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On coarse embeddability into ℓ_p-spaces and a conjecture of Dranishnikov

by

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Abstract. We show that the Hilbert space is coarsely embeddable into any ℓ_p for $1 \leq p \leq \infty$. It follows that coarse embeddability into ℓ_2 and into ℓ_p are equivalent for $1 \leq p < 2$.

Coarse embeddings were defined by M. Gromov [Gr, 7.E_2] to express the idea of inclusion in the large scale geometry of groups. G. Yu showed later that the case when a finitely generated group with a word length metric is being embedded into the Hilbert space is of great importance in solving the Novikov Conjecture [Yu], while recent work of G. Kasparov and G. Yu [KY] treats the case when the Hilbert space is replaced with just a uniformly convex Banach space. Due to these remarkable theorems coarse embeddings gain a great deal of attention, but still embeddability into the Hilbert, and more generally Banach spaces, is not entirely understood with many question remaining open.

In this context the class of ℓ_p-spaces seems to be particularly interesting. Their embeddability into the Hilbert space is known—ℓ_p admits such an embedding when $0 < p \leq 2$ but does not if $p > 2$ due to a recent result of W. Johnson and N. Randrianarivony [JR]. In this note we study the opposite situation, i.e. we show that the separable Hilbert space embeds into ℓ_p for any $1 \leq p \leq \infty$. As a consequence we obtain a new characterization of embeddability into ℓ_2, namely that the properties of embeddability into ℓ_p for $1 \leq p \leq 2$ are all equivalent.

In [GK, Section 6] the authors advertised a conjecture stated by A. N. Dranishnikov [Dr, Conjecture 4.4]: *a discrete metric space has Property A if and only if it admits a coarse embedding into the space* ℓ_1. The results presented in

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this note show that this is the same as asking whether Property A is equivalent to embeddability into the Hilbert space, and although it is a folk conjecture that such statement is not true, no example distinguishing between the two is known.

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L_p-spaces and the Mazur map. In everything what follows we consider only separable $L_p(\mu)$-spaces and we will specialize to the most interesting case of the spaces ℓ_p; the case of $L_p(\mu)$ for other, including non-separable, measures follows easily and is left to the reader. We use the standard notation $\ell_p = \ell_p(\mathbb{N})$ and we denote by $S(X)$ the unit sphere in the Banach space X.

The *Mazur map* $M_{p,q} : S(\ell_p) \to S(\ell_q)$ is defined by the formula

$$M_{p,q}(x) = \{|x_i|^{p/q} \operatorname{sign} x_i\}_{i=1}^{\infty}$$

where $x = \{x_i\}_{i=1}^{\infty} \in \ell_p$. It is a uniform homeomorphism between unit spheres of ℓ_p-spaces. More precisely, for some C depending only on p/q it satisfies the inequalities

$$\frac{p}{q}\,\|x-y\|_p \le \|M_{p,q}(x) - M_{p,q}(y)\|_q \le C\|x-y\|_p^{p/q} \tag{1}$$

for all $x, y \in S(\ell_p)$ and $p < q$, and the opposite inequalities if $p > q$ (note that $M_{p,q} = M_{q,p}^{-1}$). For the proof of these estimates and details on the Mazur map and its applications we refer the reader to [BL, Chapter 9.1].

If $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of Banach spaces, we denote by $(\sum X_n)_p$ the direct sum of X_n with the p-norm, i.e.

$$\Big(\sum_{n=1}^{\infty} X_n\Big)_p = \Big\{\mathbf{x} = \{x_n\}_{n\in\mathbb{N}} : x_n \in X_n,\ \sum_{n=1}^{\infty} \|x_n\|^p < \infty\Big\},$$

$$\|\mathbf{x}\|_p = \Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{1/p}.$$

Clearly, ℓ_p is isometric to $(\sum \ell_p)_p$.

We will also need the following classification of separable L_p-spaces.

THEOREM 1 (see e.g. [Wo, III.A]). *A separable space $L_p(\mu)$ is isometric to one of the following spaces: ℓ_p^n for $n = 1, 2, \ldots$, $L_p[0,1]$, ℓ_p, $(L_p[0,1] \oplus \ell_p^n)_p$ for $n = 1, 2, \ldots$, $(L_p[0,1] \oplus \ell_p)_p$.*

A condition for coarse embeddability. We recall the definition of a coarse embedding.

DEFINITION 1. Let X, Y be metric spaces. A map $f\colon X \to Y$ is a *coarse embedding* if there exist non-decreasing functions $\varrho_1, \varrho_2\colon [0,\infty) \to [0,\infty)$ satisfying

(1) $\varrho_1(d_X(x,y)) \le d_Y(f(x),f(y)) \le \varrho_2(d_X(x,y))$ for all $x,y \in X$,
(2) $\lim_{t\to\infty} \varrho_1(t) = +\infty$.

In [DG] M. Dadarlat and E. Guentner characterized spaces coarsely embeddable into the Hilbert $\mathcal{H}$ space in terms of existence of maps into the unit sphere $S(\mathcal{H})$. Their result is a reminiscence of a characterization of *uniform embeddability* (meaning existence of a uniform homeomorphism onto a subset) into a Hilbert space obtained by Aharoni *et al.* in [AMM].

THEOREM 2 ([DG, Theorem 2.1]). *A metric space X admits a coarse embedding into the Hilbert space $\mathcal{H}$ if and only if for every $R > 0$ and $\varepsilon > 0$ there is a map $\varphi : X \to S(\mathcal{H})$ and $S > 0$ satisfying*

(1) $\sup\{\|\varphi(x) - \varphi(y)\|_{\mathcal{H}} : x,y \in X,\ d(x,y) \le R\} \le \varepsilon$,
(2) $\lim_{S\to\infty} \inf\{\|\varphi(x) - \varphi(y)\|_{\mathcal{H}} : x,y \in X,\ d(x,y) \ge S\} = \sqrt{2}$.

We are going to use this idea to prove a similar condition for embeddings into the spaces ℓ_p. The proof relies on the original proof of Theorem 2.

THEOREM 3. *Let X be a metric space and $1 \le p < \infty$. If there is a $\delta > 0$ such that for every $R > 0$, $\varepsilon > 0$ there is a map $\varphi : X \to S(\ell_p)$ satisfying*

(1) $\sup\{\|\varphi(x) - \varphi(y)\|_p : x,y \in X,\ d(x,y) \le R\} \le \varepsilon$,
(2) $\lim_{S\to\infty} \inf\{\|\varphi(x) - \varphi(y)\|_p : x,y \in X,\ d(x,y) \ge S\} \ge \delta$,

then X admits a coarse embedding into ℓ_p.

Proof. By the assumptions for every $n \in \mathbb{N}$ there is a map $\varphi_n : X \to S(\ell_p)$ and a number $S_n > 0$ such that $\|\varphi_n(x) - \varphi_n(y)\|_p \le 1/2^n$ whenever $d(x,y) \le n$, and $\|\varphi_n(x) - \varphi_n(y)\|_p \ge \delta/2$ whenever $d(x,y) \ge S_n$. Without loss of generality we can choose the sequence of S_n's to be strictly increasing and tending to infinity as $n \to \infty$.

Choose $x_0 \in X$ and define a map $\Phi : X \to \left(\sum \ell_p\right)_p$ by the formula

$$\Phi(x) = \bigoplus_{n=1}^{\infty} \left(\varphi_n(x) - \varphi_n(x_0)\right).$$

It is easy to see that

$$\|\Phi(x)\|_p^p = \sum_{n=1}^{\infty} \|\varphi_n(x) - \varphi_n(x_0)\|_p^p < \infty,$$

which shows that Φ is well defined.

We will show that Φ is a coarse embedding. Take $k \in \mathbb{N}$ and $\sqrt[p]{k-1} \le d(x,y) < \sqrt[p]{k}$. Then

$$\|\Phi(x)-\Phi(y)\|_p^p = \sum_{n=1}^{k-1} \|\varphi_n(x)-\varphi_n(y)\|_p^p + \sum_{n=k}^{\infty} \|\varphi_n(x)-\varphi_n(y)\|_p^p$$
$$\le 2^p(k-1) + \sum_{n=k}^{\infty} \frac{1}{2^{kp}} \le 2^p(k-1)+1 \le 2^p d(x,y)^p + 1.$$

The first estimate comes from the fact that unit vectors cannot be more than distance 2 apart.

On the other hand, for $S_{k-1} \le d(x,y) < S_k$ we have

$$\|\Phi(x)-\Phi(y)\|_p^p \ge \sum_{n=1}^{k-1} \|\varphi_n(x)-\varphi_n(y)\|_p^p \ge (k-1)\left(\frac{\delta}{2}\right)^p.$$

Thus we can choose $\varrho_1(t) = \sum_{n=1}^{\infty} \delta \sqrt[p]{n}\, \chi_{[S_{n-1},S_n)}(t)$, $\varrho_2(t) = 2t+1$ and it is clear that Φ is a coarse embedding. ∎

G. Yu defined Property A [Yu], which gives a sufficient condition for embeddability of a discrete metric space into a Hilbert space. We recall a characterization of Property A given by J. L. Tu.

PROPOSITION 1 ([Tu]). *A metric space X has property A if and only if for every $R > 0$ and $\varepsilon > 0$ there is a map $\eta : X \to S(\ell_2(X))$ and $S > 0$ such that*

(1) $\|\eta(x)-\eta(y)\|_2 \le \varepsilon$ *when* $d(x,y) \le R$;
(2) $\operatorname{supp} \eta(x) \subset B(x,S)$ *for all* $x \in X$.

Theorem 2 and the above characterization exhibit the subtle relation between Property A and coarse embeddability.

The following proposition shows that the property of Theorem 3 is not sensitive to changing the index p.

PROPOSITION 2. *Let X have the property described in Theorem 3 with respect to some $1 \le p < \infty$. Then X has the same property with respect to any $1 \le q < \infty$.*

Proof. For $R > 0$ and $\varepsilon > 0$, given a map $f_p : X \to S(\ell_p)$ which satisfies conditions (1) and (2) of Theorem 3 define $f_q : X \to S(\ell_q)$ by the formula

$$f_q(x) = M_{p,q}[f_p(x)],$$

where $M_{p,q} : S(\ell_p) \to S(\ell_q)$ is the Mazur map.

If $p < q$, by inequalities (1) we have

$$\frac{p}{q}\,\|f_p(x)-f_p(y)\|_p \le \|f_q(x)-f_q(y)\|_q \le C\|f_p(x)-f_p(y)\|_p^{p/q}.$$

Consequently,

$$\sup\{\|f_q(x)-f_q(y)\|_q : x,y \in X,\, d(x,y) \le R\} \le C\varepsilon^{p/q},$$

and

$$\lim_{S\to\infty}\inf\{\|f_q(x)-f_q(y)\|_q : x,y\in X,\ d(x,y)\geq S\}\geq \frac{p}{q}\,\delta.$$

The case $p>q$ is proved similarly. ∎

In the case of Property A a statement similar to Proposition 2 was studied by Dranishnikov under the name of Property A_p in [Dr].

Corollary 4. *If X admits a coarse embedding into ℓ_2 then it admits a coarse embedding into any ℓ_p with $1\leq p\leq\infty$. In particular, the separable Hilbert space embeds into all ℓ_p.*

Proof. If X admits a coarse embedding into ℓ_2 then, by Theorem 2, X has the property from Theorem 3 for ℓ_2. By Proposition 2 it has this property also for ℓ_p, $1\leq p<\infty$, and by Theorem 3 it admits an embedding into ℓ_p.

The case $p=\infty$ is clear since ℓ_∞ is a universal space for isometric embeddings. ∎

It follows from the above proof that Theorem 3 cannot be extended to a characterization of coarse embeddability into ℓ_p if $p>2$. Indeed, in that case the procedure described in the above proof would imply that ℓ_p for $p>2$ embeds coarsely into the Hilbert space, which is not the case by a result of Johnson and Randrianarivony [JR].

In [No] it was shown that $L_p(\mu)$ for $1\leq p\leq 2$ admits a coarse embedding into the Hilbert space and that coarse embeddability into ℓ_2 is equivalent to coarse embeddability into $L_p[0,1]$ again for $1\leq p\leq 2$. This allows us to state

Theorem 5. *Let X be a separable metric space. Then the following conditions are equivalent*:

(1) *X admits a coarse embedding into the Hilbert space*;
(2) *X admits a coarse embedding into ℓ_p for some* (*equivalently all*) $1\leq p<2$;
(3) *X admits a coarse embedding into $L_p[0,1]$ for some* (*equivalently all*) $1\leq p<2$.

Note that this covers all separable $L_p(\mu)$-spaces with $1\leq p\leq 2$. This is particularly interesting since the spaces L_p for different p's are not coarsely equivalent. To see this assume they are and take $f: L_p(\mu)\to L_q(\mu)$ to be the coarse equivalence. Since L_p-spaces are geodesic, f is in fact a quasi-isometry and it induces a Lipschitz equivalence on their ultrapowers. By a theorem of Heinrich [He] ultrapowers of L_p-spaces are again L_p-spaces (possibly on a different measure), and the assertion follows from the classical fact that Lipschitz equivalence on L_p-spaces induces a linear isomorphism.

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