alpha : \alpha < \tau\}, Y = \{y_\alpha : \alpha < \tau + 1\} \subset G$ closed and homeomorphic to τ and $\tau + 1$, respectively, where τ is an infinite regular cardinal. Then for any $\lambda < \tau$ there exists $\lambda^* < \tau$ such that $x_\lambda \star y_\alpha \neq x_\beta \star y_\lambda$ for any $\alpha, \beta \geq \lambda^*$.*

Proof. Assume the contrary. Then we can select non-decreasing sequences $\langle \alpha_i \rangle_{i<\tau}$ of elements of $\tau + 1$ and $\langle \beta_i \rangle_{i<\tau}$ of elements of τ such that

(1) $x_\lambda \star y_{\alpha_i} = x_{\beta_i} \star y_\lambda$;
(2) $\langle \alpha_i \rangle_{i<\tau}$ and $\langle \beta_i \rangle_{i<\tau}$ converge to τ.

By (2), $\langle y_{\alpha_i} \rangle_{i<\tau}$ converges to y_τ. Therefore, $\langle x_\lambda \star y_{\alpha_i} \rangle_{i<\tau}$ converges to $x_\lambda \star y_\tau$. By (1), $\langle x_{\beta_i} \star y_\lambda \rangle_{i<\tau}$ also converges to $x_\lambda \star y_\tau$. By (2), we may assume that $\beta_i \neq \beta_j$ for distinct $i, j \in \tau$. Therefore, $x_\lambda \star y_\tau$ is a complete accumulation point for the set $X \star y_\lambda = \{x_\alpha \star y_\lambda : \alpha < \tau\}$, contradicting the fact that X is closed in G. ∎

LEMMA 2.4. *Let G be a topological group and $X = \{x_\alpha : \alpha < \tau\}, Y = \{y_\alpha : \alpha < \tau + 1\} \subset G$ closed and homeomorphic to τ and $\tau + 1$, respectively,* *where τ is an infinite regular cardinal. Then for any $\lambda < \tau$ there exists $\lambda^* < \tau$ such that $x_\lambda \star y_\lambda \neq x_\alpha \star y_\beta$ for any $\alpha, \beta \geq \lambda^*$.*

Proof. Assume the contrary. Then we can select non-decreasing sequences $\langle \alpha_i \rangle_{i<\tau}$ of elements of τ and $\langle \beta_i \rangle_{i<\tau}$ of elements of $\tau + 1$ such that

(1) $x_\lambda \star y_\lambda = x_{\alpha_i} \star y_{\beta_i}$;
(2) $\langle \alpha_i \rangle_{i<\tau}$ and $\langle \beta_i \rangle_{i<\tau}$ converge to τ.

By (1), the sequence $\langle x_{\alpha_i} \star y_{\beta_i} \rangle_{i<\tau}$ converges to $x_\lambda \star y_\lambda$. By (2), $\langle y_{\beta_i} \rangle_{i<\tau}$ converges to y_τ. Therefore, $\langle (x_{\alpha_i} \star y_{\beta_i}) \star y_{\beta_i}^{-1} \rangle_{i<\tau}$ converges to $(x_\lambda \star y_\lambda) \star y_\tau^{-1}$. By (2), we may assume that $\alpha_i \neq \alpha_j$ for distinct $i, j < \tau$. Therefore, $(x_\lambda \star y_\lambda) \star y_\tau^{-1}$ is a complete accumulation point for the set $\{x_{\alpha_i} : i < \tau\}$, contradicting the fact that X is closed in G. ∎

Lemma 2.5. *Let G be a topological group and $X, Y \subset G$ closed and homeomorphic to τ and $\tau+1$, respectively, where τ is an uncountable regular cardinal. Then there exist $X', Y' \subset G$ closed and homeomorphic to τ and $\tau + 1$, respectively, such that $\star$ is one-to-one on $X' \times Y'$.*

Proof. Put $X = \{x_\alpha : \alpha < \tau\}$ and $Y = \{y_\alpha : \alpha < \tau + 1\}$. We will construct $X' = \{x_{\lambda_\alpha} : \alpha < \tau\}$ and $Y' = \{y_{\lambda_\alpha} : \alpha < \tau + 1\}$ inductively.

Assume that for all $\beta < \alpha < \tau$, we have defined λ_β and λ_β^* that meet the following conditions:

(1) $\lambda_\gamma^* < \lambda_\beta^* < \tau$ for $\gamma < \beta$;
(2) $\lambda_\beta = \sup\{\lambda_\gamma^* : \gamma < \beta\}$ for $\beta > 0$;
(3) $\lambda_\beta < \lambda_\beta^*$;
(4) $x_{\lambda_\beta} \star y \neq z \star w$ if $z \in \{x_\gamma : \lambda_\beta^* \leq \gamma < \tau\}$ and $y, w \in \{y_\gamma : \lambda_\beta^* \leq \gamma < \tau + 1\}$;
(5) $x \star y_{\lambda_\beta} \neq z \star w$ if $x, z \in \{x_\gamma : \lambda_\beta^* \leq \gamma < \tau\}$ and $w \in \{y_\gamma : \lambda_\beta^* \leq \gamma < \tau + 1\}$;
(6) $x_{\lambda_\beta} \star y \neq z \star w_{\lambda_\beta}$ if $z \in \{x_\gamma : \lambda_\beta^* \leq \gamma < \tau\}$ and $y \in \{y_\gamma : \lambda_\beta^* \leq \gamma < \tau + 1\}$;
(7) $x_{\lambda_\beta} \star y_{\lambda_\beta} \neq z \star w$ if $z \in \{x_\gamma : \lambda_\beta^* \leq \gamma < \tau\}$ and $w \in \{y_\gamma : \lambda_\beta^* \leq \gamma < \tau + 1\}$.

If $\alpha = 0$ put $\lambda_\alpha = 0$. If $\alpha > 0$ put $\lambda_\alpha = \sup\{\lambda_\beta^* : \beta < \alpha\}$. Note that λ_α is limit if α is limit, and $\lambda_\alpha = \lambda_\gamma^*$ if $\alpha = \gamma + 1$. Let λ_α^* be an ordinal strictly between λ_α and τ that meets requirements (4)–(7). Such an ordinal exists by Lemmas 2.1–2.4. Put $\lambda_\tau = \tau$. Put $X' = \{x_{\lambda_\alpha} : \alpha < \tau\}$ and $Y' = \{y_{\lambda_\alpha} : \alpha < \tau + 1\}$. By (2), X' and Y' are closed in X and Y, respectively, and therefore in G. As τ is a regular cardinal, X' and Y' are homeomorphic to τ and $\tau+1$, respectively. To finish the proof we need to show that $x_{\lambda_{\alpha_1}} \star y_{\lambda_{\beta_1}} \neq x_{\lambda_{\alpha_2}} \star y_{\lambda_{\beta_2}}$ whenever $\langle \alpha_1, \beta_1 \rangle \neq \langle \alpha_2, \beta_2 \rangle$. We have a number of cases to consider.

- Case $[\alpha_1 < \min\{\beta_1, \alpha_2, \beta_2\}$ or $\alpha_2 < \min\{\alpha_1, \beta_1, \beta_2\}]$. Assume α_1 is the smallest. By (1)–(3), $\lambda_{\beta_1}, \lambda_{\alpha_2}, \lambda_{\beta_2} \geq \lambda^*_{\alpha_1}$. Now apply (4).
- Case $[\beta_1 < \min\{\alpha_1, \alpha_2, \beta_2\}$ or $\beta_2 < \min\{\alpha_1, \beta_1, \alpha_2\}]$. Apply (5).
- Case $[\alpha_1 = \beta_2 < \min\{\beta_1, \alpha_2\}$ or $\alpha_2 = \beta_1 < \min\{\alpha_1, \beta_2\}]$. Apply (6).
- Case $[\alpha_1 = \beta_1 < \min\{\alpha_2, \beta_2\}$ or $\alpha_2 = \beta_2 < \min\{\alpha_1, \beta_1\}]$. Apply (7).
- Case $[\alpha_1 = \alpha_2 \leq \min\{\beta_1, \beta_2\}$ or $\beta_1 = \beta_2 \leq \min\{\alpha_1, \alpha_2\}]$. Apply the fact that multiplication by a scalar is an automorphism. ∎

THEOREM 2.6. *Let G be a topological group and τ an uncountable regular cardinal. If τ and $\tau + 1$ are embeddable in G as closed subspaces then $\tau \times (\tau + 1)$ is embeddable in G as a closed subspace.*

Proof. By Lemma 2.5, there exist disjoint and closed $X = \{x_\alpha : \alpha < \tau\}$, $Y = \{y_\alpha : \alpha < \tau + 1\} \subset G$ such that $\star$ is one-to-one on $X \times Y$. To reach the conclusion of the theorem, it suffices to show that $\star|_{X\times Y}$ is a closed map and $X \star Y$ is closed in G. Assume the conclusion is not true. This means that there exists a point in the remainder of the Stone–Čech compactification $\beta(X \times Y) = \beta X \times Y$ that is glued up with a point in G under the continuous extension of $\star|_{X\times Y}$ over $\beta(X \times Y)$. This point from the remainder must be the complete accumulation point for $X \times \{y_\alpha\}$ in $\beta X \times Y$, for some $\alpha \leq \tau$. That is, $\star|_{X\times\{y_\alpha\}}$ is not a closed map or $X \star y_\alpha$ is not closed in G. Since multiplication by a scalar is an automorphism and X is closed in G, this cannot happen. ∎

COROLLARY 2.7. *Let G be a topological group and τ an uncountable regular cardinal. If τ and $\tau + 1$ embed in G as closed subspaces then G is not normal.*

3. Topological groups generated by $\tau + 1$. In the next section we will prove that if an Abelian topological group contains an ordinal τ, where τ is a regular cardinal or the immediate successor of such, then G contains $\tau \times \tau$ provided the order of every non-neutral element of G is greater than 3. In this section we will justify this requirement on the order. A particular case of a more general result of this section states that if an Abelian group G is algebraically generated by $\omega_1 + 1$ and $o(G) \leq 3$ then $(\omega_1 + 1) \times (\omega_1 + 1)$ is not embeddable in G.

LEMMA 3.1. *Let G be an Abelian topological group algebraically generated by $\tau + 1 \subset G$ (or by $\tau \subset G$), where τ is an uncountable regular cardinal. Suppose $X = \{x_\alpha : \alpha < \tau\} \subset G$ is homeomorphic to τ. Then there exist a non-negative integer K_X, a τ-sized $J_X \subset \tau$, and $g_X \in G$ such that for any $\alpha \in J_X$, $x_\alpha = g_X \star c_\alpha$, where c_α can be represented as the $\star$-product of K_X-many ordinals in the range $[\alpha, \tau)$.*

Proof. We will carry out the proof for the case when G is algebraically generated by $\tau+1$. For each $\alpha<\tau$ fix a triple $\langle A_\alpha, B_\alpha, C_\alpha\rangle$ of non-negative integers and a triple $\langle a_\alpha, b_\alpha, c_\alpha\rangle$ of elements of G such that

(1) $x_\alpha = a_\alpha \star b_\alpha \star c_\alpha$;
(2) a_α can be represented as the $\star$-product of A_α-many ordinals strictly less than α;
(3) $b_\alpha = \tau^{B_\alpha}$;
(4) c_α can be represented as the $\star$-product of C_α-many ordinals in the range $[\alpha,\tau)$.

Since G is Abelian and algebraically generated by $\tau+1$, such triples exist.

Since τ is regular and uncountable, there exist stationary $J_0\subset\tau$ and a triple $\langle A,B,C\rangle$ such that $\langle A_\alpha, B_\alpha, C_\alpha\rangle = \langle A,B,C\rangle$ for any $\alpha\in J_0$. For every $\alpha\in J_0$, fix an A-long sequence $\langle a_\alpha(i) : i\in A\rangle$ of ordinals less than α such that $a_\alpha = a_\alpha(0)\star\cdots\star a_\alpha(A-1)$. Such sequences exist by (2). Define a regressive function $f_0 : J_0\to\tau$ by letting $f_0(\alpha) = a_\alpha(0)$. Since J_0 is stationary, τ is an uncountable regular cardinal, and f_0 is regressive, we can apply the pressing-down lemma (see, for example, [KUN]). Applying the pressing-down lemma to J_0, τ, and f_0, we find stationary $J_1\subset J_0$ such that $a_\alpha(0) = a_\beta(0)$ for any $\alpha,\beta\in J_1$. Similarly, for each $0<i<A$, we find a stationary $J_i\subset J_{i-1}$ such that $a_\alpha(i) = a_\beta(i)$ for any $\alpha,\beta\in J_i$. Put $J_X = J_{A-1}$. It is clear that J_X is τ-sized and $a_\alpha = a_\beta = a$ for any $\alpha,\beta\in J_X$ and some fixed $a\in G$. Put $g_X = a\star\tau^B$ and $K_X = C$. By (1) and (4), g_X, K_X, and J_X are as desired. ∎

In the next statement we will use a classical fact of algebra that if G is Abelian and $o(G)=N$ (that is, the maximum order is N) then $o(g)$ divides N for every $g\in G$.

LEMMA 3.2. *Let G be an Abelian topological group algebraically generated by $\tau+1\subset G$ (or by $\tau\subset G$), where τ is an uncountable regular cardinal. Suppose $X=\{x_\alpha : \alpha<\tau\}\subset G$ is homeomorphic to τ. Then there exist a natural number K_X, an unbounded closed subset I_X of τ, and $g_X\in G$ such that $x_\alpha = g_X\star\alpha^{K_X}$ for every $\alpha\in I_X$. If, additionally, $o(G)=N$, then K_X can be chosen strictly between 0 and N.*

Proof. We will consider the case when G is algebraically generated by $\tau+1$. Let g_X, J_X, and K_X be as in the conclusion of Lemma 3.1.

CLAIM. *There exists a τ-sized $I\subset\tau$ such that $x_\lambda = g_X\star\lambda^{K_X}$ for any $\lambda\in I$.*

To prove the claim, fix any $\alpha<\tau$. We need to find $\lambda\geq\alpha$ such that $x_\lambda = g_X\star\lambda^{K_X}$. For this fix a strictly increasing sequence $\langle\alpha_n\rangle_n$ of ordinals in $J_X\cap[\alpha,\tau)$ such that $x_{\alpha_n} = g_X\star x_{n,1}\star\cdots\star x_{n,K_X}$, where $\alpha_n\leq x_{n,i}<$

α_{n+1}. Such a sequence exists by Lemma 3.1. Since $\alpha < \alpha_n < \tau$, we have $\lim_{n\to\infty} \alpha_n = \lambda > \alpha$. By continuity of $\star$, $\lim_{n\to\infty} g_X \star x_{n,1} \star \cdots \star x_{n,K_X} = g_X \star \lambda^{K_X}$. Since $x_\gamma \leftrightarrow \gamma$ is a homeomorphism of X with τ, and $\alpha_n \to \lambda$, we have $x_{\alpha_n} \to x_\lambda$. Thus, $x_\lambda = g_X \star \lambda^{K_X}$. The claim is proved.

Let I_X be the closure of I in τ. By the same argument as in the Claim, we can show that $x_\alpha = g_X \star \alpha^{K_X}$ for all $\alpha \in I_X$.

To show that $K_X > 0$, assume the contrary. Then $x_\alpha = g_X \star \alpha^{K_X} = g_X$ for all $\alpha \in I$. This contradicts the fact that $x_\alpha \leftrightarrow \alpha$ is a homeomorphism of X with τ.

Finally, let $o(G) = N$. Since $g^N = e_G$ for all $g \in G$, we can assume without loss of generality that $K_X < N$. ∎

PROPOSITION 3.3. *Let G be an Abelian topological group algebraically generated by $\tau+1 \subset G$, where τ is an uncountable regular cardinal. Suppose $o(G) \le N$ and $S_1, \ldots, S_N \subset G$ are homeomorphic to τ. If c is the complete accumulation point for each of $S_1, \ldots, S_N$ then $S_1, \ldots, S_N$ are not disjoint.*

Proof. Suppose the conclusion is true for all natural numbers less than N. Let $o(G) = N$. For each $i = 1, \ldots, N$, let $I_{S_i}, K_{S_i}, g_{S_i}$ be as in the conclusion of Lemma 3.2. Since $0 < K_{S_i} < N$ for each i, there exist distinct $i, j \in \{1, \ldots, N\}$ such that $K_{S_i} = K_{S_j} = K$. By continuity of $\star$, we have $c = g_{S_i} \star \tau^{K_{S_i}} = g_{S_j} \star \tau^{K_{S_j}}$. Therefore, $g_{S_i} = g_{S_j} = g^*$. Since I_{S_i}, I_{S_j} are τ-sized countably compact subspaces of τ, there exists $\alpha \in I_{S_i} \cap I_{S_j}$. Therefore, $g^* \star \alpha^K \in S_i \cap S_j$. ∎

COROLLARY 3.4. *Let G be an Abelian group algebraically generated by $\tau+1 \subset G$, where τ is an uncountable regular cardinal. If $o(G) \le 2^N - 1$ then $\prod_{i\in N}(\tau+1)$ is not embeddable in G.*

Proof. Observe that the topological power of N copies of $\tau+1$ contains $2^N - 1$ disjoint copies of τ that converge to the corner point $\langle \tau, \ldots, \tau \rangle$. Now apply Proposition 3.3. ∎

The results of this section give rise to the following questions.

QUESTION 3.5. *Let G be an Abelian group generated by ω_1 and $o(G) \le 3$. Is it true that $\omega_1 \times \omega_1$ is not embeddable in G?*

QUESTION 3.6. *Can Proposition* 3.3 *and/or Corollary* 3.4 *be proved without assuming commutativity?*

4. Topological powers of ordinals in topological groups. In several lemmas we will deal with the expression $P = P(\langle x_i : i \in N \rangle) = x_{N-1}^{2^{N-1}} \star \cdots \star x_0^{2^0}$. If $R \subset S \subset \{0, \ldots, N-1\}$, then by $P(S, R, \lambda, \langle \alpha_i : i \in S \setminus R \rangle)$

we denote the value of $P(\langle x_i : i \in N\rangle)$ when

$$x_i = \begin{cases} e_G & \text{if } i \in N \setminus S, \\ \lambda & \text{if } i \in R, \\ \alpha_i & \text{if } i \in S \setminus R. \end{cases}$$

For example, if $N = 4$ then $P(\{0,2,3\}, \{0,3\}, \lambda, \langle x_2\rangle) = \lambda^{2^3} \star x_2^{2^2} \star e_G^{2^1} \star \lambda^{2^0} = \lambda^8 \star x_2^4 \star \lambda$.

In the next lemma we will use the following arithmetical facts.

(1) $2^{N-1} + \cdots + 2^0 = 2^N - 1$.
(2) Let $A = a_n 2^n + \cdots + a_0 2^0$ and $B = b_n 2^n + \cdots + b_0 2^0$, where $a_i, b_i \in \{0,1\}$. If $A = B$ then $a_i = b_i$ for all i.

To prove our main results of this section (Theorems 4.7 and 4.8) we will follow the strategy of Section 2. The analogues of Lemmas 2.1–2.4 are particular cases of Lemma 4.1.

LEMMA 4.1. *Suppose G is an Abelian topological group, the order of every non-neutral element of G is greater than $2^N - 1$, and $\tau \subset G$ is an uncountable regular cardinal. Suppose $\lambda < \tau$, $S \subset \{0, \ldots, N-1\}$, and $L, R \subset S$ are distinct. Then there exists $\lambda^* < \tau$ such that*

$$P(S, L, \lambda, \langle x_i : i \in S \setminus L\rangle) \neq P(S, R, \lambda, \langle y_i : i \in S \setminus R\rangle)$$

whenever $x_i, y_j \in \tau \setminus \lambda^*$.

Proof. Assume the conclusion of the lemma is not true. Then we can find strictly increasing sequences $\langle \alpha_{n,i}\rangle_n$, $\langle \beta_{n,j}\rangle_n$ for $i \in S \setminus L$ and $j \in S \setminus R$, all converging to the same limit γ, such that

P1. $\lambda < \gamma < \tau$;
P2. $P(S, L, \lambda, \langle \alpha_{n,i} : i \in S \setminus L\rangle) = P(S, R, \lambda, \langle \beta_{n,i} : i \in S \setminus R\rangle)$ for all n.

Let $L_\lambda = i_{N-1} 2^{N-1} + \cdots + i_0 2^0$, where $i_k = 0$ if $k \notin L$ and $i_k = 1$ if $k \in L$. That is, L_λ is the exponent of λ on the left side. Similarly, we define R_λ. Let $E = j_{N-1} 2^{N-1} + \cdots + j_0 2^0$, where $j_k = 1$ if $k \in N \setminus S$ and $j_k = 0$ if $k \in S$. That is, E is the exponent of e_G.

By continuity and commutativity of $\star$, we have

$$\lim_{n\to\infty} P(S, L, \lambda, \langle \alpha_{n,i} : i \in S \setminus L\rangle) = \lambda^{L_\lambda} \star \gamma^{2^N - 1 - E - L_\lambda},$$
$$\lim_{n\to\infty} P(S, R, \lambda, \langle \beta_{n,i} : i \in S \setminus R\rangle) = \lambda^{R_\lambda} \star \gamma^{2^N - 1 - E - R_\lambda}.$$

By P2, we have $\lambda^{L_\lambda} \star \gamma^{2^N - 1 - E - L_\lambda} = \lambda^{R_\lambda} \star \gamma^{2^N - 1 - E - R_\lambda}$. By the above mentioned arithmetical facts, if $R_\lambda = L_\lambda$ then $L = R$, contradicting the hypothesis. Therefore, we may assume that $L_\lambda > R_\lambda$. Then we have $\lambda^{L_\lambda - R_\lambda} = \gamma^{L_\lambda - R_\lambda}$, that is, $(\lambda \star \gamma^{-1})^{L_\lambda - R_\lambda} = e_G$. Thus, $o(\lambda \star \gamma^{-1}) \le L_\lambda - R_\lambda \le 2^N - 1$. Therefore, $\lambda \star \gamma^{-1} = e_G$ and $\lambda = \gamma$, contradicting P1. ∎

LEMMA 4.2. *Suppose G is an Abelian topological group, the order of every non-neutral element of G is greater than 2^N-1, and $\tau+1\subset G$, where τ is an infinite regular cardinal. Suppose $\lambda<\tau$, $S\subset\{0,\dots,N-1\}$, and $L,R\subset S$ are distinct. Then there exists $\lambda^*<\tau$ such that*

$$P(S,L,\lambda,\langle x_i : i\in S\setminus L\rangle)\neq P(S,R,\lambda,\langle y_i : i\in S\setminus R\rangle)$$

whenever $x_i,y_j\in(\tau+1)\setminus\lambda^*$.

Proof. Assume the conclusion of the lemma is not true. Then we can find non-decreasing sequences $\langle\alpha_{\gamma,i}\rangle_\gamma$, $\langle\beta_{\gamma,j}\rangle_\gamma$ for $i\in S\setminus L$ and $j\in S\setminus R$, all converging to τ, such that

$$P(S,L,\lambda,\langle\alpha_{\gamma,i} : i\in S\setminus L\rangle)=P(S,R,\lambda,\langle\beta_{\gamma,i} : i\in S\setminus R\rangle)$$

for all γ. The rest of the argument is as in Lemma 4.1. ∎

We denote by $\star^{N-1}$ the map from $\prod_{i\in N}G$ to G defined by

$$\star^{N-1}(\langle x_0,\dots,x_{N-1}\rangle)=x_0\star\dots\star x_{N-1}.$$

LEMMA 4.3. *Suppose G is an Abelian topological group, the order of every non-neutral element of G is greater than 2^N-1, and $\tau\subset G$ is an uncountable regular cardinal. Then there exists $T\subset\tau$ homeomorphic to τ such that $\star^{N-1}$ is one-to-one on $\prod_{i\in N}\{t^{2^i} : t\in T\}$.*

Proof. We will construct $T=\{\lambda_\alpha:\alpha<\tau\}$ inductively. Assume that for all $\beta<\alpha$, we have defined λ_β and λ^*_β that meet the following conditions:

(1) $\lambda^*_\gamma<\lambda^*_\beta<\tau$ for all $\gamma<\beta$;
(2) $\lambda_\beta=\sup\{\lambda^*_\gamma:\gamma<\beta\}$;
(3) $\lambda_\beta<\lambda^*_\beta$;
(4) $P(S,L,\lambda_\beta,\langle x_i : i\in S\setminus L\rangle)\neq P(S,R,\lambda_\beta,\langle y_i : i\in S\setminus R\rangle)$ whenever $x_i,y_j\geq\lambda^*_\beta$, $S\subset N$, and $L,R\subset S$ are distinct.

If $\alpha=0$ put $\lambda_\alpha=0$. If $\alpha>0$ put $\lambda_\alpha=\sup\{\lambda^*_\beta:\beta<\alpha\}$. Let λ^*_α be any ordinal strictly between λ_α and τ that meets condition (4). Such an ordinal exists by Lemma 4.1.

By (1)–(3), $T=\{\lambda_\alpha:\alpha<\tau\}$ is homeomorphic to τ. To finish the proof we need to show that $P(\langle x_i : i\in N\rangle)\neq P(\langle y_i : i\in N\rangle)$ whenever $\langle x_i : i\in N\rangle\neq\langle y_i : i\in N\rangle$ and $x_i,y_i\in T$.

To prove this, let $S\subset N$ be the set of all indices i such that $x_i\neq y_i$. Put $\lambda=\min\{x_i,y_i : i\in S\}$. Let $L\subset S$ be the set of all indices i such that $x_i=\lambda$. Let $R\subset S$ be the set of all indices i such that $y_i=\lambda$. Since λ exists, R or L is not empty. Then $R\neq L$, since otherwise R and L would be in the complement of S. By the definition of S, $x_i=y_i$ for all $i\in N\setminus S$. Therefore, the equality $P(\langle x_i : i\in N\rangle)=P(\langle y_i : i\in N\rangle)$ is equivalent to $P(S,L,\lambda,\langle x_i : i\in S\setminus L\rangle)=P(S,R,\lambda,\langle y_i : i\in S\setminus R\rangle)$. By the definition of λ, if $i\in S\setminus R$ then $y_i>\lambda$. By (1)–(3), $y_i\geq\lambda^*$. Similarly, $x_i\geq\lambda^*$ for all $i \in S \setminus L$. Applying (4), we conclude that $P(S, L, \lambda, \langle x_i : i \in S \setminus L\rangle) \neq P(S, R, \lambda, \langle y_i : i \in S \setminus R\rangle)$. ∎

LEMMA 4.4. *Suppose G is an Abelian topological group, the order of every non-neutral element of G is greater than $2^N - 1$, and $\tau + 1 \subset G$, where τ is an infinite regular cardinal. Then there exists $T \subset \tau + 1$ homeomorphic to $\tau + 1$ such that $\star^{N-1}$ is one-to-one on $\prod_{i\in N}\{t^{2^i} : t \in T\}$.*

Proof. The proof is word by word that of Lemma 4.3 except that $T = \{\lambda_\beta : \beta < \tau+1\}$, where $\lambda_\tau = \tau$. For $\beta < \tau$, λ_β is constructed as in Lemma 4.3 with reference to Lemma 4.2 instead of Lemma 4.1. ∎

To state and prove our next statement we need a bit more terminology. Let τ be an uncountable regular cardinal. A set $L \subset \prod_{i\in N} \tau$ is a *τ-line* if there exist a non-empty subset $S \subset N$ and a sequence $\langle \lambda_i : i \in N \setminus S\rangle$ such that $L = \{\mathbf{x}_\alpha : \alpha < \tau\}$, where $\mathbf{x}_\alpha(i) = \alpha$ if $i \in S$ and $\mathbf{x}_\alpha(i) = \lambda_i$ if $i \in N \setminus S$. If a space X is a continuous one-to-one image of $\prod_{i\in N} \tau$ then τ-lines in X are the images of τ-lines in $\prod_{i\in N} \tau$ under the map under consideration.

PROPOSITION 4.5. *Let f be a continuous one-to-one function on $\prod_{i\in N} \tau$, where τ is an uncountable regular cardinal. Suppose f is a homeomorphism on every τ-line. Then there exists $\lambda < \tau$ such that f is a homeomorphism on $\prod_{i\in N}(\tau \setminus \lambda)$.*

Proof. Assume the contrary and let $\widetilde{f}$ be the continuous extension of f over $\prod_{i\in N}(\tau + 1)$. Assume we have defined $\alpha_{k-1} < \tau$, $\mathbf{x}_{k-1} \in [\prod_{i\in N}(\tau + 1 \setminus \alpha_{k-1})] \setminus [\prod_{i\in N}(\tau \setminus \alpha_{k-1})]$, and $\mathbf{p}_{k-1} \in \prod_{i\in N}(\tau \setminus \alpha_{k-1})$.

Step $k < \omega$. Let α_k be any ordinal below τ and above the maximum of the set $\{\mathbf{x}_{k-1}(0), \mathbf{p}_{k-1}(0), \ldots, \mathbf{x}_{k-1}(N-1), \mathbf{p}_{k-1}(N-1)\} \setminus \{\tau\}$. Since $\widetilde{f}$ is one-to-one and not a homeomorphism on $\prod_{i\in N}(\tau \setminus \alpha_k)$, there exists $\mathbf{x}_k \in [\prod_{i\in N}(\tau + 1 \setminus \alpha_k)] \setminus [\prod_{i\in N}(\tau \setminus \alpha_k)]$ such that $\widetilde{f}(\mathbf{x}_k) = f(\mathbf{p}_k)$ for some $\mathbf{p}_k \in \prod_{i\in N}(\tau \setminus \alpha_k)$.

We may assume that there exists a non-empty subset T of N such that for any k, $\mathbf{x}_k(i) = \tau$ iff $i \in T$. Put $p = \lim_{n\to\infty} \alpha_n$. By construction, $\lim_{n\to\infty} \mathbf{p}_n(i) = p$ for all $i \in N$ and $\lim_{n\to\infty} \mathbf{x}_n(j) = p$ for all $j \in N \setminus T$. We have $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$, where $\mathbf{x}(i) = \tau$ if $i \in T$ and $\mathbf{x}(i) = p$ if $i \in N \setminus T$. Put $\mathbf{p} = \langle p : i \in N\rangle$. Then $\mathbf{p}$ belongs to the τ-line $L = \{\mathbf{y}_\alpha : \alpha < \tau\}$, where $\mathbf{y}_\alpha(i)$ is τ if $i \in T$ and p otherwise. The point $\mathbf{x}$ belongs to the closure of L in $\prod_{i\in N}(\tau + 1)$. By continuity of $\widetilde{f}$, $\widetilde{f}(\mathbf{x}) = f(\mathbf{p})$. Therefore, f is not a homeomorphism on L, contradicting the hypothesis. ∎

LEMMA 4.6. *Suppose G is an Abelian topological group, the order of every non-neutral element of G is greater than $N > 1$, and $\tau \subset G$ is an uncountable regular cardinal. Then there exists $T \subset \tau$ such that $\{\alpha^N : \alpha \in T\}$ is homeomorphic to τ.*

Proof. Since the order of every non-neutral element is greater than N, $\alpha^N \neq \beta^N$ for distinct $\alpha, \beta \in \tau$. Therefore, $\{\alpha^N : \alpha < \tau\}$ is a continuous one-to-one image of the diagonal $\Delta = \{\langle \alpha : i \in N\rangle : \alpha < \tau\}$ of $\prod_{i\in N} \tau$ under $\star^{N-1}$. If $\star^{N-1}$ is a homeomorphism on Δ then we are done. Otherwise, there exists $\lambda < \tau$ such that $f(\langle \lambda : i \in N\rangle) = f(\langle \tau : i \in N\rangle)$, where f is the continuous extension of $\star^{N-1}$ over $\prod_{i\in N}(\tau + 1)$. Then $\{\alpha^N : \lambda < \alpha < \tau\}$ is naturally homeomorphic to τ. Put $T = \tau \setminus (\lambda + 1)$. ∎

THEOREM 4.7. *Let G be an Abelian topological group and $\tau \subset G$ an infinite regular cardinal. If the order of every non-neutral element of G is greater than $2^N - 1$ then $\prod_{i\in N} \tau$ is embeddable in G.*

Proof. If $\tau = \omega$ then the conclusion holds since $\prod_{i\in N} \omega$ is homeomorphic to ω. Assume τ is uncountable. By Lemma 4.3, we may assume that $\star^{N-1}$ is one-to-one on $X = \prod_{i\in N}\{\alpha^{2^i} : \alpha < \tau\}$. By Proposition 4.5, it suffices to show that $\star^{N-1}$ is a homeomorphism on every τ-line of X. If L is a τ-line in X then there exist a non-empty $K \subset N$ and $\langle g_i : i \in N \setminus K\rangle$ such that $L = \{\mathbf{x}_\alpha : \alpha < \tau\}$, where $\mathbf{x}_\alpha(i) = \alpha^{2^i}$ if $i \in K$ and $\mathbf{x}_\alpha(i) = g_i^{2^i}$ if $i \in N \setminus K$. Put

$$E = \sum_{i\in N} e_i 2^i, \qquad \text{where } e_i = 1 \text{ if } i \in K \text{ and } e_i = 0 \text{ if } i \in N \setminus K;$$

$$g^* = g_{N-1}^{2^{N-1}} \star \cdots \star g_0^{2^0}, \qquad \text{where } g_i \text{ is } e_G \text{ if } i \in K \text{ and is as above if } i \in N \setminus K.$$

Then $\star^{N-1}(L) = \{g^* \star \alpha^E : \alpha < \tau\}$. By Lemma 4.6, we may assume that $\alpha^E \leftrightarrow \alpha$ is a homeomorphic correspondence of $\{\alpha^E : \alpha < \tau\}$ with τ. Since multiplication by the scalar g^* is a homeomorphism of G with itself, we conclude that $\star^{N-1}$ is a homeomorphism on L. ∎

THEOREM 4.8. *Let G be an Abelian topological group and $\tau + 1 \subset G$, where τ an infinite regular cardinal. If the order of every non-neutral element of G is greater than $2^N - 1$ then $\prod_{i\in N} \tau$ is embeddable in G.*

Proof. By Lemma 4.4, we may assume that $\star^{N-1}$ is one-to-one on $X = \prod_{i\in N}\{\alpha^{2^i} : \alpha < \tau + 1\}$. Apply the facts that X is compact and is homeomorphic to $\prod_{i\in N}(\tau + 1)$. ∎

For our next discussion, a *torsion free group* is one in which the order of every non-neutral element is infinite.

COROLLARY 4.9. *Let G be a torsion free Abelian group. If ω_1 embeds in G then G is not hereditarily normal.*

Proof. By Theorem 4.7, $\omega_1 \times \omega_1$ embeds in G. Since the former is not hereditarily normal we are done. ∎

Our results naturally lead to the following question.

Question 4.10. *Let X be a countably compact non-compact space. Suppose X and βX embed in a topological group G as closed subspaces. Is G not normal? What if "countably compact" is replaced by "non-paracompact topological space"?*

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