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Essential tori admitting a standard tiling

by

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Abstract. Birman and Menasco (1994) introduced and studied a class of embedded tori in closed braid complements which admit a standard tiling. The geometric description of the tori from this class was not complete. Ng showed (1988) that each essential torus in a closed braid complement which admits a standard tiling possesses a staircase tiling pattern.

In this paper, we introduce and study the so-called longitude-meridional patterns for essential tori admitting a standard tiling. A longitude-meridional pattern of an essential torus can be derived from the corresponding tiled torus and carries a portion of geometric information about the embedded torus. We also study the interplay between the geometry of essential embedded tori and combinatorics of the corresponding tiled tori.

Introduction. The existing results about closed incompressible surfaces in link complements are of the following types. The first type of results are existence theorems for different classes of knots or links (satellite knots, fibered knots, closed braids etc.). In the second case, it is assumed that an essential surface in the corresponding 3-manifold is given and then one tries to transform it to one in some standard form satisfying certain properties. In many situations, it would be helpful to represent the link as a closed braid and deform the embedded surfaces to ones in some standard positions with respect to the braid axis. Closed incompressible surfaces in complements of closed 3-braids have been studied by Przytycki and Lozano [13] and Finkelstein [7]. The problem of classification of incompressible surfaces in complements of links of braid index > 3 remains unsolved.

We study incompressible surfaces in a closed braid complement via their natural (singular) foliations induced by the open book fibration of the braid structure. This approach is due to Birman and Menasco [4] (see also [2] for the details and basic machinery). The main purpose of this paper is the study of essential tori in closed braid complements which admit a standard

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tiling via their combinatorial patterns, called tiled tori. We shall also be concerned with the tiled closed orientable surfaces of genus $g > 1$.

The important role of incompressible tori in the geometry and topology of link complements has been explained by Jaco and Shalen [8], and Johannson [9]. They proved that if M is a Haken 3-manifold, then there is a finite collection $\mathcal{B}$ of incompressible and non-peripheral tori $T_1, \ldots, T_n$ in M such that each component of M split along the tori in $\mathcal{B}$ is either Seifert-fibered or a hyperbolic space.

Let L be a non-split link represented as a closed braid in S^3 with braid axis A, and let T be an incompressible and non-peripheral torus in $S^3 \setminus L$ arising in the Jaco–Shalen–Johannson decomposition of $S^3 \setminus L$. Birman and Menasco [4] showed that any such torus T may be standardized by a sequence of controlled moves on closed braid representatives of L and isotopies in closed braid complements to one in a special position. The controlled moves used in [4] take closed braids to closed braids, preserve link types and are of the following two types: braid isotopy and exchange moves. Braid isotopy means an isotopy in the complement of the braid axis which preserves braid structure at each stage, and the exchange move is defined pictorially in Fig. 1 (see also [4]).

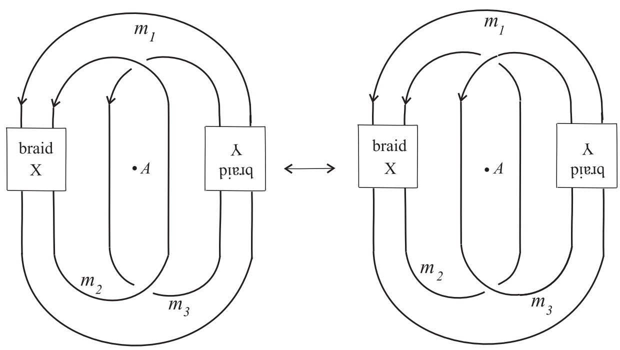


Fig. 1

Let $\mathcal{H} = \{H_\theta : \theta \in [0, 2\pi]\}$ be the open book decomposition of $\mathbb{R}^3$ into half-planes with boundary on the z-axis A. The notion of torus T in a special position in $\mathbb{R}^3$ with respect to the braid axis A is defined in terms of the natural (singular) foliation of T induced by $\mathcal{H}$ or of its combinatorial pattern $\mathcal{T}$. The description of essential tori in a special position falls into three cases (see [4]).

In the first case, a torus T is transverse to every fiber H_θ of the braid fibration and intersects it in a meridian of the solid torus bounded by T (a *torus of type* 0 in [4]). In the second case, T admits a standard mixed decomposition. Birman and Menasco [4] showed that in this case, T can be further standardized to be an embedded torus of type 1.

In the third case, T admits a standard tiling [4]. Such tori have a much more complicated geometrical description. To give it, Birman and Menasco [4] introduced and studied standard tori of types $k \geq 2$. It turned out that the embedded tori of types $k \geq 2$ do not exhaust the class of essential tori which admit a standard tiling. In [12], Ng described standard tilings of essential tori via the so-called staircase tiling patterns. A staircase tiling pattern P is parameterized by two parameters, width and height, and is enhanced with a decoration. Ng showed that every embedded torus T which admits a standard tiling possesses a staircase tiling pattern of even width $2n$ and height $k \geq 2$ (see below for details). However, the geometric description of the tori from the third class has not been completed.

Note that the special positions of essential tori in closed braid complements (the tori of type ≥ 0) have been used by Los [10] as a technical tool in his dynamical classification of knots.

In Section 2 of the present paper, we give a description of essential tori in closed braid complements which admit a standard tiling and bound a solid torus in $\mathbb{R}^3$ via the so-called "longitude-meridional" patterns. A longitude-meridional pattern of an essential torus can be derived from the corresponding tiled torus and carries a portion of geometric information about the embedded torus. A key role in its definition is played by a combinatorial meridian of the corresponding tiled torus, chosen in a suitable way. Note that a longitude-meridional pattern for an embedded tiled torus contains a little more geometric information than the staircase pattern suggested by Ng [12]. Algorithm 2.1 in Section 2 allows one to find a slice (combinatorial) meridian on an embedded torus which admits a standard tiling or on the corresponding tiled torus. For tiled tori with a standard tiling, we also introduce several integral parameters which characterize their combinatorial properties.

Ng [12] described a series of examples of $2n$ by k staircase patterns with decoration which can be geometrically realized as embedded essential tori in some closed braid complements. We treat several examples of tiled essential tori that have embeddings in more detail. Finally, we show that each tiled torus with a standard tiling that has an embedding is essential in some link complement.

We also pose and discuss some open questions relating to the interplay between the combinatorics of tiled tori and the geometry of the corresponding embedded tori.

Section 1 presents some preliminaries: we review the singular foliations of incompressible surfaces in closed braid complements. We also consider the combinatorial patterns for foliated closed oriented surfaces which admit a tiling.

1. Foliated surfaces in closed braid complements and their combinatorial patterns. All surfaces embedded in $\mathbb{R}^3$ are assumed to be smooth of class C^r where $r \geq 2$. Throughout Sections 1 and 2, K and L denote non-split links in S^3, unless otherwise specified. We start by reviewing the Birman and Menasco approach to the study of incompressible surfaces in closed braid complements. For details see [4] and [2].

Let $\mathcal{L}$ denote the link type of the link L in S^3. Choose cylindrical coordinates (r, θ, z) in 3-space $\mathbb{R}^3$ (S^3 thought of as $\mathbb{R}^3$ together with a point at ∞). Let $\mathcal{H} = \{H_\theta : \theta \in [0, 2\pi]\}$ be the open book decomposition of $\mathbb{R}^3$ into half-planes with boundary on the z-axis A. A link L is a closed braid with A as *braid axis* if each component intersects every half-plane H_θ, $\theta \in [0, 2\pi]$, transversally. Let S' be an incompressible closed orientable surface in $\mathbb{R}^3 \setminus L$. Assume that S' is in general position with respect to $\mathcal{H}$. The intersections of the H_θ's with S' induce a smooth foliation $\mathcal{F}$ of S' of class C^{r-1}. This foliation is, in general, singular. Its singular points are the points where S' meets the braid axis A or is tangent to the half-planes H_θ, and are called *vertices* and *singularities*, respectively. By general position, the number of singular points is finite and all singularities can be assumed to be of saddle type or center type. Since S' does not intersect L, each non-singular leaf is either an arc with both endpoints on A, or a circle. As in [4], we shall call the leaves in the first case *b-arcs*, while in the second case the non-singular leaves are called *c-circles*. After an appropriate isotopy in $S^3 \setminus L$, S' can be replaced with an *essential*, smoothly embedded surface S, so that the natural foliation on S has the following properties (see Theorem 1.1 of [2]):

(i) The foliation is radial at the vertices. All singularities are of saddle type. The singularity together with its leaves (branches) is then called a *singular leaf* of the foliation.
(ii) The singularities fall into three types *bb*, *bc* and *cc* (see below).
(iii) The vertices are (cyclically) ordered by their order on the braid axis A. Moreover, after an appropriate isotopy, distinct singular leaves are on distinct singular half-planes (singular fibers) H_θ, so they are also cyclically ordered.

In the following, we shall assume that an embedded essential surface S is chosen so that its natural foliation has properties (i)–(iii).

Consider now the different types of singularities which can occur in the foliation of an essential torus T. There are only two types of non-singular leaves, so at most three types of singularities can occur, called *bb*, *bc* and *cc*-singularities, according to the types of leaves which are surgered when passing through a singularity. If T is an incompressible torus in $\mathbb{R}^3 \setminus L$, the singularities of type *cc* can be eliminated from the foliation [4]. If the foliation of T consists only of c-circles, then T has no singularities and possesses a *circular foliation* [4]. If the foliation of T involves bc-singularities and maybe bb-singularities, we say that T has a *mixed foliation*. Finally, if each singularity of the foliation is a bb-singularity, we say that T admits a *tiling*. By Lemma 4 of [4], one may assume that each incompressible and non-peripheral torus T in a link complement has either a circular foliation, or a mixed foliation, or a tiling. In the first case, T is of type 0.

Birman and Menasco [4] showed that any essential torus T in a closed braid complement which has a mixed foliation can be replaced, via a sequence of exchange moves on closed braids and isotopies in closed braid complements, by a torus T' with a standard mixed foliation. Moreover in this case, any essential torus T' in a closed braid complement can be assumed to be of type 1.

In the rest of this section, we review the combinatorial patterns for essential closed orientable surfaces S of genus $g \geq 0$, the induced foliation of which includes only bb-singularities (see also [6]). Theorems 1.1 and 1.2 stated here will be used in Section 2 in the study of tiled tori in an essential way. The graph of singular leaves of the foliation of S, enhanced with information about its embedding in the surface and some additional data, defines a combinatorial pattern for the foliated surface S. The dual point of view is to consider, instead of the foliation of S and its singular leaves, a decomposition of S into *tiles*, one for each saddle point (see Fig. 1(a)), by cutting the surface along appropriate b-arcs (see also Theorem 1.2 of [2]). This yields a combinatorial pattern for a foliated surface in a closed braid complement $\mathbb{R}^3 \setminus L$, called a *tiled surface* (see also [3]).

A *tiled surface* $\mathcal{S}^{\mathcal{F}}$ is a triple $(S, G, \mathcal{E})$, where S is a closed connected oriented surface, G is a graph which is embedded in S, and $\mathcal{E}$ is some combinatorial data which we call a *decoration*. The graph G is bipartite with each node a *vertex* or a *singularity*. The branches of singular leaves of the foliation determine the edges of G. Each component $D \subset S \setminus G$ is a disc and ∂D contains exactly four edges. The vertices are cyclically ordered according to the order on the axis A and the singularities are cyclically ordered according to increasing θ. Each vertex has a sign and the singularities are also marked by signs. The sign of a vertex v is $+$ (respectively, $-$) if the positive normal vector to S at v agrees (respectively, disagrees) with the orientation of the axis A. The sign of a singularity s or the tile containing s is $+$ (respectively, $-$) if the positive normal to T at s agrees (respectively, disagrees) with the normal to H_θ which points in the direction of increasing θ. Any two vertices v and w which are adjacent to the same singularity s so that v, s and w lie on the boundary of a face D have different signs.

Note also that if $\mathcal{S}^{\mathcal{F}} = (S, G, \mathcal{E})$ is a tiled surface, then its underlying surface S admits a singular foliation so that the graph G is the union of singular leaves of this foliation (see also [3]). In Section 2, we shall keep however the dual point of view. So, instead of the decorated graph G of a tiled torus $\mathcal{T}$ we shall consider the corresponding decomposition of the embedded torus T into tiles, enhanced with a decoration. In this case, we define the underlying graph of the tiling to be a graph H embedded in T which is formed by the set of *vertices* of $\mathcal{T}$ and b-arcs bounding the tiles, the *edges* of H. It follows from the definition that the sign of vertices in each tile τ of the tiling alternates when encountering the boundary of τ. Therefore the graph H is bipartite. Moreover the sign of the singularity in the tile τ is determined by the signs of its vertices and the remaining decoration data inherited from $\mathcal{T}$ (which does not include information about signs). A change of orientation on the surface S leads to changes of the signs of all vertices and singularities of $\mathcal{T}$. For this reason, in Section 2, we shall usually omit the sign information in the decoration of $\mathcal{T}$.

Each foliated bb-tile with decoration admits a canonical (unique up to foliation preserving isotopy) embedding in oriented 3-space with respect to the z-axis A ([2]; see Fig. 2(b)).

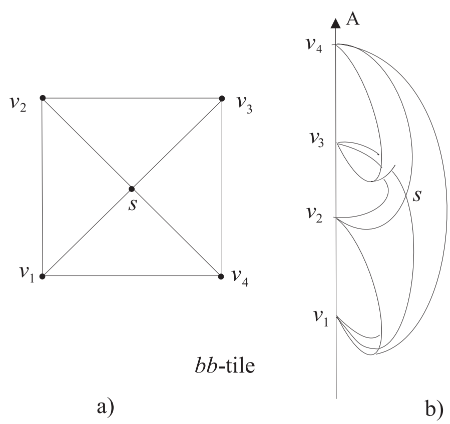


Fig. 2

Definition 1.1. We shall say that a tiled surface $\mathcal{S}^{\mathcal{F}}$ *has an embedding* (in 3-space) if it can be represented as a bipartite (embedded) graph, formed by singular points and singular leaves of some foliated embedded surface S (with respect to the axis A) and enhanced with an appropriate decoration.

Let $\mathcal{S}^{\mathcal{F}}$ be a tiled surface as before. Suppose the singular leaves occur at angles $\theta_1, \ldots, \theta_l$ given in circular order. Each θ-interval (θ_{i-1}, θ_i), in particular, (θ_l, θ_1), is uniquely determined and is meant in the oriented (ordered) sense. Denote by $b(v_i, v_j)$ a b-arc joining the vertices v_i and v_j and let (θ_m, θ_n) be the maximal open interval in which there is a b-arc between the vertices

v_i and v_j which is homotopic to $b(v_i, v_j)$ rel endpoints. Denote by $[b(v_i, v_j)]$ the equivalence class given by these b-arcs. We say that the b-arc $b(v_i, v_j)$ *exists in the θ-interval* (θ_{p-1}, θ_p) if $(\theta_{p-1}, \theta_p) \subset (\theta_m, \theta_n)$, i.e. if some representative of the equivalence class $[b(v_i, v_j)]$ exists between θ_{p-1} and θ_p. In [3, Theorem 3.5], J. Birman and M. Hirsch give a test for embeddability of a tiled surface $\mathcal{S}^{\mathcal{F}}$ where the underlying surface S of $\mathcal{S}^{\mathcal{F}}$ has a boundary. By a slight modification, this test can be adopted for checking the embeddability of a closed oriented tiled surface in 3-space with respect to the braid axis A.

PROPOSITION 1.1. *Let $\mathcal{S}^{\mathcal{F}}$ be a (closed) tiled surface and let the vertices and regions of $\mathcal{S}^{\mathcal{F}}$ be labelled in the manner described above. Then $\mathcal{S}^{\mathcal{F}}$ has an embedding if and only if:*

(i) *The singularities about each positive (respectively, negative) vertex in the foliation are positively (respectively, negatively) cyclically ordered with respect to increasing θ.*
(ii) *The vertices about each positive (respectively, negative) singularity are positively (respectively, negatively) cyclically ordered on the oriented braid axis.*
(iii) *The endpoints of any b-arc in the θ-interval (θ_{i-1}, θ_i) never separate the endpoints of a b-arc in the same interval.*

Proof. The proof follows the one of Theorem 3.5 of [3]. The only difference is that instead of the so-called gb-arcs which can occur when the surface has boundary, we take into account only the (true) b-arcs. These arcs exhaust all possible non-singular leaves of a foliation in our case. All the other arguments in the proof of Theorem 3.5 in [3] carry over to the case of (closed) tiled surfaces without any changes. ∎

PROPOSITION 1.2. *Let $\mathcal{S}^{\mathcal{F}}$ be a (closed) tiled surface that has an embedding and has a decoration $\mathcal{E}$ which includes the cyclic ordering of its vertices, the cyclic ordering of its singularities, and the signs of the vertices and singularities. Let h be an embedding of its underlying surface S into the oriented space $\mathbb{R}^3$, determined by $\mathcal{S}^{\mathcal{F}}$. The embedding h is unique up to foliation preserving isotopy.*

Proof. The proof is similar to that of Theorem 4.1 of [2] (see also Theorem 2.2 of [3]). It follows from the above discussion that each decorated bb-tile admits a unique (up to foliation preserving isotopy) embedding in $\mathbb{R}^3$, with respect to the z-axis A. Now the embedding of the tiled surface in $\mathbb{R}^3$ is determined by specifying how the boundary components of the embedded tiles are glued together. But the two b-arc boundary components of tiles D_1 and D_2 are glued together if they have two common vertices and the singularities belonging to D_1 and D_2 occur consecutively in the fibration. Therefore the decoration $\mathcal{E}$ of $\mathcal{S}^{\mathcal{F}}$ determines uniquely how the boundary components of the tiles are identified. It follows that the combinatorial data contained in $\mathcal{S}^{\mathcal{F}}$ determines the embedding h of the surface S uniquely up to foliation preserving isotopy. ∎

Definition 1.2. Let K be a knot or a link represented as a closed braid with the braid axis A and let S be a surface embedded in $S^3 \setminus K$, with foliation induced by the open book decomposition of $S^3 \setminus A$. Let β be a b-arc of the foliated surface S which is contained in a fiber H_θ. Then β is called *essential* if both sides H_θ split along β are pierced by K, otherwise it is called *inessential* (see Fig. 3). S is called an *essential tiled surface* if all b-arcs on the foliated surface S are essential.

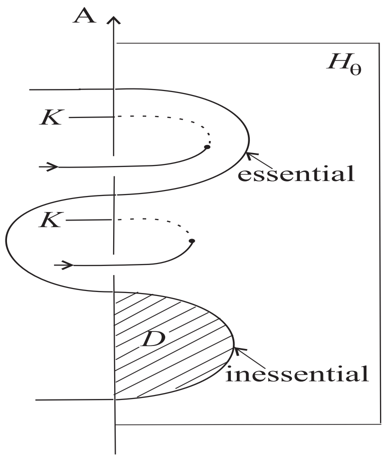


Fig. 3

In the same way we define an *essential tiled surface* $\mathcal{S}^{\mathcal{F}}$ that has an embedding in the link complement $S^3 \setminus K$ with respect to the axis A.

The combinatorial conditions imposed on the graph and the decoration of a tiled surface that has an embedding are not independent. We shall not concentrate on this point here.

2. Combinatorial description of essential tori which admit a standard tiling. Let T be an essential torus in a closed braid complement which admits a tiling. Choose an orientation on T. We shall say that a tiling of T is *standard* if each of its vertices is of valence 4 and for any vertex v the four tiles adjacent to v occur cyclically with signs $+, -, +, -$, when traveling on T around v. By Lemma 7 of [4], each incompressible torus T in the complement of a closed braid L which admits a *tiling* can be transformed, via a sequence of braid isotopies and exchange moves (on closed braids) and isotopies in closed braid complements, to an essential torus T'

in the complement of a closed braid L' so that T' admits a *standard tiling*. It follows that L' and L have the same link type $\mathcal{L}$. Note that T' can be chosen to be smooth.

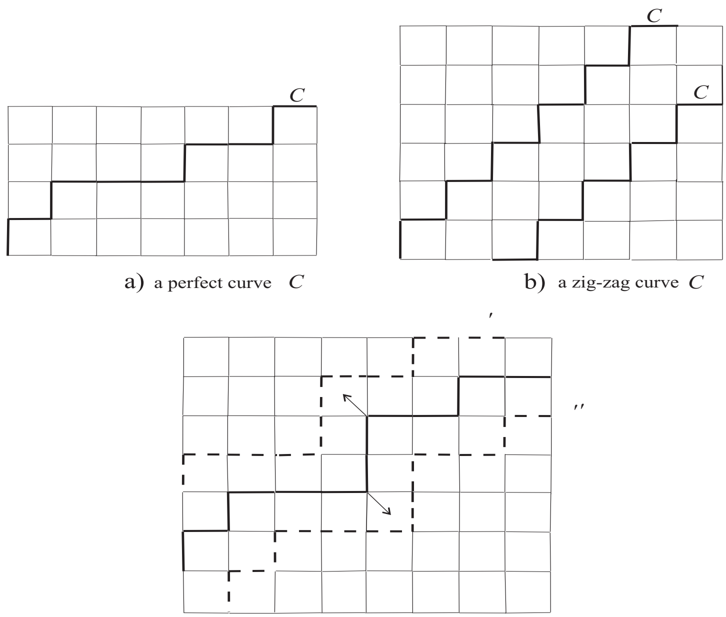
By a curve on the torus T we shall mean a piecewise smooth curve C which consists of a finite number of (non-singular) closed b-arcs, unless otherwise specified. To each such curve on T there corresponds a combinatorial curve $\mathcal{C}$ on the tiled torus $\mathcal{T}$ in a natural way. Note also that a (closed) combinatorial curve $\mathcal{C}$ on $\mathcal{T}$ can be considered as a (closed) path in the graph of $\mathcal{T}$. Then by the *length* of $\mathcal{C}$ we shall mean the length of the path corresponding to $\mathcal{C}$ in the graph and use the notation $l(C)$. In the future, we shall often identify a curve C on T with its combinatorial analogue $\mathcal{C}$ on $\mathcal{T}$ and use the same notation C for both. Under this convention, the length of C on T is also well defined.

Let $\mathcal{C}$ be a simple curve on $\mathcal{T}$. Fix an orientation on $\mathcal{C}$ and fix also an edge e (a b-arc) on $\mathcal{C}$. Let u and v be the initial and terminal vertices of e with respect to the given orientation on $\mathcal{C}$. Let v_1, v_2 and v_3 be the other three vertices on $\mathcal{T}$ which are adjacent to v, and let e_1, e_2 and e_3 be the edges joining v to v_1, v_2 and v_3, respectively. Since T is oriented it makes sense to talk about the left and right edges among the three outcoming edges e_1, e_2 and e_3 at the vertex v. Suppose e_1 is the left edge (b-arc) and e_3 the right edge (b-arc) at v. Then the remaining edge (b-arc) e_2 is called *straight* with respect to e. We shall say that the oriented curve (path) $\mathcal{C}$ makes a *right turn*, a *left turn* or has a *straight pass* at the vertex v if the edge f of $\mathcal{C}$ appearing just after e and leaving v is e_1, e_3 and e_2, respectively.

A (combinatorial) simple curve C on T (or $\mathcal{T}$) is called *perfect* if for some choice of orientation of C the left and right turns along C alternate. It follows that, in addition to the left and right turns, a perfect curve C can also contain "straight" passes (see Fig. 4(a) for a local picture of a perfect curve on $\mathcal{T}$). A perfect curve on $\mathcal{T}$ is called a *zig-zag curve* if it does not contain any straight passes, so the sequence $e_1, e_2, \ldots$ that forms $\mathcal{C}$ consists only of left and right turns that alternate (see Fig. 4(b)). A perfect curve is called *straight* if it does not contain locally any left and right turns.

Consider a tiled plane $\mathbb{R}^2$ as the universal cover of a tiled torus $\mathcal{T}$ and enhance it with the decoration inherited from the one of $\mathcal{T}$. Fix a covering $\varphi\colon \mathbb{R}^2 \to T$. Now, taking into account the signs of the tiles in $\mathcal{T}$, we may consider $\mathbb{R}^2$ as a tiled plane with a chessboard coloring. Since the definitions of perfect and zig-zag combinatorial curves on $\mathcal{T}$ are of local character, we can extend them in a direct manner to the chessboard plane $\mathbb{R}^2$.

Let $\mathcal{C}$ be a closed perfect curve on $\mathcal{T}$ and let $\lambda = \varphi^{-1}(\mathcal{C})$ be a perfect curve on $\mathbb{R}^2$. Then $\mathcal{C}$ is homotopically non-trivial in the torus T (or $\mathcal{T}$), and λ is unbounded in $\mathbb{R}^2$. First assume that λ is a straight line on $\mathbb{R}^2$, i.e. $\mathcal{C}$ does not contain any turns. It then makes sense to talk about two elementary



c) the two shifts of the curve

Fig. 4

"horizontal" shifts on the chessboard plane $\mathbb{R}^2$, in directions transversal to λ on $\mathbb{R}^2$. There are also four diagonal elementary shifts of $\mathbb{R}^2$. They all preserve the tiling on the plane and are understood in the combinatorial sense. Passing again from $\mathbb{R}^2$ to the tiled torus $\mathcal{T}$, we have essentially two elementary (non-diagonal) shifts of $\mathcal{T}$ that are transversal to a straight line $\mathcal{C}$ on $\mathcal{T}$. Similarly, if the perfect curve λ is not straight (i.e. contains turns) it makes sense to talk about two elementary diagonal shifts (by one) of $\mathbb{R}^2$ in directions transversal to λ (see Fig. 4(c) for a local picture of such shifts). We say that combinatorial curves c and c' on the plane $\mathbb{R}^2$ or on the tiled torus $\mathcal{T}$ are *parallel* if one is obtained from the other by a sequence of elementary (diagonal, vertical and horizontal) shifts of the tiled plane or the tiled torus.

Next, every torus T which admits a standard tiling can be cut open to obtain a planar tiled fundamental domain. We say that T has a (d, k)-*staircase tiling pattern* or *d by k staircase tiling pattern* P if a standard tiling of T has a staircase-tiling fundamental domain P with k rows and d tiles across each row, and its two opposite zig-zag sides are identified on T with a possible shift in the order of vertices, while the top and bottom sides are identified so that the second vertex on the bottom side coincides with the first vertex on the top side. It follows from the definition of standard tiling that $k \geq 2$ and d is even. As discussed in [12], any embedded torus T which admits a standard tiling has essentially two staircase patterns, dual to each other in some sense. In Fig. 5(a) we indicate an essential torus T which admits a standard tiling and has a 2 by 3 staircase pattern P.

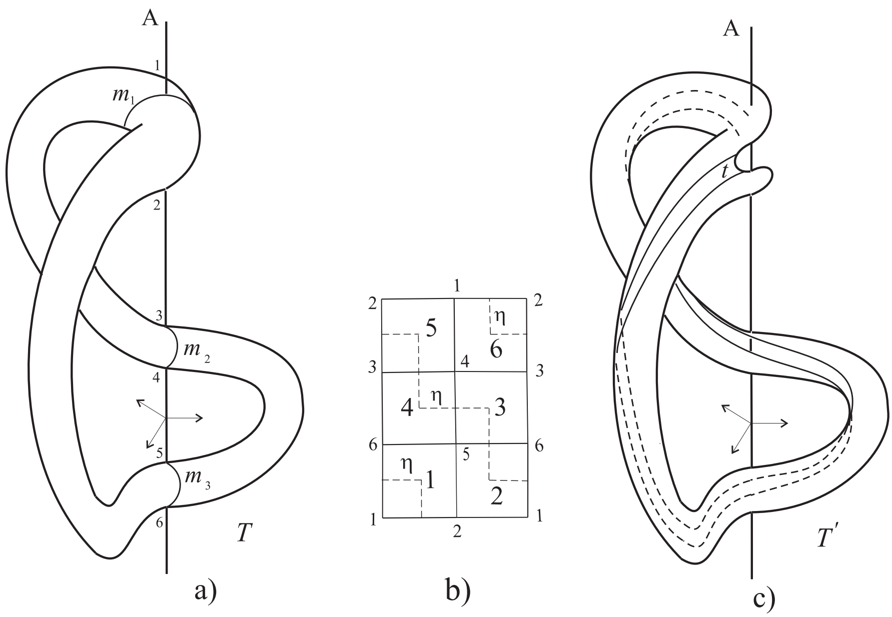


Fig. 5

The combinatorial patterns of embedded tori in closed braid complements considered in [12] and [4] represent actually tiled tori in the sense of Definition 1.1 (in the first case the torus T is cut along two cycles). It follows that the question of whether a given (decorated) staircase pattern can be realized geometrically can be answered by passing the tests in Proposition 1.1. By Proposition 1.2, if a tiled torus $\mathcal{T}$ has an embedding, its geometric realization (in $\mathbb{R}^3$) is unique in a certain sense.

Note that the fundamental domain of any embedded torus T that admits a standard tiling and has a type $k \geq 2$ can be represented by a rectangle of size 2 tiles by k tiles, where the opposite edges on the sides with k arcs are identified without any shift in the order [4]. For $k = 3$, see the example in Figs. 5(a) and 5(b). Ng [12] showed that for any natural numbers n and k, $k \geq 2$, there is a $(2n, k)$-staircase pattern P which is geometrically realizable. That is, there is a decoration of the pattern P so that the corresponding tiled torus $\mathcal{T}$ has an embedding. The latter can be checked directly by using Proposition 1.1. To describe geometrically the corresponding embedded tori, let us recall the definition of making tracks on tori of type $k \geq 2$.

Let T be a torus of type $k \geq 2$ which is made up of k cylinders C_i by consecutively gluing them along the corresponding boundary components (see Fig. 5(a) for $k = 3$ and Fig. 6(a) for $k = 2$). Each closed curve $m_i = C_i \cap C_{i+i}$ is a meridian of the torus T which intersects the axis A in exactly two points, say x_i and y_i, $i = 1, \ldots, k$. The points x_i and y_i decompose m_i into two arcs, α_i and β_i. The torus T admits a 2 by k rectangular pattern P (see Fig. 5(b) in the case $k = 3$). Consider on T a zig-zag longitude η, as in Figs. 6(a) and 5(b). Note that η intersects each meridian m_i, $i = 1, \ldots, k$, on T once and does not meet the axis A. Now we push the surface T along η in the direction of the inward normal to the surface at the points of η until each arc α_i (β_i, respectively) which intersects η is isotoped rel endpoints to a new arc α_i' (β_i', respectively) which intersects A in two more points. We shall say that the resulting torus T' is obtained from T by *making a track along* η on T (see Fig. 5(c)).

Let λ be a zig-zag longitude on T, isotopic to η, which is obtained from η by an elementary shift (see Fig. 6(a)). In the same way as before, one

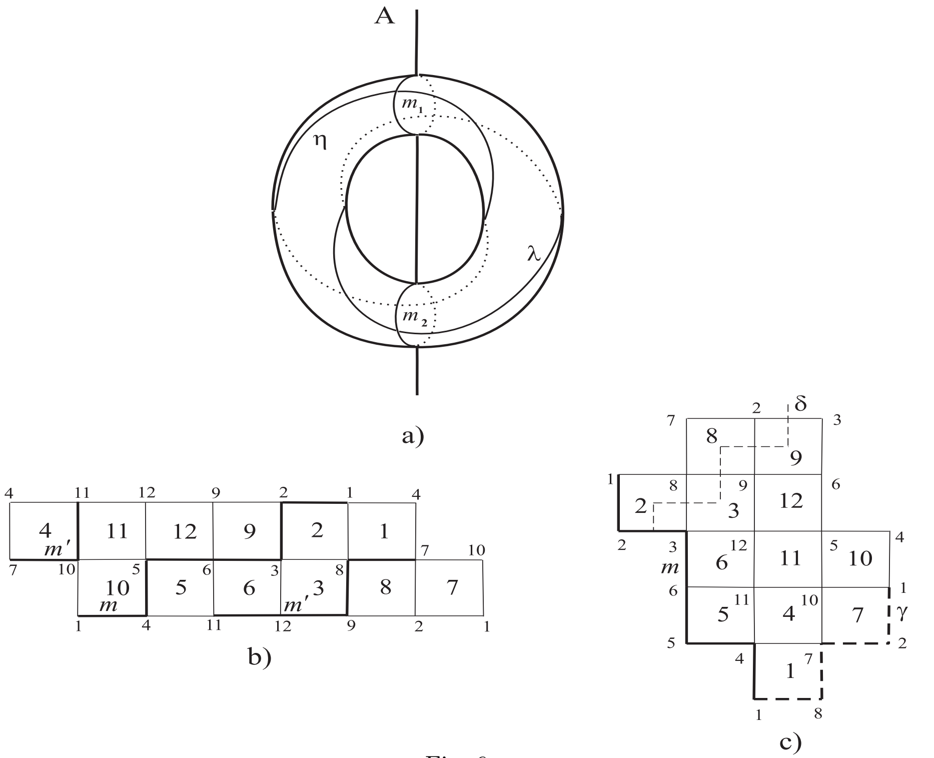


Fig. 6

can define the operation of making s parallel tracks along η and t parallel tracks along λ on T (see also [12]). Note that each meridian m_i is deformed by means of this procedure to a meridian m_i' on T' which intersects A in exactly $2(t+s+1)$ points. Making $n-1$ parallel tracks on T in an appropriate manner, we get a new torus T' which is smoothly embedded in S^3 and has a $2n$ by k staircase pattern P. For example, for the embedded torus which is obtained from the one in Fig. 6(a) by making one track along the longitude η and another track along λ, we get the 6 by 2 staircase tiling pattern pictured in Fig. 6(b). We then say that T' has the *generalized type* k, $k \geq 2$. Ng also showed that in some link complement each such torus is essential.

Ng [12] posed the problem of finding a complete set of well defined moves on embedded tori of type $k \geq 2$ such that any embedding of a torus which admits a standard tiling can be obtained from one of type $n \geq 2$ by a sequence of such moves.

We shall see that applications of the operation of making tracks alone to tori of type $k \geq 2$ do not suffice to obtain all essential tori which admit a standard tiling. The corresponding example of an embedded torus is indicated in Fig. 7. To describe this example, first take arcs c_1 and c_2 in

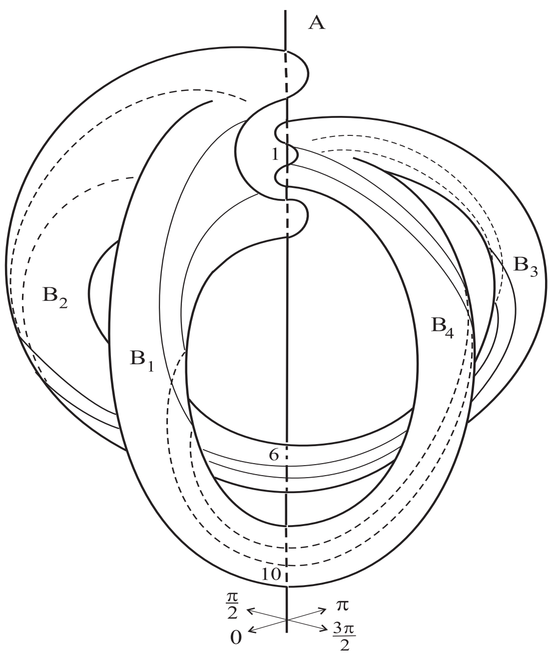


Fig. 7

the half-planes H_0 and $H_{\pi/2}$, with ends $(1, 10)$ and $(1, 6)$, respectively, on the axis A. Next take solid cylinders B_1 and B_2 in $\mathbb{R}^3$ with cores c_1 and c_2, respectively. Then B_1 and B_2 are glued together along a common meridional disc D. The other slice meridional disc bounding B_i on the opposite side is denoted by D_i, $i = 1, 2$. The cylinders B_1 and B_2 may be chosen to have D large, and D_1 and D_2 small and disjoint. Denote by B the solid cylinder obtained by gluing B_1 and B_2. Put $S = \mathrm{cl}(\partial B \setminus (D_1 \cup D_2))$. Let l be a longitude on S that makes a full twist along a meridian. Next, in the manner described before, we make a track along l on B so that the point 1 on the axis will be inside the track (see Fig. 7). The resulting solid cylinder is denoted by B'. The discs D_1 and D_2 are then deformed into new discs D_1' and D_2', respectively. Now take an arc c_3 in H_π with ends 1 and 6 and an arc c_4 in $H_{3\pi/2}$ with ends 1 and 10, all the ends being on the axis A. Consider thin tubes B_3 and B_4 about the arcs c_3 and c_4 which have the following properties:

1) $B_3 \cap B_4$ is a small meridional disc D' which passes through the vertex 1 on the axis A.
2) $B_3 \cup B_4$ intersects B' along the two meridional discs, D_2' and D_1', so that $B_3 \cup B_4 \cup B'$ forms a solid torus $\mathbb{T}'$ in $\mathbb{R}^3$.

Extend the track on the cylinder B' to the whole solid torus along a longitude of the cylinder $B_3 \cup B_4$ so that the resulting track wraps twice along a meridian of $\mathbb{T}'$. The resulting solid torus is denoted by $\mathbb{T}$. The boundary of $\mathbb{T}$ is the desired torus T. It is clear from the construction that T admits a standard tiling. It is also not difficult to show that T is essential in the complement of some closed braid b which determines a 2-component link. Note, however, that the closed braid b then admits exchange moves after which the torus T can be replaced with a new one which will be of the type described in previous examples.

Recall that each torus T in a 3-sphere S^3 bounds a solid torus $\mathbb{T}$ at least on one side [5]. Let T be a torus in a closed braid complement $\mathbb{R}^3 \setminus L$ which admits a standard tiling and bounds a solid torus $\mathbb{T}$ in $\mathbb{R}^3$. For a given $\theta \in [0, \pi]$ define the *slice* T_θ of the torus T at the angle θ to be the set $T \cap (H_\theta \cup H_{\theta+\pi})$ (where we identify 0 and 2π). Similarly, the slice $\mathbb{T}_\theta$ of the solid torus $\mathbb{T}$ at θ is $\mathbb{T} \cap (H_\theta \cup H_{\theta+\pi})$. For $\theta \in (\pi, 2\pi]$ we define T_θ to be equal to $T_{\theta-\pi}$ and $\mathbb{T}_\theta$ to be equal to $\mathbb{T}_{\theta-\pi}$. The slice T_θ is called *non-singular* if it contains no singularities of $\mathcal{T}$. In that case, the slice $\mathbb{T}_\theta$ is the disjoint union of a finite number of plane discs and annuli, and T_θ is the boundary of $\mathbb{T}_\theta$.

Now our aim is to define a longitude-meridional pattern for the torus T (or $\mathcal{T}$). Let m be a meridian of minimal length on $\mathcal{T}$ that bounds a meridional disc D in $\mathbb{T}$. Fix orientations on m and T. Note that each minimal meridian m on $\mathcal{T}$ is perfect. This becomes clear when we pass from the torus $\mathcal{T}$ to its universal cover $\mathbb{R}^2$. Indeed, the existence of two consecutive "left" or "right" turns on the curve $\eta = \varphi^{-1}(m) \subset \mathbb{R}^2$, with possible "straight" passes between them, immediately leads to a new meridian m' on $\mathcal{T}$ with length $l(m') \leq l(m) - 2$, from which the assertion follows.

Fix an edge e on the oriented meridian m. The set E of edges on m can be decomposed into two subsets, E_v and E_h, where each arc a from E_v is parallel to the arc e on $\mathcal{T}$, while every arc f from E_h is transverse to it. Put $l_1 = |E_v|$ and $l_2 = |E_h|$. The unordered pair $\{l_1, l_2\}$ is independent of the choice of a minimal meridian on T, so it is an intrinsic characteristic of the tiled torus $\mathcal{T}$ that has an embedding. Cutting T along m, we obtain a cylinder R bounded by the two copies of the meridian m, c and c'. Let p be the zig-zag curve of minimal length on the tiled cylinder R that joins a vertex u on the curve c to some vertex u' on c'. The number $s_m = [(l(p)+1)/2]$ is called the *width* of the cylinder R. Cutting R along p, we obtain a tiled fundamental domain D for the given embedded torus T. The domain D is bounded by two parallel zig-zag lines d and d', the copies of p, and the parallel perfect curves that correspond to the boundary components of the cylinder R (which we also denote by m and m'). The torus T is obtained from R by gluing m and m' together, with a possible shift t. Taking into account the orientation of m and T, this shift is characterized by two parameters t_h and t_v, the "horizontal" and "vertical" components. Let w be the vertex on c' that corresponds to the vertex u via the identification of the circles c and c'. Consider a curve q on c' which joins u' to w and so that the curve $\gamma = p \cup q$, which joins u to w, is perfect. Then on T, the closed curve γ represents a perfect longitude. The tiled plane region D, enhanced with decoration, will be called *a longitude-meridional pattern* for the embedded torus T (or the tiled torus $\mathcal{T}$) bounding the solid torus $\mathbb{T}$ in $\mathbb{R}^3$. In Fig. 6(c) we indicate a longitude-meridional pattern for T which is represented by a staircase pattern shown in Fig. 6(b).

Let $s = \min s_m$ where the minimum is taken over all minimal combinatorial meridians m on $\mathcal{T}$. It follows from the above discussion that the unordered pair $\{l_1, l_2\}$ and the number s are intrinsic characteristics of the embedded torus T that admits a standard tiling and bounds a solid torus in $\mathbb{R}^3$ (or for the corresponding tiled torus $\mathcal{T}$ that has an embedding).

In the exceptional case when $l_1 = 0$ or $l_2 = 0$, the meridian m contains only straight passes.

For the tiled torus indicated in Fig. 6(c) we have $l_1 = 4$, $l_2 = 2$ and $s = 2$, $t = (0,0)$.

PROPOSITION 2.1. *The embedded graph of a tiled torus $\mathcal{T}$ with a standard tiling is uniquely determined up to isomorphism by the triple $\langle m, s_m, t_m\rangle$ where m is a minimal meridian on $\mathcal{T}$, s_m is the width of the corresponding cylinder R and t_m is the shift that determines the identification of the boundary components of R.*

Proof. Note that a tiled fundamental domain D of any tiled torus $\mathcal{T}$, together with the information about identification of the pairs of edges on the boundary of D, determines uniquely (up to isomorphism) the underlying graph H of $\mathcal{T}$. In particular, a longitude-meridional pattern Q of a tiled torus

$\mathcal{T}$ determines the graph H uniquely. In the exceptional case when l_1 or l_2 is zero, the assertion is obvious. So we consider only the non-exceptional case.

Let $\mathcal{T}$ and $\mathcal{T}_1$ be two embedded tiled tori with standard tilings that are characterized by the same triple $\langle m, s_m, t_m \rangle$. Denote by H and H_1 the underlying graphs of $\mathcal{T}$ and $\mathcal{T}_1$, respectively. Let p be a zig-zag curve on $\mathcal{T}$ that determines the width s_m of the corresponding cylinder R. We shall show that, given the triple $\langle m, s_m, t_m \rangle$, there is a canonical way to obtain a tiled fundamental domain of the tiled torus, from which the assertion follows. Fix a covering φ: $\mathbb{R}^2 \to T$. Let m' be a lift of the minimal meridian m to $\mathbb{R}^2$ with ends a and a', where $l(m) = l(m')$. Moreover let p' be the lift of p with one end at a, where $l(p') = l(p)$. We have $s_m = [(l(p')+1)/2]$. The fundamental domain for the tiled torus $\mathcal{T}$ can be obtained as follows. Shift first m' along the zig-zag line p' on $\mathbb{R}^2$, i.e. perform s_m elementary diagonal shifts defined by p'. The resulting perfect curve $\widehat{m}$ is also a lift of m to $\mathbb{R}^2$. Next, shift $\widehat{m}$ along a perfect curve that is characterized by the pair $t_m = (t_h, t_v)$, i.e. perform t_h horizontal and t_v vertical shifts. The resulting perfect curve on $\mathbb{R}^2$ is denoted by $\widetilde{m}$. Under the above shifts, m' fills up a tiled domain D that is actually a plane domain for the tiled torus $\mathcal{T}$. The domain D is bounded by two pairs of parallel (in combinatorial sense) curves. It follows from the construction that D is also a tiled fundamental domain for $\mathcal{T}_1$. The way of identification of parallel sides on D to obtain a tiled torus is the same in both cases. This completes the proof. ∎

Note that, for a given tiled torus $\mathcal{T}$ that has an embedding, the width of the cylinder R_m depends, in general, on the choice of a minimal meridian m on $\mathcal{T}$. For an example, we indicate in Fig. 8 two longitude-meridional patterns D_1 and D_2 for a tiled torus $\mathcal{T}$ that correspond to the choice of the minimal meridians m_1 and m_2 on $\mathcal{T}$. It is easy to check that the corresponding cylinders R_1 and R_2 have width 1 and 2, respectively.

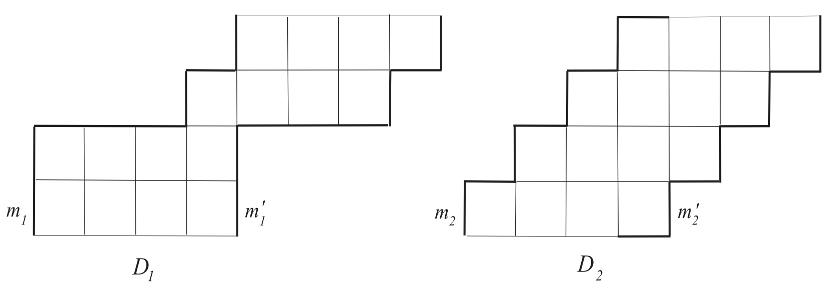


Fig. 8

Remark 2.1. The decorated longitude-meridional and staircase patterns for the torus T that admits a standard tiling and bounds a solid torus in $\mathbb{R}^3$ represent the same tiled torus $\mathcal{T}$. The only difference is in the choice of a fundamental domain of T.

QUESTION 2.1. Given a configuration of the combinatorial meridian m that is characterized by the parameters l_1, l_2, where $l_1 + l_2$ is an even number, and a natural number s, does there exist a tiled torus $\mathcal{T}$ that has an embedding and possesses a longitude-meridional pattern Q with the meridian m and width s? What are the admissible values of the shift t in that case?

It is known that the answer to the first part of Question 2.1 is affirmative. The proof of this fact will be given separately.

Let $\mathcal{T}$ be a tiled torus with a standard tiling that has an embedding, and γ a perfect longitude on $\mathcal{T}$. Moreover let H be the underlying graph of $\mathcal{T}$. The graph H is naturally embedded in the torus T. Consider now a topological graph G, the dual of the embedded graph H on T. The vertices of G are simply the singularities of the foliation $\mathcal{F}$ on T. The edges of G are the arcs which join the singularities in adjacent tiles and are transverse to the leaves of $\mathcal{F}$. The combinatorial curve γ on $\mathcal{T}$ can be considered also as a cycle in the topological graph H. It now makes sense to consider the perfect longitudes δ on T or on $\mathcal{T}$ that are obtained from γ by shifting it by $1/2$. From the combinatorial point of view, δ is considered as a cycle in the topological graph G. The most interesting case is when δ is obtained by such a shift from a zig-zag longitude γ on $\mathcal{T}$. Fix a minimal meridian m on $\mathcal{T}$ and an orientation on it. There is a natural orientation on δ so that the orientation on T agrees with the one defined by the pair (m, δ) of oriented curves. We define $w(\delta)$ to be the winding number of δ around the axis A. Note that $w(\delta)$ can be directly read off from the tiled torus $\mathcal{T}$ or from the longitude-meridional pattern Q for T. For example, consider the embedded torus T' in Fig. 5(c). It is obtained from the torus T of Fig. 5(a) by making a track t along the longitude η shown in Fig. 5(b). Let δ be a zig-zag longitude on $\mathcal{T}'$ along the track t on T'. We then have $w(\delta) = \pm 1$. Next, shifting horizontally by 1 the longitude η on the tiled torus $\mathcal{T}$, we obtain another zig-zag longitude λ on T. Let T_1 be the torus obtained from T by making a track along λ on T, and let δ' be a zig-zag longitude on T_1 along this track. We then have $w(\delta') = \pm 2$. For the tiled torus indicated in Fig. 6(c) we have $w(\delta) = \pm 1$.

QUESTION 2.2. Given a tiled torus that has an embedding, how to recognize in combinatorial terms a meridian of the solid torus bounded by the embedded torus with the given tiling?

The answer to this question is important for solving many other related problems. Below we give a simple method for finding a (minimal) meridian on the torus T which admits a standard tiling. This method can also be used for finding combinatorial meridians on tiled tori that have an embedding. The corresponding algorithm works due to Theorem 2.2.

Before formulating the next result, we first introduce some notions.

By a *θ-sector* in S^3 or in $\mathbb{R}^3$ we shall mean a part of the 3-sphere or 3-space bounded by two half-planes. More precisely, for a given pair $\langle\varphi,\psi\rangle$ of angles denote by $H_{\varphi,\psi}$ the θ-sector bounded by the half-planes H_φ and H_ψ, $H_{\varphi,\psi} = \bigcup_{\theta\in[\varphi,\psi]} H_\theta$, where the (closed) θ-interval $[\varphi,\psi]$ is defined as in Section 1. The plane in $\mathbb{R}^3$ which contains the half-plane H_θ will be denoted by P_θ. Recall also that for each $\theta \in [0,2\pi]$ we denote by T_θ the slice of T at angle θ. Any non-singular slice T_θ has, in general, several connected components C each of which is homeomorphic to a circle.

THEOREM 2.2. *Let $\mathcal{T}$ be a tiled torus with a standard tiling that has an embedding T in $\mathbb{R}^3$. Then there exist $\theta \in [0,2\pi)$ and a component C of the non-singular slice T_θ so that the circle C bounds a meridional disc in a solid torus $\mathbb{T}$ bounded by T.*

Proof. Let $s_0, s_1, \ldots, s_{2r-1}$ be the cyclic order of all the singularities of $\mathcal{T}$ numbered according to increasing $\theta \in [0,2\pi)$. Let θ_i be the polar angle so that $s_i \in H_{\theta_i}$, $i = 0,1,\ldots,2r-1$, where the values θ_i are regarded modulo 2π and the integers i are considered modulo $2r$. We may suppose that each plane P_{θ_i} contains at most one singularity. Choose a sequence $0 \le \varphi_0 < \varphi_1 < \cdots < \varphi_{2r-1} < 2\pi$ of polar angles so that for each $0 \le i \le 2r-1$ there is exactly one singularity in $H_{\varphi_i,\varphi_{i+1}}$. We may assume for simplicity that $\varphi_0 = 0$. To continue, we need the following

LEMMA 2.1. *There exist a plane P_{φ_i}, where $0 \le i \le 2r-1$, and a component C of the slice $T_{\varphi_i} = T \cap P_{\varphi_i}$ so that the closed curve C is homotopically non-trivial in T.*

Proof of the lemma. Suppose the contrary, i.e. that all components of any slice T_{φ_i} are homotopically trivial in T. Then each such component bounds a disc on T. In particular, T_{φ_0} is the union of disjoint closed curves $C_1,\ldots,C_k$ so that each C_i bounds a disc D_i^0 on T. Note that these curves cover all the vertices of the tiled torus $\mathcal{T}$. Let $\mathcal{D} = \{D_{j_1}^0,\ldots,D_{j_l}^0\}$ be the set of all outer discs on the torus T from the above collection. Then the discs in $\mathcal{D}$ are disjoint. Consider a disc $D_{j_i}^0$. For $\varepsilon > 0$ small enough, the regular ε-neighborhood $N(C_{j_i})$ of C_{j_i} in $D_{j_i}^0$ lies completely on the same side of the plane P_{φ_0}. We color the disc $D_{j_i}^0$ green if $N(C_{j_i}) \subset H_{0,\pi}$, and red if $N(C_{j_i}) \subset H_{\pi,2\pi}$. Denote by $\mathcal{D}^-$ ($\mathcal{D}^+$, respectively) the set of discs $D_{j_i}^0 \in \mathcal{D}$ colored green (red, respectively). Suppose $\mathcal{D}^+ \neq \emptyset$. We claim that $\mathcal{D}^-$ is then empty. Indeed, suppose $\mathcal{D}^- \neq \emptyset$. Then, in a fragment of the torus where the discs from $\mathcal{D}^+$ neighbor on the ones of $\mathcal{D}^-$, we have essentially

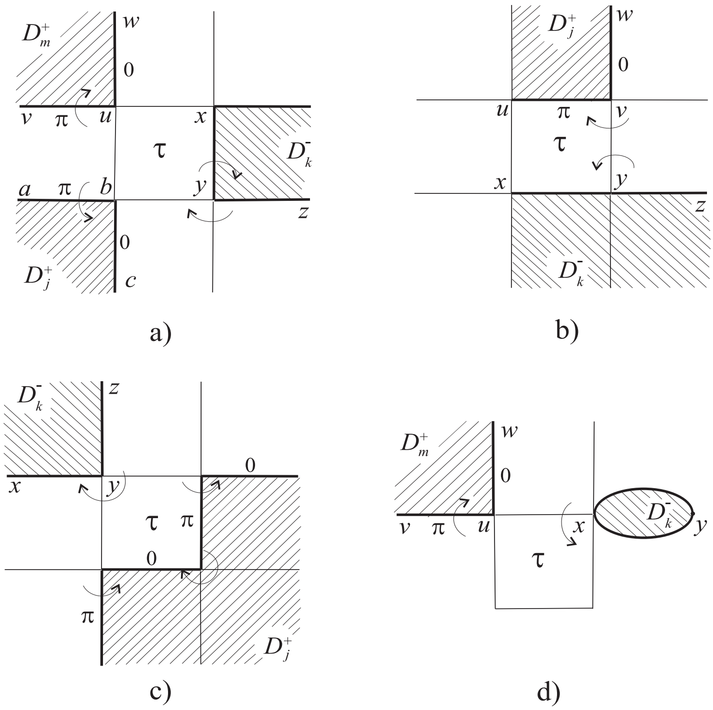


Fig. 9

one of the four pictures in Fig. 9. Here each tile which is not contained in any disc from $\mathcal{D}$ is colored white. Now a direct inspection of possible decorations (by the parameter θ) of the tiles and their edges in the indicated fragments shows that none of them is admissible. It follows that $\mathcal{D}^- = \emptyset$. Let $H_{\psi_0}, H_{\psi_1}, \dots, H_{\psi_l}$ be the enumeration of all half-planes from the θ-sector $H_{0,\pi}$, where $0 = \psi_0 < \psi_1 < \cdots < \psi_{l-1} < \psi_l = \pi$, each of which is contained in some plane P_{φ_i}.

Now we define inductively the sequences $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_l$ and $G_0, G_1, \dots$ $\dots, G_{l-1}$, where each $\mathcal{D}_i$ and each G_i is a disjoint union of connected regions on the torus T. We set $\mathcal{D}_0 = \bigcup \mathcal{D}^+$. Put $G_i' = T \cap H_{\psi_i,\psi_{i+1}}$, $i = 0, 1, \dots, l-1$. Each G_i' is then the disjoint union of connected regions d_s^i on T of the following two types.

(a) Each region of type (a) is bounded on T by two b-arcs, $b^1(i,j)$ and $b^2(i,j)$, with common vertices i and j so that $b^1(i,j)$ exists at $\theta = \psi_i$, $b^2(i,j)$ exists at $\theta = \psi_{i+1}$ and both arcs represent the same edge of adjacent tiles (so $[b^1(i,j)] = [b^2(i,j)]$).

(b) Each region of type (b) is a tile τ which contains a unique singularity s and is bounded by two arcs b_1 and b_3 which exist at $\theta = \psi_i$ and two b-arcs which exist at $\theta = \psi_{i+1}$.

Denote by G_0 the subset of G_0' which is the union of the regions d_s^0 that are not contained in $\mathcal{D}_0$. Suppose we have already defined the sets G_i and $\mathcal{D}_i$. Define G_{i+1} to be the subset of G_{i+1}' which is the union of the regions d_s^{i+1} that are not contained in $\mathcal{D}_i$. Then $\mathcal{D}_{i+1}$ is defined to be the subspace $\mathcal{D}_i \cup G_i$ of the torus T. Denote by $\mathcal{B}_i$ the set of connected components of $\mathcal{D}_i$.

The induction process finishes at step l. Note that, by the definition of $\{\varphi_i\}$, each set G_i, $i = 0, 1, \ldots, l-1$, contains at most one region of type (b). Moreover, by the definition of $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_l$, the set $\mathcal{D}_l$ coincides with the whole torus T. It follows directly from the definition that each $\mathcal{D}_i$ is a submanifold of T contained in the θ-sector H_{π,ψ_i} and each edge (b-arc) of any boundary component of $\mathcal{D}_i$ is contained in $H_\pi \cup H_{\psi_i}$. The following question arises. What are the components of the submanifolds $\mathcal{D}_i$? We claim that for each $i \leq l-1$ every component (region) R_j^i of $\mathcal{D}_i$ is a disc with a finite number of holes. That is, each R_j^i is bounded by a finite number of cycles $C_{j_1}, \ldots, C_{j_p}$, one of which, say C_{j_1}, is outer and contains a disc D_{j_i} on T, while $C_{j_2}, \ldots, C_{j_p}$ bound disjoint discs $D_{j_2}, \ldots, D_{j_p}$ on T lying inside D_{j_1}. The case when the number of holes in a disc is 0 is not excluded.

The proof is by induction on i. For $i = 0$ the assertion follows directly from the definition of $\mathcal{D}_0$. Suppose the assertion holds for all $i < m$ where $m < l$. By definition, $\mathcal{D}_m$ is obtained from $\mathcal{D}_{m-1}$ by gluing a finite number of regions d_s^m of type (a) and, possibly, a region of type (b).

Consider first the case when G_{m-1} contains no region of type (b). Adding a finite number of regions of type (a) to G_{m-1} does not change the topological or combinatorial properties of $\mathcal{D}_{m-1}$ and its components. In particular, $|\mathcal{B}_m| = |\mathcal{B}_{m-1}|$. Moreover if $R_j^{m-1} \in \mathcal{B}_{m-1}$ is a disc with k holes on T, then, after adding a finite number of regions of types (a), it remains a disc on T with the same number of holes.

Now consider the second possibility. We may think of $\mathcal{D}_m$ as made of $\mathcal{D}_{m-1}$ in two steps. In the first step, we glue to $\mathcal{D}_{m-1}$ a finite number of regions of type (a) from G_{m-1}, as before. In the second step, we glue a (unique) tile τ to the resulting manifold $\mathcal{D}_{m-1}'$ on T. As already noted, the first step does not lead to any changes of topological or combinatorial properties of the submanifold $\mathcal{D}_{m-1}$. We now have to inspect what happens if we glue a tile τ to $\mathcal{D}_{m-1}'$ in an appropriate manner. The treatment of this procedure also falls into two cases.

In the first case, there are components R_j^{m-1} and R_n^{m-1} of $\mathcal{D}_{m-1}'$ and pair of opposite edges b_1 and b_3 of the tile τ, with $b_1 \in H_{\psi_{m-1}}$ and $b_3 \in H_{\psi_{m-1}}$, so that b_1 lies on the outer boundary component of R_j^{m-1} and b_3 lies on the outer boundary component of R_n^{m-1}. Suppose R_j^{m-1} is a disc on T with q_j holes and R_n^{m-1} is one with q_n holes. Gluing the tile τ to R_j^{m-1} and R_n^{m-1}

along the common edges results in a disc on T with $q_j + q_n$ holes. Thus $|\mathcal{B}_m| = |\mathcal{B}_{m-1}| - 1$ in this case.

In the second case, there is a component R_j^{m-1} of $\mathcal{D}'_{m-1}$ and a pair of opposite edges b_1 and b_3 of the tile τ, with $b_1 \in H_{\psi_{m-1}}$ and $b_3 \in H_{\psi_{m-1}}$, so that b_1 and b_3 lie on the same (not necessarily outer) boundary component of R_j^{m-1}. Denote by C the corresponding boundary component of R_j^{m-1}. Suppose R_j^{m-1} is a disc on T with q holes. If C is not outer, then it bounds a disc lying inside another one bounded by the outer boundary component of R_j^{m-1}. In this situation, gluing τ to R_j^{m-1} along the common edges obviously yields a new disc R_j^m on T with $q+1$ holes. The other components of $\mathcal{D}'_{m-1}$ remain unchanged. As a result, the cycle C is transformed into two new cycles each of which bounds a disc on T.

Now suppose that C is the outer boundary component of R_j^{m-1}. Gluing the tile τ to R_j^{m-1}, we obtain a new connected region, R_j^m, on T. All the other components of $\mathcal{D}'_{m-1}$ remain unchanged. As a result of gluing τ to R_j^{m-1}, the cycle C is transformed into two new cycles, say C_1 and C_2, the boundary components of the submanifold $\mathcal{D}_m$ of T. By construction, the closed curves C_1 and C_2 are isotopic in T. Moreover there is a component C' of the slice T_{ψ_m} which lies completely in $\mathcal{D}_m$ and is isotopic to C_1 in T. By assumption, C' also bounds a disc on T. It follows that so do C_1 and C_2. Note, however, that after gluing τ to R_j^{m-1}, the boundary components of the new region R_j^m may encircle on T, in principle, the other components of the submanifold $\mathcal{D}_m$. In any case, $\mathcal{D}_m$ is a disc with $q+1$ holes on T. We also have $|\mathcal{B}_m| = |\mathcal{B}_{m-1}|$.

At the last lth step of the inductive procedure only two types of gluing regions from G_{l-1} to components of $\mathcal{D}_{l-1}$ are possible. These are indicated in Fig. 10.

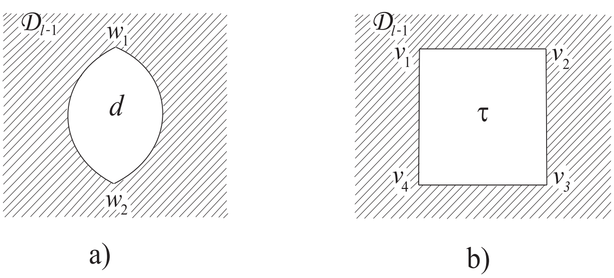


Fig. 10

In the first case, the region d of type (a) from G_{l-1} is glued to a component R_i^{l-1} of $\mathcal{D}_{l-1}$ along two common edges $b^1(s,t)$ and $b^2(s,t)$, where $b^1(s,t)$ exists at $\theta = \psi_{l-1}$ and $b^2(s,t)$ exists at $\theta = \psi_l = \pi$. In the second case, a tile τ is glued to R_i^{l-1} along common edges b_1, b_2, b_3, b_4, where b_1, b_3 is a pair of opposite edges of τ which exist at $\theta = \psi_{l-1}$, and b_2, b_4 is a pair of opposite edges of τ which exist at $\theta = \psi_l = \pi$. In both cases, this procedure consists in sealing up the holes bounded by the cycles C_j^{l-1}, the boundary components of the regions R_j^{l-1} of $\mathcal{D}_{l-1}$. Since the initial closed curves $C_1, \dots, C_k$ bounding on T the discs $D_1^0, \dots, D_k^0$, respectively, cover all the vertices of the tiled torus $\mathcal{T}$, the submanifold $\mathcal{D}_{l-1}$ of T has to be connected, i.e. $\mathcal{D}_{l-1}$ is a disc with a finite number of holes on the torus. Passing from $\mathcal{D}_{l-1}$ to $\mathcal{D}_l$, we only seal up the holes of $\mathcal{D}_{l-1}$ on T. As a result, at the lth step we obtain a closed submanifold $\mathcal{D}_l$ of T with Euler characteristic two, i.e. a sphere. This is a contradiction. This completes the proof of the lemma. ∎

Let us return to the proof of Theorem 2.2. Let P_{φ_i}, $0 \le i \le 2r-1$, be a plane from the above collection so that the slice $T \cap P_{\varphi_i}$ contains a cycle which is homotopically non-trivial in T. Let C' be an innermost (on the plane P_{φ_i}) homotopically non-trivial component of the slice $T \cap P_{\varphi_i}$. The cycle C' bounds a disc D' on the plane P_{φ_i}. The disc D' can contain in its interior only the components of $T \cap P_{\varphi_i}$ which are homotopically trivial in T. Let $C_1, \dots, C_t$ be all the components of $T \cap P_{\varphi_i}$ which are contained in the interior of D'. Each C_i bounds a disc D_i on the torus T and the disc B_i on the plane P_{φ_i}. For each i choose an open ε-neighborhood U_i of B_i in P_{φ_i} with a small ε. Remove all open discs U_i, $i = 1, \dots, t$, from D'. Denote by $\widetilde{D}$ the remaining part of D'. Cutting S^3 open along the torus T, we obtain two connected 3-manifolds, T_1 and T_2, with common boundary T. Let T_k be the one which contains $\widetilde{D}$. By the above reasoning, in the interior of T_k there are closed discs W_i with the following properties:

(a) $W_i \cap W_j = \emptyset$ if $i \ne j$;
(b) $W_i \cap \widetilde{D} = \partial W_i = \mathrm{Fr} U_i$, $i = 1, \dots, t$.

Put $D = \widetilde{D} \cup \bigcup_{i=1}^t W_i$. It is clear from the above construction that the disc D is contained in T_k and $\partial D = C'$. Now, by using the standard cut-and-paste technique, it is not difficult to show that T_k is a solid torus and D is its meridional disc. This completes the proof of the theorem. ∎

A meridional disc D of a solid torus $\mathbb{T}$ bounded by the slice component C' and obtained by surgery on a plane disc, as described in the proof of Theorem 2.2, will be called *good*. Note that if the torus T under the assumptions of Theorem 2.2 bounds solid tori on both sides in S^3, say $\mathbb{T}_1$ and $\mathbb{T}_2$, the above theorem asserts only that there is a component C which is a meridional curve for one of them. Let us consider the torus T of type 3 indicated in Fig. 5. It bounds a solid torus $\mathbb{T}_1$ in $\mathbb{R}^3$. Note that T also bounds in S^3 a solid torus $\mathbb{T}_2$ on the opposite side. It is easy to see that the slice meridian m_1 on T bounds a meridional disc D of $\mathbb{T}_1$. On the other hand, no component of any slice of T bounds a meridional disc of $\mathbb{T}_2$.

The following algorithm for finding a meridional disc on an essential torus that admits a standard tiling is based on the proofs of Theorem 2.2 and Lemma 2.1. Similarly, we may consider instead of T a tiled torus $\mathcal{T}$ with a standard tiling that has an embedding. Let T be an essential torus in a closed braid complement which admits a standard tiling. Let $s_0, s_1, \ldots, s_{2k-1}$ be the cyclic order of all the singularities of the foliation of T according to increasing $\theta \in [0, 2\pi)$. For each $i = 0, 1, \ldots, 2k-1$ let ψ_i be the polar angle so that $s_i \in H_{\theta_i}$.

ALGORITHM 2.1

Step 1. Choose a cyclic sequence $\varphi_0 < \varphi_1 < \cdots < \varphi_{2k-1}$ so that no ψ_j is equal to any φ_i or $\varphi_i + \pi$ and for each i there is at most one singularity on T which lies in $H_{\varphi_i, \varphi_{i+1}}$, $i = 0, 1, \ldots, 2k-1$.

Step 2. Construct a cyclic sequence $T_{\varphi_0}, T_{\varphi_1}, \ldots, T_{\varphi_{2k-1}}$ of the slices of the torus T (or $\mathcal{T}$).

Step 3. Inspect consecutively all the components of $T_{\varphi_0}, T_{\varphi_1}, \ldots, T_{\varphi_{2k-1}}$ until a (homotopically) non-trivial cycle C' in some T_{φ_i} occurs. Theorem 2.2 provides the existence of such a slice of T and of its component C'. The triviality of the cycle C' on T can be easily checked by passing to the chessboard tiled plane $\mathbb{R}^2$, the universal cover of the embedded tiled torus T.

Step 4. Check if there are other homotopically non-trivial cycles in T which lie inside the disc D' bounded by C' on the corresponding plane. Choose an innermost one with respect to the disc D'. The resulting curve C is the desired meridian on the torus T that bounds a solid torus $\mathbb{T}$.

Let T be a torus in a closed braid complement $\mathbb{R}^3 \setminus L$ which admits a standard tiling $\mathcal{T}$. By Theorem 2.2 and symmetry, we may assume that T bounds in $\mathbb{R}^3$ a solid torus $\mathbb{T}$ and has a slice meridional curve C. Moreover C bounds a good meridional disc in $\mathbb{T}$. Fix an orientation on T. Using the combinatorial presentation of the curve C on $\mathcal{T}$, one can isotope it to a minimal meridian m on T (or $\mathcal{T}$). Fix an orientation on m. Among all the collections of disjoint minimal meridians in $\mathcal{T}$ that include m consider the one, say $\mathcal{M}$, with a maximal number of elements. It is easy to show that all the meridians in $\mathcal{M}$ may be assumed to be parallel to m in the combinatorial sense (see definition above).

Disjoint minimal meridians m and m' on $\mathcal{T}$ that are obtained from each other by an elementary shift bound in a natural way a cylinder R_m of width 1. The tiled cylinder R_m enhanced with the data inherited from $\mathcal{T}$ is then called an *elementary building block* for the tiled torus $\mathcal{T}$ and the embedded torus T that bounds a solid torus $\mathbb{T}$. Note that any elementary building block R_m consists of $|l(m)|$ tiles. In $\mathcal{M}$ there may be other cylinders R' of width 1. Each such R' is bounded by two disjoint parallel minimal meridians m_1 and m_2. We then say that R' is a *defective building block* with respect to m. A defective building block R' is characterized by the number $d(m) = r(m) - l(m)$, where $r(m)$ is the number of tiles in R'.

We have the following

PROPOSITION 2.3. *Let Q be a longitude-meridional pattern for $\mathcal{T}$ that is characterized by the meridian m, width s and shift t as defined above. Moreover let $\mathcal{M}$ be the maximal collection of disjoint minimal meridians on $\mathcal{T}$ that contains m and $|\mathcal{M}| = r$. Then $r = s$ and all the meridians in $\mathcal{M}$ may be chosen to be parallel to m. Moreover $\mathcal{M}$ decomposes T into s building blocks R_i; all of them are elementary except maybe one which is defective.*

Proof. The assertion follows directly from the definitions, Proposition 2.1 and the above discussion. ∎

The defective building block R_j for $\mathcal{T}$ can contain internal vertices, so the minimal meridians from a collection $\mathcal{M}$ may not cover all vertices of $\mathcal{T}$. However, if m is a straight minimal meridian or a zig-zag minimal meridian (i.e. $l_1 = l_2$) on $\mathcal{T}$, then any maximal collection $\mathcal{M}$ that contains m decomposes $\mathcal{T}$ into elementary building blocks, so it covers all the vertices of $\mathcal{T}$. In general, this is not the case. To see this, let us consider the pattern in Fig. 5(b). The corresponding embedded torus T bounds a solid torus $\mathbb{T}$ in $\mathbb{R}^3$ and is unknotted in $\mathbb{R}^3$. It follows that T bounds a solid torus $\mathbb{T}_1$ on the opposite side in S^3. One of the two combinatorial curves coded by $(1, 6), (6, 3), (3, 2), (2, 1)$ represents a minimal meridian m' on the embedded tiled torus $\mathcal{T}$ that bounds $\mathbb{T}_1$. The maximal number r of disjoint minimal meridians on $\mathcal{T}$ is 1. It follows that m' decomposes the (embedded) tiled torus $\mathcal{T}$ into one defective building block with respect to $\mathbb{T}_1$.

On the other hand, the maximal number r of disjoint minimal meridians on the embedded torus T (or the tiled torus $\mathcal{T}$) that bounds $\mathbb{T}$ equals 3, and these meridians decompose $\mathcal{T}$ into three elementary building blocks and cover all the vertices of $\mathcal{T}$.

Let R_i be any building block for $\mathcal{T}$ from the meridional decomposition of $\mathcal{T}$ by a collection $\mathcal{M}$ of minimal meridians that contains m. Let m_1 and m_2 be the two minimal meridians that bound R_i and let V_1 and V_2 be the sets of vertices on m_1 and m_2, respectively. Suppose first that m is a straight curve on $\mathcal{T}$. Then m_2 is obtained from m_1 by an elementary "horizontal" shift (by one). This gives a natural bijection $h\colon V_1 \to V_2$.

Suppose now that m contains turns. If the block R_i is elementary, then there is a natural map $h\colon V_1 \to V_2$ induced by a diagonal shift of m_1 by one. If R_i is defective there is also a shift of $\mathcal{T}$ along a perfect curve (see above) that moves m_1 to m_2. This shift also induces a natural bijection $h\colon V_1 \to V_2$.

In any case, h can be extended to a PL-homeomorphism of the meridians m_1 and m_2 which maps edges of m_1 into edges of m_2.

A characteristic feature of a decomposition of $\mathcal{T}$ by minimal combinatorial meridians is that the latter has a geometric meaning. Indeed, each meridian m_i from $\mathcal{M}$ bounds a meridional disc D_i in the solid torus $\mathbb{T}$. Moreover, using the standard cut-and-paste technique and general position, we may achieve that the discs D_i, $i = 1, \ldots, s$, are disjoint. This gives a decomposition of $\mathbb{T}$ into k solid cylinders $\mathcal{R}_i$, the building blocks for $\mathbb{T}$, where each $\mathcal{R}_i$ is bounded by the embedded cylinder R_i and the meridional discs D_i and D_{i+1}, $i = 1, \ldots, s-1$.

Our interest in decomposition of embedded tiled tori into building blocks and its analysis is motivated by an attempt to extend the class of standard embedded tori of type ≥ 2 and generalized type ≥ 2 to a wider class of geometric tori of standard position. In this way, we hope to obtain a geometric description for the whole class of essential tori that admit a standard tiling. For standard embedded tori of type ≥ 2 and generalized type ≥ 2, the building blocks that form a decomposition are elementary and have a geometric description. In the general case, the geometric description is not complete. Note, however, that the natural maps $h\colon m_1 \to m_2$ defined above establish some coherence in the order of vertices on the meridians m_1 and m_2, with respect to the axis A, and in the cyclic order of the arcs on these meridians, with respect to the polar angle θ.

We would like to standardize the geometric position of the minimal meridians and building blocks for the embedded tiled tori T. Unfortunately, there is no canonical way of choosing such meridians on an embedded tiled torus. One tries also to find, among the minimal meridians on T that form a decomposition of T into building blocks, those which are close in a certain sense to the slice meridional curves. In this connection the following question arises:

QUESTION 2.3. Does any embedded torus T which admits a standard tiling and bounds a solid torus in $\mathbb{R}^3$ possess a slice minimal meridian?

We shall see that the answer to this question is negative.

REMARK 2.2. Let T be an essential torus which admits a standard tiling $\mathcal{T}$ and bounds a solid torus $\mathbb{T}$. Moreover let H be the underlying graph of $\mathcal{T}$ embedded in T and let G be the dual of the embedded graph H on T. Note that as graphs, G and H are isomorphic. Each edge of the embedded graph G joins two singularities in $\mathcal{T}$ and is transverse to the leaves of the foliation $\mathcal{F}$ of T, so it has the natural orientation according to increasing θ. Denote by G' the corresponding orgraph. The set E' of oriented edges (arcs) of G' can be covered by a collection $\mathcal{B}$ of closed (oriented) zig-zag paths l_j, l'_j, $j = 1, \ldots, q$, which have the following properties. For a fixed oriented path l_n from $\mathcal{B}$, each l_j is parallel to l_n on T, while each l'_j is antiparallel to it, $j = 1, \ldots, q$ (see Fig. 11).

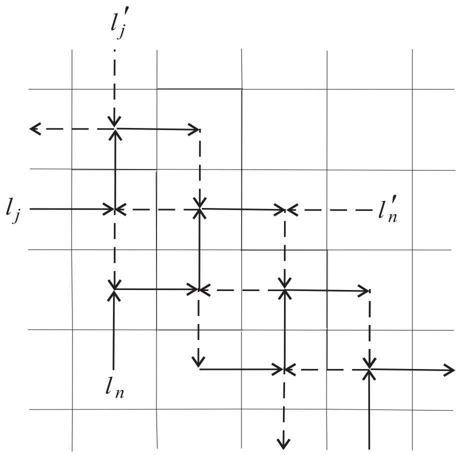
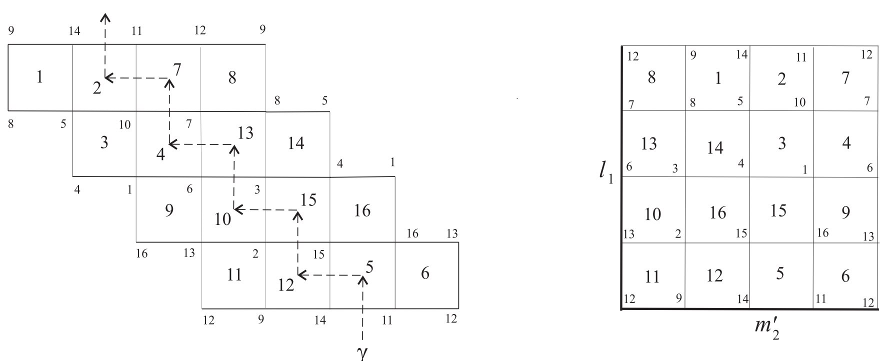


Fig. 11

We shall use the construction of the dual graph G below.

Let us consider the tiled torus $\mathcal{T}$ represented by a staircase pattern in Fig. 12(a). This remarkable example is due to Ng [12]. It is not difficult to verify that the tiled torus $\mathcal{T}$ passes the tests from Proposition 1.1, so it has an embedding. By Proposition 1.2, the embedding T of the tiled torus $\mathcal{T}$ in $\mathbb{R}^3$ is unique in a certain sense. Now we apply Algorithm 2.1 to $\mathcal{T}$.



a) b)

Fig. 12

First consider the slices T_{θ_i} of the embedded torus T at the angles $\theta_1 = \frac{1}{16}\pi, \theta_2 = \frac{3}{16}\pi, \ldots, \theta_8 = \frac{15}{16}\pi$ (see Fig. 13).

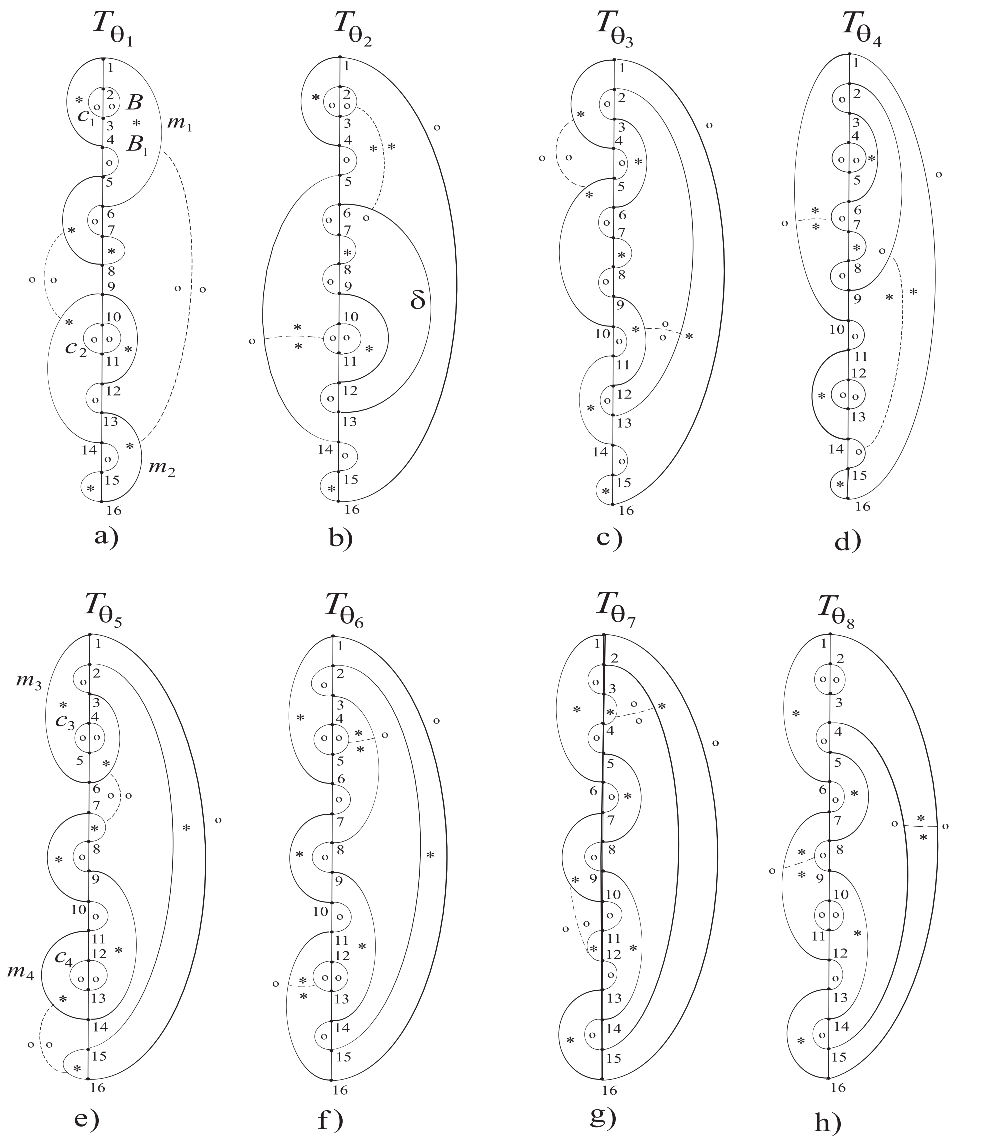


Fig. 13

We conclude that T bounds in $\mathbb{R}^3$ a solid torus $\mathbb{T}_1$. The corresponding slice meridians are the curves m_1, m_2 in Fig. 13(a) and the curves m_3, m_4 in Fig. 13(e). The combinatorial meridians m_1, m_2, m_3, m_4 on $\mathcal{T}$ can be coded by the cyclic sequences of vertices as follows: $m_1 = (1, 4, 5, 8, 7, 6, 1)$, $m_2 = (9, 14, 15, 16, 13, 12, 9)$, $m_3 = (1, 6, 3, 2, 15, 16, 1)$, $m_4 = (7, 10, 11, 14, 9, 8, 7)$. Note that no meridian $m_i, i = 1, 2, 3, 4$, bounds a slice meridional disc of $\mathbb{T}_1$, but each such meridian bounds in $\mathbb{T}_1$ a good meridional disc.

For example, let B_1 be the disc on the plane P_{θ_1} bounded by the curve m_1 of Fig. 13(a). Then there is a disc B inside B_1 which is bounded by a closed curve c_1, which is decomposed by vertices 2 and 3 into two b-arcs $b(2,3)$. Clearly B lies outside the solid torus $\mathbb{T}_1$. Moreover c_1 is obviously homotopically trivial on T. Similarly, the closed curves c_2, c_3, c_4 in Figs. 13(a) and 13(e) are also homotopically trivial on T.

The meridians m_1, m_2, m_3, m_4 can be replaced with the minimal meridians m_1', m_2', m_3', m_4', respectively, where $m_1' = (1,6,3,4,1)$, $m_2' = (9,14,11,12,9)$, $m_3' = (2,15,16,13,2)$, $m_4' = (7,8,5,10,7)$. The latter give a combinatorial decomposition of the tiled torus $\mathcal{T}$ (or the embedded torus T) into (embedded) building blocks. Next, the closed curve $\delta = (6,7,8,9,12,13)$ in Fig. 13(b) is non-trivial on T and bounds a disc D in the exterior of $\mathbb{T}_1$. It follows that T bounds in S^3 a solid torus $\mathbb{T}_2$ on the opposite side and D is a meridional disc of $\mathbb{T}_2$. Thus $\mathbb{T}_1$ is "unknotted" in $\mathbb{R}^3$.

Note also that the torus T, considered as the boundary of the solid torus $\mathbb{T}_2$, contains no perfect meridian in any slice T_θ. The minimal meridian l_1 which bounds a meridional disc in $\mathbb{T}_2$ has the following combinatorial presentation: $l_1 = (12,13,6,7,12)$. The longitude-meridional patterns for T corresponding to the meridian m_2' and the zig-zag longitude l, and the straight longitude l_1, are indicated in Figs. 12(a) and 12(b), respectively. Let γ be the oriented longitude on T indicated in Fig. 12(a). By direct computation, we have $w(\gamma) = \pm 3$.

Let us consider T as the boundary of the solid torus $\mathbb{T}_1$. The minimal meridians m_1', m_2', m_3', m_4' bounding the building blocks for T have a similar geometric configuration with respect to the parameter θ and the cyclic order of the vertices on the axis A. This can be shown by using the natural maps h defined above. Similarly, if T is considered as the boundary of $\mathbb{T}_2$, the minimal meridians l_1, l_2, l_3, l_4 on T are similar in a certain sense. The same is true for the tiled tori that have embeddings and generalized type $k \geq 2$.

As opposed to the embedded tori considered before, the tiled torus $\mathcal{T}$ in Fig. 12 has no slice perfect meridian. Finally, T is essential. This is shown in Fig. 13. Here we denote by $\circ$ the intersections of the component of L lying inside $\mathbb{T}_1$ with the fibers H_θ, while $*$ stands for the intersections of the remaining components (lying outside $\mathbb{T}_1$) with these fibers. It turns out that this result can be extended to the whole class of embedded tiled tori.

Let $\mathcal{T}$ be a closed tiled torus and let T be an embedding of the tiled surface in Euclidean 3-space $\mathbb{R}^3$, with respect to the z-axis A. This embedding is unique in a certain sense. Denote by $\mathcal{F}$ the natural singular foliation of T which yields in turn the given tiled surface $\mathcal{T}$. In the following, by a curve on T we shall mean any smooth or PL-curve on T. Let $\mathcal{U}$

be a finite collection consistings of simple closed curves C_i, $i = 1, \dots, t$, on T so that no C_i intersects the axis A. We shall say that $\mathcal{U}$ is *essential* on T (or $\mathcal{T}$) if each C_i is transverse to the leaves of the foliation $\mathcal{F}$ and for any b-arc $b(k, j)$ on T there is a curve C_n from $\mathcal{U}$ which intersects $b(k, j)$.

THEOREM 2.4. *Any embedded tiled torus T' is essential in the complement of some non-split link L represented as a closed braid with braid axis A.*

Proof. Let $\mathcal{T}'$ be a tiled torus that has an embedding, and let T' be an embedding in 3-space with respect to the axis A. Then T' is a Riemannian manifold with metric inherited from $\mathbb{R}^3$. We may assume that T' bounds a solid torus $\mathbb{T}$ in $\mathbb{R}^3$ and a positive normal to T' points inwards with respect to $\mathbb{T}$. Let H be the underlying graph of $\mathcal{T}'$ and let G be the dual of the graph H embedded in T'. Then G is a regular graph of valence 4 embedded in T'. The edges of G are transverse to the leaves of the natural foliation $\mathcal{F}$ on T' and can be oriented in a natural way (see above). Let G' denote the corresponding orgraph. The arcs of G' can be covered by a collection $\mathcal{B}'$ of oriented closed zig-zag paths l_i, $i = 1, \dots, 2q$, with the properties given in Remark 2.2. Note also that $\mathcal{B}'$ can be divided into two subcollections $\mathcal{B}'_1$ and $\mathcal{B}'_2$ of zig-zag lines, where $|\mathcal{B}'_1| = |\mathcal{B}'_2| = q$, which have the following properties.

(i) Any two paths from the same subcollection $\mathcal{B}'_i$, $i = 1, 2$, are parallel on $\mathcal{T}'$ (i.e. can be obtained from each other by an elementary diagonal shift on $\mathcal{T}'$).

(ii) Any two paths from distinct subcollections are antiparallel on $\mathcal{T}'$.

There is another collection $\mathcal{D}'$, dual to $\mathcal{B}'$ in some sense, of closed oriented paths in the embedded graph G which also cover the arcs of the orgraph G'. The collection $\mathcal{D}'$ consists of oriented zig-zag lines a_j, $j = 1, \dots, 2t$, on G and can be divided into two subcollections, $\mathcal{D}'_1$ and $\mathcal{D}'_2$, with the same properties as the ones of $\mathcal{B}'$. The intersection index $i(l_k, a_j) = w$ of the curves l_k and a_j on T', where $l_k \in \mathcal{B}'$, is non-zero and can change only in sign when passing to other representatives within the same class. All the curves from the collections $\mathcal{B}'$ and $\mathcal{D}'$ may be assumed to be smooth outside the singularities of $\mathcal{F}$ on T', which they pass through.

By definition, no curve l_i, $i = 1, \dots, 2q$ and a_j, $j = 1, \dots, 2t$, intersects the axis A. Moreover each oriented curve from $\mathcal{B}'$ and $\mathcal{D}'$ is transverse to the non-singular leaves of $\mathcal{F}$ and every edge is directed according to increasing θ. It follows that both $\mathcal{B}'$ and $\mathcal{D}'$ are essential on T'. Let K be the union of all curves from $\mathcal{B}'$. Let $d = \mathrm{dist}(K, A)$. We may push each curve $l_i, i = 1, \dots, 2q$, off singularities s in small δ-neighborhoods of s on T', where $\delta < d/4$, to obtain a new essential collection $\mathcal{B}$ of disjoint oriented closed curves β_i,

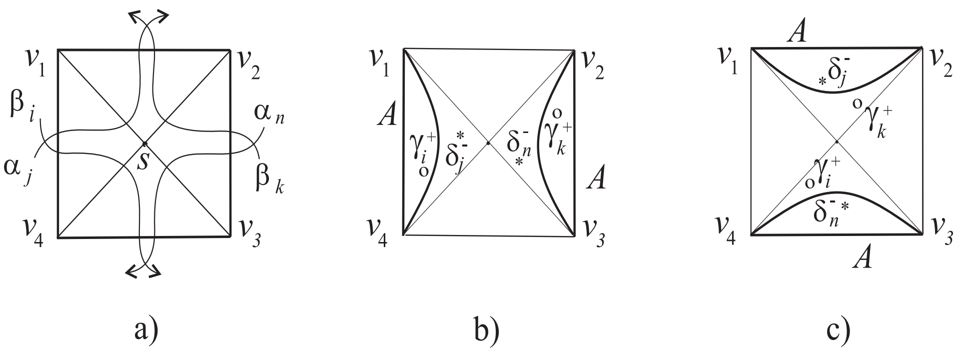


Fig. 14

$i = 1, \ldots, 2q$, on T' or $\mathcal{T}'$ (see Fig. 14(a)). Similarly, by pushing the curves a_j off singularities s on T', in small neighborhoods of these singularities, we obtain a new essential collection of disjoint curves $\mathcal{D} = \{\alpha_1, \ldots, \alpha_{2t}\}$ on T'. Moreover, we may assume that each oriented curve β_i, $i \leq 2q$, and α_j, $j \leq 2t$, is smooth and transverse to the fibers of $\mathcal{F}$ and is directed according to increasing θ. It follows that the curves β_i and α_j are all transverse to the half-planes H_θ, $\theta \in [0, 2\pi)$.

It may occur that α_j or β_i bounds a meridional disc D in a solid torus bounded by T' in S^3. In any case, one of the following two possibilities holds:

1) β_i does not bound a meridional disc in $\mathbb{T}$ and α_j determines a nontrivial homology class in $H_1(S^3 \setminus \mathbb{T}, \mathbb{Z})$;
2) α_j does not bound a meridional disc in $\mathbb{T}$ and β_i determines a nontrivial homology class in $H_1(S^3 \setminus \mathbb{T}, \mathbb{Z})$.

Suppose, for instance, that 1) holds. Consider a normal tubular ε-neighborhood $N(T')$ of T' in $\mathbb{R}^3$, where $\varepsilon > 0$ is chosen small enough. It can be considered as the image of a smooth embedding $g : T' \times (-\varepsilon, \varepsilon)$, so that $g(T' \times \{0\})$ is identified with T'. Moreover, we may assume that the manifold $N(T')^+ = g(T' \times \{+\varepsilon/2\})$ is the $\varepsilon/2$-push-off of the torus T' in the direction of the positive normal to T' and $N(T')^- = g(T' \times \{-\varepsilon/2\})$ is the $\varepsilon/2$-push-off of T' in the opposite direction. Put $B_1 = g(T' \times (0, \varepsilon))$ and $B_2 = g(T' \times (-\varepsilon, 0))$. By assumption, $N(T')^+ \subset B_1 \subset \mathbb{T}$ and $N(T')^- \subset B_2 \subset \mathbb{R}^3 \setminus \mathbb{T}$. Let β_i^+ (α_j^-, respectively) be the $\varepsilon/2$-push-off of β_i (α_j, respectively) in the direction of the positive (negative, respectively) normal to T', $i = 1, \ldots, 2q$, $j = 1, \ldots, 2t$, and let p_i: $\beta_i \to \beta_i^+$ and q_j: $\alpha_j \to \alpha_j^-$ be the corresponding homeomorphisms. Then $\beta_i^+ \subset N(T')^+$ and $\alpha_j^- \subset N(T')^-$ for all i and j. Define g_i: $\beta_i \to \mathbb{T}$ as follows. Let x be any point on the curve β_i and H_{θ_i} the half-plane which passes through x, and let u be the orthogonal projection of $p_i(x)$ on H_{θ_i}. We put $g_i(x) = u$ and $c_i = g_i(\beta_i)$, $i = 1, \ldots, 2q$. By construction, all maps g_i, $i = 1, \ldots, 2q$, are continuous. Each curve c_i has the natural orientation inherited from the one of β_i. In a similar way we define $f_j\colon \alpha_j \to \mathbb{R}^3 \setminus \mathbb{T}$ and oriented continuous curves d_j, $i = 1, \dots, 2t$. If $\varepsilon > 0$ is small enough, both families of curves $\{c_1, \dots, c_{2q}\}$ and $\{d_1, \dots, d_{2t}\}$ are disjoint. It follows that all the curves c_i and d_j are transverse to the half-planes H_θ, where $\theta \in [0, 2\pi)$, and are oriented in the direction of increasing θ.

Using the standard technique, we may choose ϵ small enough and isotope each curve c_i in its ϵ-neighborhood in B_1 so that the resulting oriented curves γ_i are smooth and preserve all the properties of c_i mentioned above.

In the same way, by isotoping the curves d_j, $j = 1, \dots, 2t$, we obtain a collection of oriented smooth closed curves δ_j, $j = 1, \dots, 2t$, all positioned in B_2. Let L be the $2(q+t)$-component link in $\mathbb{R}^3$ consisting of the curves γ_i, $i = 1, \dots, 2q$, and δ_j, $j = 1, \dots, 2t$. By our assumption and the definition of γ_i and δ_j, we have $\mathrm{lk}(\gamma_i, \delta_j) \neq 0$ for all i and j. The link L is then represented as a closed braid with braid axis A and is contained in the complement of T'. By the definition of γ_i and δ_j, the torus T' is essential in the complement of the closed braid L. In Figs. 14(b) and 14(c), we indicate by $\circ$ the intersections of the components γ_i of L with the non-singular fibers H_θ, while $*$ stands for intersections of the components δ_j with these fibers, when passing through a singular fiber H_{θ_0}. Moreover L is a non-split link. By our construction, the components of L inside the solid torus $\mathbb{T}$ intersect each meridional disc of $\mathbb{T}$ at least twice. If T' is knotted it follows that T' is incompressible. If T' bounds in S^3 a solid torus $\mathbb{T}_1$ on the opposite side, then each meridional disc of $\mathbb{T}_1$ is pierced at least twice by the components of L lying inside $\mathbb{T}_1$. In this case, T' is also incompressible in $S^3 \setminus L$. ■

The further development of the combinatorics of tiled surfaces of genus $g > 1$ and of the geometry of embedded tiled surfaces will be given in a forthcoming paper.

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