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Model-theoretic consequences of a theorem of Campana and Fujiki

by

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Abstract. We give a model-theoretic interpretation of a result by Campana and Fujiki on the algebraicity of certain spaces of cycles on compact complex spaces. The model-theoretic interpretation is in the language of canonical bases, and says that if b, c are tuples in an elementary extension $\mathcal{A}^*$ of the structure $\mathcal{A}$ of compact complex manifolds, and b is the canonical base of $\mathrm{tp}(c/b)$, then $\mathrm{tp}(b/c)$ is internal to the sort $(\mathbb{P}^1)^*$. The Zilber dichotomy in $\mathcal{A}^*$ follows immediately (a type of U-rank 1 is locally modular or nonorthogonal to the field $\mathbb{C}^*$), as well as the "algebraicity" of any subvariety X of a group G definable in $\mathcal{A}^*$ such that $\mathrm{Stab}(X)$ is trivial.

1. Introduction. This paper concerns the interaction between complex-geometric notions and model-theoretic notions in the structure theory of compact complex spaces. It has been known for some time that model-theoretic ideas yield a rather striking dichotomy for simple compact complex manifolds M: either M is algebraic, or else there is no "2-parameter" family of finite-to-finite analytic correspondences between M and itself. But, up to now, the only proof of this of which I was aware went through the results on Zariski geometries and their validity for compact complex manifolds, together with some other ingredients (see [6], [7] and [11]). It turns out that the dichotomy above and more are almost immediate consequences of a theorem proved independently by Campana [1] and Fujiki [3]. They prove, roughly speaking, that if S is a compact space of cycles $(Z_s : s \in S)$ on a compact complex space X then the natural morphism from the graph $(\{x, s) : x \in Z_s,\ s \in S\})$ of S to X is a Moishezon map. Via a translation established by Moosa ([9], [10]), this yields the following striking statement in the language of canonical bases (to be read in a saturated elementary extension $\mathcal{A}^*$ of the many-sorted structure $\mathcal{A}$ of compact complex spaces):

(∗) for any b, c, $\mathrm{tp}(\mathrm{Cb}(\mathrm{tp}(c/b))/c)$ is "algebraic", that is, internal to $\mathbb{C}^*$.

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The statement (∗) (or the original Campana–Fujiki statement) yields the dichotomy for U-rank 1 types in $\mathcal{A}^*$: they are modular or "algebraic". Also the algebraicity of suitable subvarieties of meromorphic groups and homogeneous spaces follows directly, subsuming results of Ueno [15] as well as results from [8]. I guess that the benefit of the model-theoretic translation (∗) lies in being able to work directly with bimeromorphic objects without worrying about specific compactifications. As the reader might surmise, the statement (∗), when suitably re-interpreted, is also directly provable (using suitable jet spaces) in other algebraic/model-theoretic structures of interest, such as differential and difference fields. (See [14].)

The details of the observations above are given in the next section.

2. Results. For the theory of compact complex spaces see [4]. For the model-theoretic treatment of this subject see [11].

Let us work towards stating the Campana–Fujiki theorem. Let X be a reduced, irreducible, compact complex space. There are two notions of cycle spaces on X. The Douady space $D(X)$ parametrizes pure-dimensional analytic subsets of X. The Barlet space $C(X)$ parametrizes integral linear combinations of irreducible analytic subsets (of the same dimension) of X. Campana works with Barlet spaces and Fujiki with Douady spaces. A morphism $f : Y \to X$ of compact complex spaces is said to be *projective* if there is a coherent analytic sheaf $\mathcal{F}$ over X and an embedding (over X) h of Y into the projective linear space $\mathcal{P}(\mathcal{F})$ over X associated with $\mathcal{F}$ such that $\pi \circ h = f$, where $\pi : \mathcal{P}(\mathcal{F}) \to X$ is the map realizing $\mathcal{P}(\mathcal{F})$ as a fibre space over X. A morphism $f : Y \to X$ is said to be *Moishezon* if it is bimeromorphic (over X) to a projective morphism $f' : Y' \to X$. Campana proves:

FACT 2.1. *Let S be an irreducible, compact, analytic subset of $C(X)$. Let Z_s denote the cycle parameterized by $s \in S$. Assume that for general $s \in S$, Z_s is irreducible. Let $Y = \{(x,s) \in X \times S : x \in Z_s,\ s \in S\}$, and let $f : Y \to X$ be the projection to the first coordinate. Then f is a Moishezon map.*

In fact the above statement comes from [2] (Theorem 3.6). The original theorem in [1] states that f factors through an embedding in a suitable Grassmannian of $\mathcal{F}$ over X, where $\mathcal{F}$ is the coherent analytic sheaf of germs of differential operators of suitably bounded order. Such an f is Moishezon. In Fujiki's statement, $C(X)$ is replaced by $D(X)$.

We now work towards the model-theoretic interpretation. $\mathcal{A}$ is the many-sorted structure of compact complex spaces X_i with predicates for analytic subvarieties of Cartesian products $X_{i_1} \times \ldots \times X_{i_n}$. Note that with this "language", all elements of all sorts in $\mathcal{A}$ are named by constants. Th($\mathcal{A}$) has quantifier elimination, elimination of imaginaries and is stable with finite Morley rank (sort-by-sort). $\mathcal{A}^*$ will be a very saturated elementary extension of $\mathcal{A}$. Among the sorts in $\mathcal{A}$ is the projective line $\mathbb{P}_1$ over $\mathbb{C}$. There is no harm (via Chow's theorem) in identifying this sort with the complex field $\mathbb{C}$ equipped with addition, multiplication, and constants for all elements (they are bi-interpretable). $\mathbb{C}^*$ denotes the "extension" of this sort in $\mathcal{A}^*$. Let b, c be tuples from $\mathcal{A}^*$. Following [9], [10], we will say that $\mathrm{tp}(c/b)$ is *Moishezon* if $\mathrm{tp}(c/\mathrm{acl}(b))$ is internal to $\mathbb{C}^*$, or equivalently if there is some finite tuple b' including b and independent of c over b such that $c \in \mathrm{dcl}(d, b')$ for some tuple d from $\mathbb{C}^*$. (Note that $\mathrm{tp}(c/b)$ is Moishezon just if every stationarization of it is Moishezon.) Moosa [10] observes the following:

FACT 2.2. *Suppose that X, Y are irreducible compact complex spaces and that $f : Y \to X$ is a Moishezon map. Let $c \in Y^*$ be a generic point of Y (over $\mathcal{A}$). Then $\mathrm{tp}(c/f(c))$ is Moishezon.*

We can now obtain:

THEOREM 2.3. *Let b, c be finite tuples from $\mathcal{A}^*$. Assume that $\mathrm{tp}(c/b)$ is stationary and that $b = \mathrm{Cb}(\mathrm{tp}(c/b))$. Then $\mathrm{tp}(b/c)$ is Moishezon.*

Proof. Let X, S, Z be irreducible compact complex spaces of which c, b and (c, b) respectively are generic points (over $\mathcal{A}$). Then Z_b is irreducible with generic point c, and moreover b is a canonical parameter for Z_b (by quantifier elimination). Replacing S by a suitable modification, we may assume that the projection $\pi : Z \to S$ is flat. The universal properties of the Douady space $D(X)$ of X yield a morphism $p : S \to D(X)$ such that $p(s)$ corresponds to Z_s. Then $p(S)$ is a compact irreducible analytic subset of $D(X)$. Also, by compactness, there is a Zariski open subset U of S such that for $s_1, s_2 \in U$, $Z_{s_1} = Z_{s_2}$ iff $s_1 = s_2$. Thus $p : S \to p(S)$ is a modification. The end result is that, after replacing b by something interdefinable with it (which we can do), we may assume that S is a compact analytic subspace of $D(X)$ and that Z is the associated subspace of $X \times S$. By Fact 2.1 (working with Douady spaces), the canonical morphism $f : Z \to X$ is Moishezon. By 2.2, $\mathrm{tp}(b, c/c)$ is Moishezon, and thus $\mathrm{tp}(b/c)$ is Moishezon.

REMARK 2.4. *Let b, c be as in the above theorem. Then for any tuple a from $\mathcal{A}^*$, $\mathrm{tp}(b/ca)$ is Moishezon.*

Proof. The conclusion of Theorem 2.3 tells us that there is some set C of parameters containing c and such that b is independent of C over c and $b \in \mathrm{dcl}(C, d)$ for some tuple d from $\mathbb{C}^*$. Without loss of generality (that is, by automorphism) C is independent of a over bc, so independent of bca over c, and so independent of b over ca.

Let us now give some applications.

COROLLARY 2.5. *Let $p(x)$ be a stationary type of U-rank 1 over some set in $\mathcal{A}^*$. Then p is either modular or nonorthogonal to (the generic type of) $\mathbb{C}^*$.*

Proof. As $\mathcal{A}^*$ has finite Morley rank (sort-by-sort) we may assume that $p(x)$ is over a finite tuple a of parameters. If p were not modular then by 2.2.6 of [12] there would be a tuple c of realizations of p (in fact a pair is sufficient) and some tuple b such that $b = \mathrm{Cb}(\mathrm{tp}(c/ba))$ and $b \notin \mathrm{acl}(ca)$. By Remark 2.4, $\mathrm{tp}(b/ca)$ is Moishezon. In particular $\mathrm{tp}(b/ca)$ is nonorthogonal to the generic type of $\mathbb{C}^*$. But b lives in p^{eq}, and thus p is nonorthogonal to $\mathbb{C}^*$.

The next application concerns definable subsets of groups and homogeneous spaces. Definable groups in $\mathcal{A}$ have naturally the structure of "meromorphic groups" (see [13] and [8]). A definable group in $\mathcal{A}^*$ can be considered as the generic fibre of a meromorphic family of meromorphic groups.

Let us start with a general lemma about stable groups.

LEMMA 2.6. *Let G be a connected group (type)-definable in a saturated stable structure $\overline{M}$. Let $c \in G$ be such that $p(x) = \mathrm{tp}(c)$ is stationary. Let H be the left stabilizer of p. Let $a \in G$ be generic over c. Let c/H denote Hc as an element of the right coset space $H\backslash G$. Then c/H is interdefinable over a with $\mathrm{Cb}(\mathrm{tp}(a/ca))$. (Similarly with left and right interchanged.)*

Proof. So $H\backslash G$ denotes the space $\{Hg : g \in G\}$ of right cosets of H in G, and as above we write Hg as g/H when we want to treat it as an element rather than a definable set. Some more notation: given a stationary type q over $\emptyset$ of some element of G, and given $a \in G$, by qa we mean the restriction to a of the translate $q'a$ of q' by a where q' is the global nonforking extension of q. Note that qa is stationary and $q'a$ is its global nonforking extension. Moreover $qa = \mathrm{tp}(da/a)$ for d realizing $q|a$.

Now let us make some observations.

(i) a realizes $p^{-1}ca$.

This is because c^{-1} realizes p^{-1}, and (as a is generic in G over c), ca is generic over c^{-1} so independent of c^{-1}.

(ii) ca/H is interdefinable with $\mathrm{Cb}(p^{-1}ca)$.

Indeed, H is the left stabilizer of p so the right stabilizer of p^{-1}, which clearly yields (ii): an automorphism f fixes ca/H iff $p^{-1}ca$ is parallel to $f(p^{-1}ca) = p^{-1}f(ca)$.

Note that c/H and ca/H are interdefinable over a. So by (i) and (ii) we deduce that c/H and $\mathrm{Cb}(\mathrm{tp}(a/ca))$ are interdefinable over a.

REMARK 2.7. Note that Lemma 2.6 includes the old result [5] that in modular (or 1-based) groups any stationary type is a translate of the generic type of a subgroup: if $\mathrm{Cb}(\mathrm{tp}(a/ca)) \in \mathrm{acl}(a)$ then $c/H \in \mathrm{acl}(\emptyset)$ and so $\mathrm{tp}(c)$ is a translate of the generic type of H.

Lemma 2.6 applies to groups definable in $\mathcal{A}^*$. So together with Theorem 2.3 we obtain:

Corollary 2.8. *Let G be a group definable in $\mathcal{A}^*$. Work over some algebraically closed set of parameters A over which G is definable. Let $c \in G$ and let H be the left stabilizer of* $\mathrm{tp}(c/A)$. *Then* $\mathrm{tp}((c/H)/A)$ *is Moishezon.*

For groups definable in $\mathcal{A}$, that is, *meromorphic groups*, Corollary 2.9 below gives a more geometric looking statement. As a matter of notation, if X and Y are complex (not necessarily compact) spaces definable in $\mathcal{A}$, we will say that X and Y are *meromorphically isomorphic* if there is a bihomolorphic map between X and Y which is also definable in $\mathcal{A}$.

Corollary 2.9. *Let G be an arbitrary meromorphic group* (*not necessarily commutative*). *Let X be an irreducible meromorphic subvariety of G, and let $H = \{g \in G : g \cdot X = X\}$ be the set-theoretic left stabilizer of X in G. Let $\overline{H\backslash X}$ denote the Zariski closure of $H\backslash X$ in the meromorphic homogeneous space $H\backslash G$ of right cosets Hg $(g \in G)$. Then $\overline{H\backslash X}$ is meromorphically isomorphic to an algebraic variety.* (*Likewise with left and right interchanged.*)

Proof. Let $p = \mathrm{tp}(c/\mathcal{A})$ be the generic type of X. (So $c \in X^* \subseteq G^*$.) Then H (or rather its extension H^*) as defined in the statement to be proved identifies with the left stabilizer of p. By Corollary 2.8, $\mathrm{tp}((c/H^*)/\mathcal{A})$ is Moishezon. As $\mathcal{A}$ is a model, this is witnessed over $\mathcal{A}$, namely $c/H^* \in \mathrm{dcl}(d)$ for some tuple from $\mathbb{C}^*$. It follows that c/H^* is interdefinable with some tuple in $\mathbb{C}^*$. But c/H^* is a generic point over $\mathcal{A}$ of $(\overline{H\backslash X})^* \subseteq (H\backslash G)^*$. As $H\backslash G$ is a homogeneous space it follows that $\overline{H\backslash X}$ is meromorphically isomorphic to an algebraic variety.

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