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Centers of a dendroid

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Abstract. A bottleneck in a dendroid is a continuum that intersects every arc connecting two non-empty open sets. Piotr Minc proved that every dendroid contains a point, which we call a center, contained in arbitrarily small bottlenecks. We study the effect that the set of centers in a dendroid has on its structure. We find that the set of centers is arc connected, that a dendroid with only one center has uncountably many arc components in the complement of the center, and that, in this case, every open set intersects uncountably many of these arc components. Moreover, we find that a map from one dendroid to another preserves the center structure if each point inverse has at most countably many components.

1. Introduction. The point p is a *center* of the dendroid D if there are two points c and d in D such that for every $\varepsilon > 0$ there is a continuum C containing p of diameter less than ε and there are two open sets, U containing c and V containing d, such that C intersects every arc from U to V. The concept of a dendroid center was introduced by Piotr Minc and he proved that every dendroid contains at least one center [5, Theorem 3.6]. We used the results of Minc in [1] to show that no continuum maps exactly 2-to-1 onto a dendroid. Minc called the continuum C a *bottleneck*, the points c and d *basin points*, and the two open sets U and V *basins*.

We can put all dendroids into distinct and useful classes with strong structural elements determined by the behavior of the centers or the only center. We show that the set of centers of a dendroid is arc connected and a dendroid with more than one center has uncountably many strong centers. We find that every non-endpoint is a strong center in a locally connected dendroid (also called a dendrite), and that the strong centers are dense in a Suslinean dendroid. On the other hand, some dendroids have only one center. For example, the cone point in the cone over a Cantor set is the only center and it is a strong center. A significantly more complicated dendroid

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with a single center that is not a strong center was constructed in 1977 by Krasinkiewicz and Minc in [2]. This dendroid was shown by Minc [5] to be the image of an at most 3-to-1 map from an indecomposable continuum. We show that any dendroid that is the image of an indecomposable continuum with countable inverses has only one center, and it cannot be strong. Minc's proof in [5] that such an image cannot be in the plane is very complicated and in Section 4 we use new techniques to give a very short proof. We also show in Section 4 that, similar to these two prototype dendroids, any dendroid with a single center has uncountably many arc components in the complement of the center, each without interior, and that this complexity is shared by any open set in the dendroid. Finally, in Section 5, we show that if g is a continuous function between dendroids D and D' such that every point inverse has only countably many components, then g preserves much of the center structure. In particular, if D has only one center then it is mapped by such a map onto the only center in D'.

2. Some definitions. When working with centers in a dendroid it is sometimes helpful to name a pair of basin points. We use the term *bc-center* to denote a center such that b and c are basin points. Another special case is a point p in a dendroid D that we call a *strong center* if there are open sets U and V such that every arc in D from U to V contains p.

In this paper a topological space is a *continuum* if it is connected, compact, and metric. A continuum is *non-unicoherent* if it is the union of two subcontinua whose intersection fails to be connected. A locally connected continuum is a *dendrite* if it is hereditarily unicoherent, and an arc connected continuum is a *dendroid* if it is hereditarily unicoherent. A point x in a dendroid D is an *endpoint* if $D \setminus \{x\}$ has only one arc component. A function is a *map* if it is continuous.

3. Structure theorems for dendroid centers. In this section we study the sets of centers in a dendroid and their pairs of basin points. We show that the set of centers in a dendroid is arc connected. And there is some uniformity of the basin points in that if x and y are both centers then there are basin points b and c that work for both; furthermore, every other point on the arc xy is a strong center with basins that contain b and c. We show further that the basin points b and c can always be chosen to be endpoints of the dendroid. Note that endpoints can never be centers: if p is a bc-center then it follows from the definition that p must belong to the arc $bc \setminus \{b, c\}$.

We use the notation bc to denote the unique arc in a dendroid from b to c.

LEMMA 1. *If a dendroid D is locally connected at x, and x is not an endpoint, then x is a center of D.*

Proof. Let b and c be two points such that x is in the arc $bc \setminus \{b, c\}$. If x is not a bc-center then there are two sequences, $\{b_i\}$ converging to b and $\{c_i\}$ converging to c, and a connected open set C containing x, such that the arcs $b_i c_i$ do not intersect the continuum $\overline{C}$. We may assume the arcs $\{b_i c_i\}$ converge to a subdendroid D' containing b and c. But D' has an arc from b to c, and there is only one arc from b to c in D, and it contains x; so x is in D'. Thus x is in the limiting set of the $\{b_i c_i\}$ and C must intersect most of the arcs; a contradiction. ∎

LEMMA 2. *Suppose xy is an arc in $ac \setminus \{a, c\}$ in a dendroid. Then the set of ac-centers in xy is closed.*

Proof. Suppose t in xy is a limit point of the ac-centers in xy, $\varepsilon > 0$, and N denotes the $\varepsilon/4$-neighborhood of t. There is an ac-center t' in xy so that $t't$ is an arc in N; and there is, in N, a bottleneck C' for t' and basins U and V containing a and c respectively. Since every arc from U to V intersects C', and since C' is contained in the closure C of the arc component of N that contains t, C is a bottleneck of diameter less than ε that contains t, with basins U and V. Hence t is an ac-center. ∎

LEMMA 3. *If x is an ab-center in a dendroid D, y is a cd-center, and $x \neq y$, then every point on the arc xy is an ac-center, or every point of xy is a bc-center, or every point of xy is an ad-center, or every point of xy is a bd-center.*

Proof. Since x must be in ab, one of a and b, say a, satisfies $ax \cap xy = \{x\}$; and since y must be in cd, one of c and d, say c, satisfies $xy \cap yc = \{y\}$. We have $ac = ax \cup xy \cup yc$.

Suppose every point of xy, except possibly x or y, is an ac-center. From Lemma 2 we know that x and y are also ac-centers. So, if the theorem statement is not true there is a point z in $xy \setminus \{x, y\}$ that fails to be an ac-center. From the negation of the definition of an ac-center, there is a sequence of arcs in the complement of z, $\{a_i c_i\}$, with $\{a_i\}$ converging to a and $\{c_i\}$ converging to c. Now we will define a contact point, t_i, for each arc $a_i c_i$. If the arc $a_i c_i$ intersects ac, the intersection is an arc A_i that does not contain z and, in particular, A_i will be contained in zc or in az; in this case let t_i denote any point of A_i. If $a_i c_i$ does not intersect ac, let t_i denote the endpoint of the irreducible arc from $a_i c_i$ to ac that lies in ac. By taking subsequences we may assume that either every contact point t_i lies in zc or every contact point t_i lies in az.

Note here that the only properties of the arc sequence $\{a_i c_i\}$ that we will use are

(1) the endpoints $\{a_i\}$ converge to a, and the endpoints $\{c_i\}$ converge to c,
(2) either (i) none of the arcs $\{a_i t_i\}$ intersect $az \setminus \{z\}$, or (ii) none of the arcs $\{t_i c_i\}$ intersect $zc \setminus \{z\}$.

Notice that case (i) implies that the arc $a_i t_i$ does not intersect any continuum containing x that does not contain z (thanks to the unique arc property of D). And case (ii) implies that $t_i c_i$ does not intersect any continuum containing y that does not contain z.

We will assume that the sequence $\{a_i c_i\}$ satisfies case (i). The argument for the rest of the proof, in the case that every t_i lies in az, is symmetric to the one that follows.

We will show that the arc bx intersects ac only in the point x. The arc bx cannot contain a point of ax other than x since x has to be in ab, as does every ab-center. Suppose bw is an arc that intersects xc only in w, where w is not x. Let C denote a bottleneck containing x that contains neither a nor w nor z, and let U and V denote its basins about a and b. But there is a bypass around C that contradicts its bottleneck definition, namely the subdendroid $a_i t_i \cup t_i w \cup wb$, for some a_i in U. This bypass dendroid contains an arc from U to $b \in V$ that misses C. So bx intersects ac only in the point x.

Now we notice that if the theorem statement is false there is a point z' in xy different from x and y such that z' is not a bc-center. As before, then, there is a sequence of arcs $\{b'_i c'_i\}$ with contact points $\{t'_i\}$ satisfying

(1) $\{b'_i\}$ converges to b and $\{c'_i\}$ converges to c,
(2) either (i) the arcs $\{b'_i t'_i\}$ do not intersect $bz' \setminus \{z'\}$, or (ii) the arcs $\{t'_i c'_i\}$ do not intersect $z'c \setminus \{z'\}$.

In case (i), the fact that x is an ab-center is contradicted using the bypass dendroid $a_i t_i \cup t_i t'_i \cup t'_i b'_i$, for some i.

Therefore case (ii) must hold, and we see first that the arc dy intersects ac only in the point y by using the same argument used to show that the arc bx intersects ac only in the point x. Now, for the final contradiction, we note that there is a third point z'' in xy different from x and y that is not a bd-center, and so there is another sequence of points $\{b''_i d''_i\}$ with contact points $\{t''_i\}$ satisfying

(1) $\{b''_i\}$ converges to b and $\{d''_i\}$ converges to d,
(2) either (i) none of the subarcs $\{b''_i t''_i\}$ intersect $bz'' \setminus \{z''\}$, or (ii) none of the subarcs $\{t''_i d''_i\}$ intersect $z''d \setminus \{z''\}$.

At this point we have three arc sequences; the first is assumed to satisfy case (i), the second satisfies case (ii), and the third satisfies either case (i) or case (ii). But each of these two possibilities provides a bypass: case (i) for the third sequence implies a bypass $a_i t_i \cup t_i t''_i \cup t''_i b''$ around the ab-center x for some i, and case (ii) for the third sequence implies a bypass $c_i't_i' \cup t_i't_i'' \cup t_i''d_i''$ around the cd-center y for some i. ∎

THEOREM 1. *The collection of centers in a dendroid is arc connected.*

THEOREM 2. *If x and y are distinct bc-centers of a dendroid D, and z is a point in $xy \setminus \{x, y\}$, then z is a strong center of D using basins that contain b and c.*

Proof. Let C_1 and C_2 be disjoint bottlenecks for x and y respectively such that z is not in $C_1 \cup C_2$. If U_1 and V_1 denote the basins for C_1, and U_2 and V_2 denote the basins for C_2, let $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$. Then every arc from U to V contains z since every arc from C_1 to C_2 contains z. ∎

COROLLARY 1. *Suppose g is a map from an indecomposable continuum I onto a dendroid D such that each point inverse has only countably many components. Then D has exactly one center, and it cannot be strong.*

Proof. We prove this corollary using essentially that same argument used by Minc in [5] to show that any map from an indecomposable continuum onto a planar dendroid has at least one uncountable point inverse. If D has more than one center then by Theorem 2, it has a strong center, say p, with basins U and V. Since the inverse of p has only countably many components each of which must be in some composant of I, and since I has uncountably many composants, some composant K has an image that misses p. Since K is dense in I, $g(K)$ is dense in D, and so $g(K)$ must intersect both of the basins, say $b \in g(K) \cap U$ and $c \in g(K) \cap V$. If $x \in K$ maps to b and $y \in K$ maps to c, then a proper subcontinuum in K containing x and y maps to a subdendroid of D that contains an arc from b to c that misses p. This contradicts the fact that p is a strong center. ∎

THEOREM 3. *If $x \in ab \subset cd$ in a dendroid, and x is an ab-center, then x is a cd-center.*

Proof. Let C be a bottleneck containing x with basins U and V, containing a and b. Then every arc from U to V intersects C. If there do not exist basins containing c and d with the same property, then there is a convergent sequence of arcs $\{c_i d_i\}$, with $\{c_i\}$ and $\{d_i\}$ converging to c and d, such that every $c_i d_i$ misses U (or every $c_i d_i$ misses V). But, since $\{c_i d_i\}$ converges to a subdendroid that contains c and d, the subdendroid has to contain the arc cd and hence the point a. This contradicts the fact that a is in an open set that misses every arc in the sequence $\{c_i d_i\}$. ∎

COROLLARY 2. *If x is a center for a dendroid D, then there are endpoints c and d of D such that x is a cd-center.*

COROLLARY 3. *If x and y are centers for a dendroid D, then there are endpoints c and d of D such that both x and y are cd-centers.*

Proof. This follows from Lemma 3 and Theorem 3. ∎

4. Dendroids with only one center. All dendroids fall into categories with explicit structure determined by the centers. There are dendroids in which every non-endpoint is a center (dendrites for example), there are dendroids in which the centers are dense (Suslinean dendroids for example), there are dendroids in which the centers are not dense and the closure of the set of centers is a continuum with uncountably many arc components in its complement, and finally there are dendroids with only one center, a strong center, and dendroids with only one center, a center that is not strong. The latter cannot be embedded in the plane.

For each x and y in a dendroid D, we define $Q_x(y) = \{z \in D \mid y \in zx\}$.

If x is a center of D, we look at the arc components of the complement of x. If y is a point in one of these arc components we find that checking whether or not the set $Q_x(y)$ has interior is a surprisingly important tool for studying the structure of D. For instance, if x is the only center in D, then we will see that no $Q_x(y)$ has interior. This fact is used often in the proofs in Sections 4 and 5.

LEMMA 4. *Suppose x is a center for a dendroid D, $y \in D \setminus \{x\}$, and $Q_x(y)$ has interior. Then y is a strong center for D.*

Proof. Let O be an open set in $Q_x(y)$. Since x is a center, there is a bottleneck C containing x and missing y, with basins U and V such that C, U, and V are disjoint.

First note that no arc in $Q_x(y)$ can intersect C, and that any arc that contains a point of $Q_x(y)$ but does not contain y is an arc in $Q_x(y)$.

Now, if there is an arc A from U to a point z_1 of O that misses y, and if there is an arc B from V to a point z_2 of O that misses y, then the union $A \cup B \cup yz_1 \cup yz_2$ contains an arc from U to V that misses C. This contradicts the bottleneck property. So one of A or B cannot exist. Hence y is a strong center with respect to the basins O and U, or y is a strong center with respect to the basins O and V. ∎

LEMMA 5. *If x is a point in a dendroid D, and z is a center of D, then for every point y in $xz \setminus \{x, z\}$, $Q_x(y)$ has interior.*

Proof. Suppose $y \in xz \setminus \{x, z\}$. Let C be a bottleneck containing z that misses y, with basins U and V. If there is a point u in $U \setminus Q_x(y)$ and a point v in $V \setminus Q_x(y)$ then neither of the arcs xu nor xv contains y, and hence their union contains an arc from u to v that misses C; this contradicts the fact that C is a bottleneck. ∎

THEOREM 4. *The point x is the only center of a dendroid if and only if, for every $y \neq x$, $Q_x(y)$ does not have interior.*

Proof. If x is the only center for a dendroid D, then from Lemma 4, no $Q_x(y)$ can have interior for any $y \neq x$. If x is not the only center, that is, if either x is not a center or it is not the only center, then there is a center z of D with $z \neq x$. From Lemma 5, if y is a point in $xz \setminus \{x, z\}$, then $Q_x(y)$ has interior. ∎

LEMMA 6. *If x is a point in a dendroid D, and K is an arc component of $D \setminus \{x\}$, then K is the countable union of continua each of which is contained in $Q_x(y)$ for some y in K. If K contains no center of D then each continuum has empty interior.*

Proof. Suppose K is an arc component of $D \setminus \{x\}$ that does not contain a center for D, and e is an endpoint in K. For each positive integer n, let C_n denote the complement of the open ball about x of radius $1/n$, and let z_n denote the point in xe such that xz_n is an irreducible arc from x to C_n. (We assume here that the distance from x to e is at least 1.) For each pair, n and k, of positive integers such that $n < k$, let $K_{nk} = \{y \in K \mid yz_n \subset C_k\}$. We will show the collection of all K_{nk} satisfies the conclusion of the theorem.

Since $xe \setminus \{x\}$ is the union of the arcs $z_n e$ for $n \in \mathbb{N}$, we know that K is the union of all of the sets $Q_x(z_n)$. Furthermore, for each n, $Q_x(z_n)$ is contained in the union, for all k, of the sets K_{nk}. Hence K is the union of all of the K_{nk}.

To see that K_{nk} is a continuum, first note that it is connected since it is a union of arcs in K, all with the same endpoint, z_n. To see that K_{nk} is closed, suppose $\{t_i\}$ is a sequence of points in K_{nk} that converges to the point t. By the definition of K_{nk}, the arc $z_n t_j$ is in the closed set C_k, so the closure of the union of these arcs is in C_k. But the arc $z_n t$ is in the closure of the union of the arcs and hence is a subset of C_k. So t is in K_{nk}, and K_{nk} is closed.

We know from the Minc result in [5] that there is a center in D. So if K contains no center then either x is a center or there is a center in an arc component of $D \setminus \{x\}$ other than K. Either way, since each K_{nk} is a subset of $Q_x(z_k)$, and z_k is not a center for D, we conclude that K_{nk} cannot have interior by Lemma 4. ∎

THEOREM 5. *If x is the only center for a dendroid D, then*

(i) *each arc component of $D \setminus \{x\}$ is the countable union of continua, each with empty interior*,

(ii) *$D \setminus \{x\}$ has uncountably many arc components*,

(iii) *every open set in D intersects uncountably many arc components of $D \setminus \{x\}$.*

Proof. This follows immediately from Lemma 6 and the Baire Theorem. ∎

THEOREM 6. *Let x be a point in a dendroid D. Then the following are equivalent*:

(i) *For every $y \neq x$ in D, $Q_x(y)$ does not have interior.*
(ii) *No arc component of $D \setminus \{x\}$ has interior.*
(iii) *No set that is the countable union of arc components of $D \setminus \{x\}$ has interior.*
(iv) *D has only one center, the point x.*

Proof. Suppose property (i) is true and suppose K is a component of $D \setminus \{x\}$. From Lemma 6, K is the union of countably many continua, each of which is contained in $Q_x(y)$ for some y in K. Since no $Q_x(y)$ has interior, the continua do not have interior, and it follows from the Baire Theorem that K has no interior. If no arc component of $D \setminus \{x\}$ has interior, it follows, again from the Baire Theorem, that property (iii) holds. Assume property (iii) holds, and suppose the point z, different from x, is a center for D. It follows from Lemma 5 that for every y on the arc xz, the set $Q_x(y)$ has interior. Thus the arc component of $D \setminus \{x\}$ containing z has interior, contrary to assumption. From the Minc result [5] there is at least one center in D, so x must be the only center of D. From Theorem 4, if x is the only center for D, then no $Q_x(y)$ has interior, for any y different from x. ∎

In [3], Minc and Krasinkiewicz showed that in any dendroid D there are two endpoints that are contained in arbitrarily small "co-connected" open sets. (A set is *co-connected* if its complement is connected.) If U is a co-connected open set in a dendroid, then, since $D \setminus U$ is closed and connected in D, $D \setminus U$ is a dendroid, and hence is arc connected. So no arc can go into U and go back out. That is, the intersection of any arc in D with U is either empty or its closure is an arc with one endpoint in U.

COROLLARY 4 (proved in [5]). *Every planar dendroid has a strong center.*

Proof. Suppose D is a planar dendroid. From Theorem 2 we know that if D has two centers then every point between them is a strong center, and we know from [5] that D has at least one center. So we suppose that D has a single center x and we will show that x is a strong center.

Since x is a center it cannot be an endpoint. So, from [3] there are co-connected open sets, U and V, in D whose closures are disjoint and do not contain x.

Since U is open in the subspace D of the plane, there is an open set W in the plane so that $W \cap D = U$. Let E be one of the countably many components of W. Any arc from D that enters E must stay inside E (else it enters U and leaves U). Since $E \cap U$ is itself open in D and co-connected, the set $E \cap U$ has the same properties as U. That is, we may assume that

W is connected. Similarly, let W' be a connected open set in the plane such that $W' \cap D = V$. From Theorem 5 we know that uncountably many of the arc components of $D \setminus \{x\}$ intersect each of U and V, and from Moore's theorem [6] only countably many of these arc components can have triods. Let $\mathcal{A}$ and $\mathcal{B}$ denote the two uncountable collections of arcs without triods from x to U and V respectively that are arc components of $D \setminus \{x\}$.

We need to identify three arcs in $\mathcal{A}$, say A_1, A_2, and A_3, that satisfy (i) $A_2 \setminus (U \cup x)$ is in a bounded component C_U of the complement (in the plane) of $A_1 \cup A_3 \cup W$, and (ii) W' is not in C_U. If we start with any four elements of $\mathcal{A}$, it is possible that no three of them satisfy both properties. But if we start with five such arcs, three of them will work. Since there are uncountably many candidate arcs, the arcs, A_1, A_2, and A_3, that satisfy (i) and (ii) exist. Similarly, we identify three arcs B_1, B_2, and B_3 from $\mathcal{B}$ such that (i) $B_2 \setminus (V \cup x)$ is in a bounded component C_V of the complement (in the plane) of $B_1 \cup B_3 \cup W'$, and (ii) W is not in C_V. Let U' and V' denote open sets whose closures are in $C_U \cap D$ and $C_V \cap D$, respectively, that contain points from the middle arc, A_2 or B_2, and so are not empty. Since no arc can go through W or W', the outside arcs $A_1 \cup A_3$ and $B_1 \cup B_3$ channel all arcs from U' to V' through x and so x is a strong center for D with basins U' and V'. ■

5. Preserving centers with maps. It is straightforward to construct a map (continuous function) from, for instance, the cone over a Cantor set onto another dendroid that greatly increases the number of centers, or whose image of the center point is not a center point of the image dendroid. We show that this cannot happen if the map has point inverses with only countably many components. With such a map, the image of the only center is the only center of the image dendroid. We show that, like the complement of a single center, the complement of the closure of the centers has uncountably many arc components.

THEOREM 7. *If D is a dendroid whose centers are all contained in a proper subcontinuum C and f is a map from D onto a non-degenerate dendroid D' such that for each point z in $D' \setminus f(C)$, the inverse of z has only countably many components, then $f(C)$ contains all of the centers of D'.*

Proof. Suppose some point z not in $f(C)$ is a center for D'. Let zt be an irreducible arc from z to $f(C)$. From Lemma 5 we know that, for some point y in the arc $zt \setminus \{t, z\}$, $Q_t(y)$ contains an open set U. Since $y \in zt \setminus \{z, t\}$, y is not in $f(C)$. Let C_i denote the countable collection of continua that are the components of $f^{-1}(y)$ in D. For each i, let $x_i z_i$ denote an irreducible arc from C to C_i. Notice that each C_i is a subset of $Q_{x_i}(z_i)$. For each i, the arc component K_i of $D \setminus \{x_i\}$ that contains z_i has no center since it does not intersect C; so by Lemma 6, K_i is the countable union of closed nowhere dense sets. By the Baire Theorem, the open set $f^{-1}(U)$ is not a subset of $\bigcup_{i=1}^{\infty} K_i$ and hence cannot be a subset of $\bigcup_{i=1}^{\infty} Q_{x_i}(z_i)$; so there is a point w in $f^{-1}(U)$ whose irreducible arc to a point x in C does not contain any point that maps to y. Hence $f(wx)$ contains an arc from U to $f(x)$ in $f(C)$ that does not contain y, contradicting the fact that U is a subset of $Q_t(y)$. ∎

Corollary 5. *If $g : D \to D'$ is a map from a dendroid D onto a non-degenerate dendroid D' whose point inverses have only countably many components, and if x is the only center for D, then $g(x)$ is the only center of D'.*

Proof. Let the set C in the proof of Theorem 7 be the set $\{x\}$, where x is the only center of D. ∎

Corollary 6. *If a continuum C in a dendroid D contains all of the centers of D, and C is not all of D, then D/C is a dendroid whose only center is $\{C\}$.*

Proof. The space D/C is a dendroid since the monotone image of a dendroid is a dendroid [4]; the rest follows from Theorem 7. ∎

Corollary 7. *If D is a dendroid and C is a proper subcontinuum of D that contains all of the centers of D, then $D \setminus C$ has uncountably many arc components and any open set in $D \setminus C$ intersects uncountably many of the arc components of $D \setminus C$.*

Proof. This corollary follows from Theorems 5 and 7. ∎

Corollary 8. *If an open set U in a dendroid D does not contain an uncountable collection of pairwise disjoint non-degenerate arcs then U contains a strong center of D.*

Proof. We know from Theorem 5 that D cannot contain a point that is the only center of D. If C denotes the closure of the centers in the dendroid D, then we know by Corollary 7 that U must intersect C. Since U intersects C, and since, by Theorems 1 and 2, the strong centers are dense in C (if C has more than one point), we know that there must be a strong center in U. ∎

Theorem 8. *Suppose T is a subset of D with interior whose closure misses the closure of the centers in D. Then T has uncountably many components.*

Proof. From Corollary 7 we know that the interior of T intersects uncountably many arc components of $D \setminus C$. Now, suppose one component E of T intersects two of these arc components, A and B. The closure, $\overline{E}$, is a continuum in the closure of T and so $\overline{E}$ misses C. But any continuum in a dendroid is arc connected so there is in $\overline{E}$ an arc from a point of A to a point of B in the complement of C. That contradicts the definition of arc component of $D \setminus C$. Hence T must have uncountably many components. ∎

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