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On universality of countable and weak products of sigma hereditarily disconnected spaces

by

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Abstract. Suppose a metrizable separable space Y is sigma hereditarily disconnected, i.e., it is a countable union of hereditarily disconnected subspaces. We prove that the countable power X^ω of any subspace $X \subset Y$ is not universal for the class $\mathcal{A}_2$ of absolute $G_{\delta\sigma}$-sets; moreover, if Y is an absolute $F_{\sigma\delta}$-set, then X^ω contains no closed topological copy of the Nagata space $\mathcal{N} = W(I,\mathbb{P})$; if Y is an absolute G_δ-set, then X^ω contains no closed copy of the Smirnov space $\sigma = W(I,0)$.

On the other hand, the countable power X^ω of any absolute retract of the first Baire category contains a closed topological copy of each σ-compact space having a strongly countable-dimensional completion.

We also prove that for a Polish space X and a subspace $Y \subset X$ admitting an embedding into a σ-compact sigma hereditarily disconnected space Z the weak product $W(X,Y) = \{(x_i) \in X^\omega : \text{almost all } x_i \in Y\} \subset X^\omega$ is not universal for the class $\mathcal{M}_3$ of absolute $G_{\delta\sigma\delta}$-sets; moreover, if the space Z is compact then $W(X,Y)$ is not universal for the class $\mathcal{M}_2$ of absolute $F_{\sigma\delta}$-sets.

A topological space X is called $\mathcal{C}$-*universal*, where $\mathcal{C}$ is a class of spaces, if for every space $C \in \mathcal{C}$ there is a closed embedding $f : C \to X$. It is well known that the Hilbert cube $Q = [0,1]^\omega$ is $\mathcal{M}_0$-universal, whereas its pseudointerior $s = (0,1)^\omega$ is $\mathcal{M}_1$-universal, where $\mathcal{M}_0$ and $\mathcal{M}_1$ are the Borel classes of compact and Polish spaces, respectively (all spaces considered in this paper are metrizable and separable, all maps are continuous). Let us remark that both Q and s are countable products of finite-dimensional spaces. This raises the following question: can the countable power X^ω of a finite-dimensional space X be $\mathcal{C}$-universal for a higher Borel class $\mathcal{C}$? Taking into account results of [BR] and [Ca_1], it was conjectured in [Ba] that the

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countable power X^ω of any finite-dimensional (resp. strongly countable-dimensional) space X is not $\mathcal{A}_1$-universal (resp. $\mathcal{A}_2$-universal). Here $\mathcal{A}_1$ and $\mathcal{A}_2$ are the Borel classes of σ-compact and absolute $G_{\delta\sigma}$-spaces, respectively.

In this paper we confirm this conjecture. We define a space X to be *sigma hereditarily disconnected* provided X can be written as a countable union $X = \bigcup_{n=1}^{\infty} X_n$ of hereditarily disconnected spaces. Recall that a space X is *hereditarily disconnected* if it contains no connected subset containing more than one point (see [En, 1.4.2]).

For a class $\mathcal{C}$ of spaces we denote by $\mathcal{C}$(c.d.) and $\mathcal{C}$(s.c.d.) the subclasses of $\mathcal{C}$ consisting of countable-dimensional and strongly countable-dimensional spaces $C \in \mathcal{C}$, respectively. Let us remark that each strongly countable-dimensional space is countable-dimensional and each countable-dimensional space is sigma hereditarily disconnected.

Theorem 1. (1) *If a space X has a sigma hereditarily disconnected completion*, *then the countable power X^ω is not $\mathcal{A}_1$*(s.c.d.)*-universal.*

(2) *If a space X embeds into a sigma hereditarily disconnected absolute $F_{\sigma\delta}$-space*, *then X^ω is not $\mathcal{A}_2$*(c.d.)*-universal.*

(3) *If a space X is sigma hereditarily disconnected*, *then X^ω is not $\mathcal{A}_2$-universal.*

For a class $\mathcal{C}$ of spaces let $\mathcal{C}$(s.c.d.c.) denote the subclass of $\mathcal{C}$ consisting of spaces with a strongly countable-dimensional completion. The class $\mathcal{A}_1$(s.c.d.) from the first statement of Theorem 1 is the best possible in the following sense.

Theorem 2. *If X is an absolute retract of the first Baire category*, *then the countable power X^ω is $\mathcal{A}_1$*(s.c.d.c.)*-universal.*

Clearly, there exist finite-dimensional σ-compact absolute retracts of the first Baire category, for example the space $X = D \setminus E$, where D is a dendrite with a dense set E of end-points.

Countable powers are partial cases of *weak products*

$$W(X, A) = \{(x_i) \in X^\omega : x_i \in A \text{ for all but finitely many indices } i\},$$

where A is a subset of a space X.

The most known and important weak products are the Smirnov space $\sigma = W(I, \{0\})$ and the Nagata space $\mathcal{N} = W(I, \mathbb{P})$, where $I = [0, 1]$ and $\mathbb{P}$ is the set of irrational numbers in I. Note that both σ and $\mathcal{N}$ are subsets of the Hilbert cube $Q = I^\omega$. It is well known that the Smirnov space σ is $\mathcal{A}_1$(s.c.d.)-universal [Mo_1] and the Nagata space $\mathcal{N}$ is $\mathcal{A}_2$(c.d.)-universal [Mo_2]. Let us remark that according to Theorem 1 the Smirnov space σ admits no sigma hereditarily disconnected completion, while the Nagata space $\mathcal{N}$ admits no embedding into a sigma hereditarily disconnected absolute $F_{\sigma\delta}$-space. This answers Question 1.3 of [Mo_2]. Recently T. Radul

[Ra] (see also [BRZ, §4.1, Ex. 3]) has shown that the weak product $W(Q,\sigma)$ is universal for the additive Borel class $\mathcal{A}_3$ of absolute $F_{\sigma\delta\sigma}$-spaces. Can the weak product $W(X,Y)$ be $\mathcal{C}$-universal for a higher Borel class, if Y is finite-dimensional or strongly countable-dimensional? In particular, can $W(X,Y)$ be universal for the multiplicative Borel classes $\mathcal{M}_2$ and $\mathcal{M}_3$ of absolute $F_{\sigma\delta}$- and $G_{\delta\sigma\delta}$-spaces, respectively?

We recall that a space X is defined to be *σ-complete* if X can be written as a countable union $X=\bigcup_{i=1}^\infty X_i$, where each X_i is complete-metrizable and closed in X.

THEOREM 3. *Let Y be a subspace of a Polish space X.*

(1) *If Y has a sigma hereditarily disconnected completion, then the weak product $W(X,Y)$ is not $\mathcal{M}_2$-universal*;

(2) *If Y embeds into a σ-complete sigma hereditarily disconnected space, then $W(X,Y)$ is not $\mathcal{M}_3$-universal.*

The proofs of our theorems rely on simple homological arguments, so we need to recall some standard notations from homology theory. For every integer $q\ge 0$ let $H_q(X)$ denote the qth singular homology group of a space X (reduced in dimension zero so that $H_0(X)=0$ if and only if X is path-connected) and let $H_*(X)=\bigoplus_{q=0}^\infty H_q(X)$. For closed subsets $B\subset A$ of the Hilbert cube Q we denote by j_B^A the homomorphism of $H_*(Q\setminus A)$ into $H_*(Q\setminus B)$ induced by inclusion. A closed subset A of Q is defined to be an *irreducible barrier for an element* $\alpha\in H_q(Q\setminus A)$ if $\alpha\ne 0$ but $j_B^A(\alpha)=0$ for any closed proper subset $B\subset A$; and A is an *irreducible barrier in Q* if either $A=Q$ or A is a closed irreducible barrier for some (non-trivial) element $\alpha\in H_q(X\setminus A)$, $q\ge 0$.

The following lemma plays a crucial role in the proof of Theorems 1, 3 and seems to have an independent value.

MAIN LEMMA. *For every countable cover $\{X_n\}_{n\in\mathbb{N}}$ of an irreducible barrier A in the Hilbert cube Q, one of the sets X_n contains a connected subset $C\subset X_n$ whose closure $\overline{C}$ is an irreducible barrier in Q.*

Proof of Main Lemma. We need the following two homological lemmas proven in [Ca_2].

LEMMA 1. *Suppose A is a closed subset of the Hilbert cube Q such that $H_q(Q\setminus A)\ne 0$ for some $q\ge 0$. Then A contains an irreducible barrier B for some $\alpha\in H_q(Q\setminus B)$.*

LEMMA 2. *If A is an irreducible barrier in Q then for every closed subset $B\subset A$ separating A we have $H_*(Q\setminus B)\ne 0$.*

To prove the Main Lemma assume on the contrary that $\{X_n\}_{n=1}^\infty$ is a countable cover of an irreducible barrier $A \subset Q$ such that no X_n contains a connected subset C whose closure is an irreducible barrier in Q. To get a contradiction we will construct a decreasing sequence $A = A_0 \supset A_1 \supset \ldots$ of irreducible barriers in Q such that $A_n \cap X_n = \emptyset$ for every $n \geq 1$. Then by compactness of A we will find a point $a \in \bigcap_{n=1}^\infty A_n \subset A$ that does not belong to $\bigcup_{n=1}^\infty X_n \supset A$, a contradiction.

The construction of $\{A_n\}$ is inductive. Set $A_0 = A$ and suppose that for an $n \geq 0$ irreducible barriers $A_0 \supset \ldots \supset A_n$ satisfying $A_k \cap X_k = \emptyset$ for $1 \leq k \leq n$ have been constructed. By our hypothesis, $A_n \cap X_{n+1}$ is either disconnected or not dense in A_n. In both cases, one may easily construct a closed subset B separating A_n and missing X_{n+1}. By Lemma 2, we have $H_*(Q \setminus B) \neq 0$, and by Lemma 1, B contains an irreducible barrier A_{n+1} in Q. Evidently, A_{n+1} is as required because $A_{n+1} \cap X_{n+1} = \emptyset$. ■

Some auxiliary results. By a *subcube* of the Hilbert cube $Q = I^\omega$ we understand a subset of the form $\prod_{n\in\omega} I_n$, where each I_n is a closed non-degenerate interval in I and $I_n = I$ for all but finitely many indices n.

We define a subset X of Q to be *q-dense* in Q, for a non-negative integer q, if every map $f : K \to Q$ of an at most q-dimensional compactum K can be uniformly approximated by maps into X; and X is *∞-dense* if it is q-dense in Q for every $q \in \mathbb{N}$.

We will need another two homological lemmas proven in [Ca$_2$] (Lemmas 3 and 4).

Lemma 3. *If $A \subset Q$ is an irreducible barrier for some $\alpha \in H_q(Q \setminus A)$, then for any subcube P of Q whose interior meets A we have $H_q(P \setminus A) \neq 0$.*

Lemma 4. *If A is an irreducible barrier in $Q \times Q$ and Y is an ∞-dense subset in Q, then there is a point $y \in Y$ such that $A \cap (\{y\} \times Q)$ contains an irreducible barrier B in $\{y\} \times Q$.*

For any $q \geq 0$ let $\mathcal{N}_q = \{(t_i)_{i\in\omega} \in Q :$ at most q coordinates t_i are rational$\}$ denote the analog of the Nöbeling space in the Hilbert cube. It is easily seen that $\mathcal{N}_q$ is a G_δ-set in Q and $\mathcal{N} = \bigcup_{q=0}^\infty \mathcal{N}_q$.

Lemma 5. *For every $q \geq 0$ the sets σ, s, $Q \setminus s$, $\mathcal{N}$, and $\mathcal{N}_q$ are q-dense in Q.*

Proof. The q-density of σ, s, $Q \setminus s$ in Q is easily seen and well known. The q-density of $\mathcal{N}_q$ in Q can be proven by analogy with the proof of the universality of the Nöbeling space (see [En, 1.11.5]). Finally, the q-density of $\mathcal{N}$ in Q follows from the q-density of $\mathcal{N}_q$ in Q and the inclusion $\mathcal{N}_q \subset \mathcal{N}$. ■

Lemma 6. *If $A \subset Q$ is an irreducible barrier for some $\alpha \in H_q(Q \setminus A)$ then $A \cap X$ is dense in A for every $(q+1)$-dense subset $X \subset Q$.*

Proof. Assume on the contrary that for some $(q+1)$-dense set $X \subset Q$ the intersection $A \cap X$ is not dense in A. Then there is an open set $U \subset Q$ such that $U \cap A \neq \emptyset$ and $\overline{U} \cap A \cap X = \emptyset$. Let $B = A \setminus U$. Then $B \neq A$ and thus $j^A_B(\alpha) = 0$. Fix a q-dimensional polyhedron K, a function $f : K \to Q \setminus A$, and an element $\beta \in H_q(K)$ with $f_*(\beta) = \alpha$. Since $j^A_B(\alpha) = 0$, there exists a $(q+1)$-dimensional polyhedron L containing K and a function $g : L \to Q \setminus B$ such that $g|K = f$ and $i_*(\beta) = 0$, where i is the embedding of K into L (see [Ma, p. 293]). If $h : L \to X$ is sufficiently near to g, then $h(L) \subset Q \setminus B$ and $h|K$ is homotopic to f in $Q \setminus A$. This yields $f_*(\beta) = (h|K)_*(\beta)$ and from $h(L) \subset Q \setminus B$, we get $h(L) \cap A \subset (A \setminus B) \cap X = U \cap A \cap X = \emptyset$. Then in $H_q(Q \setminus A)$ we have $\alpha = f_*(\beta) = (h|K)_*(\beta) = h_* \circ i_*(\beta) = 0$, a contradiction. ∎

In what follows we will need the following modification of the Main Lemma.

LEMMA 7. *Suppose X is an ∞-dense G_δ-set in Q and A is an irreducible barrier in Q. If $\{X_n\}_{n\in\mathbb{N}}$ is a countable cover of the set $A \cap X$, then one of the sets X_n contains a connected subset $C \subset X_n$ whose closure $\overline{C}$ is an irreducible barrier in Q.*

Proof. Since X is a G_δ-set in Q, we may write $A \setminus X = \bigcup_{n\in\mathbb{N}} A_n$, where each A_n is compact. Then we have a countable cover $\{A_n, X_n\}_{n\in\mathbb{N}}$ of the irreducible barrier A. By the Main Lemma, there is a connected set $C \subset Q$ such that $\overline{C}$ is an irreducible barrier in Q and either $C \subset A_n$ or $C \subset X_n$. The case $C \subset A_n$ is impossible. Indeed, by the compactness of A_n, $\overline{C} \subset A_n$. Thus $C \cap X = \emptyset$, a contradiction with Lemma 6. ∎

Finally, we need the following particular case of [BRZ, 3.1.1]:

LEMMA 8. *Let X be a Polish space and $Y \subset X$.*

(1) *If Y is $\mathcal{A}_2$-universal, then there is an embedding $\varphi : Q^\omega \to X$ such that $\varphi^{-1}(Y) = W(Q, s)$.*

(2) *If Y is $\mathcal{M}_2$-universal, then there is an embedding $\varphi : Q^\omega \to X$ such that $\varphi^{-1}(Y) = Q^\omega \setminus W(Q, s)$.*

(3) *If Y is $\mathcal{M}_3$-universal, then there is an embedding $\varphi : Q^\omega \to X$ such that $\varphi^{-1}(Y) = Q^\omega \setminus W(Q, \sigma)$.*

Proof of Theorem 1. (1) Suppose X^ω is $\mathcal{A}_1$(s.c.d.)-universal and X has a sigma hereditarily disconnected completion Y. Since $\sigma \in \mathcal{A}_1$(s.c.d.), we may fix a closed embedding $\varphi : \sigma \to X^\omega$. By Lavrent'ev's Theorem, this embedding extends to an embedding $\overline{\varphi} : G \to Y^\omega$ of some G_δ-set $G \subset Q$ containing σ. Since $\varphi(\sigma)$ is closed in X^ω and dense in $\overline{\varphi}(G)$, we have $\overline{\varphi}^{-1}(X^\omega) = \sigma$. For $m \geq 0$ denote by $\varphi_m : G \to Y$ the composition of φ with the coordinate projection $\mathrm{pr}_m : Y^\omega \to Y$.

Using the fact that $Y \setminus X$ and σ are sigma hereditarily disconnected, write $Y\setminus X = \bigcup_{n=1}^{\infty} Y_n$ and $\sigma = \bigcup_{n=1}^{\infty} Z_n$, where Y_n and Z_n are hereditarily disconnected. Write also $Q\setminus G = \bigcup_{n=1}^{\infty} G_n$, where each G_n is compact. Since $\sigma \subset G$, we have $\sigma \cap G_n = \emptyset$ for $n \geq 1$.

Thus, the Hilbert cube $Q = \sigma \cup (Q \setminus G) \cup (G \setminus \sigma)$ has the countable cover $\{Z_n, G_n, \varphi_m^{-1}(Y_n)\}_{n,m\in\mathbb{N}}$. By the Main Lemma, there is a connected set $C \subset Q$ such that $\overline{C}$ is an irreducible barrier in Q and either $C \subset Z_n$, $C \subset G_n$, or $C \subset \varphi_m^{-1}(Y_n)$ for some $n, m \in \mathbb{N}$.

Since all Z_n's are hereditarily disconnected, no Z_n can contain the (connected) set C. Next, assuming that $C \subset G_n$ for some n, we derive from the compactness of G_n that $\overline{C} \subset G_n$ and thus $\overline{C} \cap \sigma = \emptyset$, a contradiction with Lemmas 6 and 5.

Thus $C \subset \varphi_m^{-1}(Y_n)$ for some $n, m \in \mathbb{N}$. Then $\varphi_m(C)$, being a connected subset of a hereditarily disconnected space, is a single point $y \in Y_n \subset Y\setminus X$. Since $\varphi_m^{-1}(y)$ is a closed subset in G missing σ, it follows that $\overline{C}$ is an irreducible barrier in Q missing σ, contrary to Lemmas 6 and 5 again.

(2) Suppose X^ω is $\mathcal{A}_2$(c.d.)-universal and X embeds into a sigma hereditarily disconnected $F_{\sigma\delta}$-space Y. Since $\mathcal{N} \in \mathcal{A}_2$(c.d.), we may fix a closed embedding $\varphi : \mathcal{N} \to X^\omega$. It follows easily from the Lavrent'ev Theorem that this embedding extends to an embedding $\overline{\varphi} : G \to Y^\omega$ of some $F_{\sigma\delta}$-set $G \subset Q$ containing the Nagata space $\mathcal{N}$. As in the preceding case, observe that $\overline{\varphi}^{-1}(X^\omega) = \mathcal{N}$. For $m \geq 0$ let $\varphi_m = \mathrm{pr}_m \circ\varphi : G \to Y$.

Using the fact that $Y \setminus X$ and $\mathcal{N}$ are sigma hereditarily disconnected, write $Y\setminus X = \bigcup_{n=1}^{\infty} Y_n$ and $\mathcal{N} = \bigcup_{n=1}^{\infty} Z_n$, where Y_n and Z_n are hereditarily disconnected. The complement $Q \setminus G$, being a $G_{\delta\sigma}$-subset of Q^ω, can be written as $Q\setminus G = \bigcup_{n=1}^{\infty} G_n$, where each G_n is a G_δ-set in Q^ω. Observe that $\mathcal{N} \cap G_n = \emptyset$ for $n \geq 1$.

Thus, Q has the countable cover $\{Z_n, G_n, \varphi_m^{-1}(Y_n)\}_{n,m\in\mathbb{N}}$. By the Main Lemma, there is a connected set $C \subset Q$ such that $\overline{C}$ is an irreducible barrier for some non-trivial $\alpha \in H_q(Q \setminus \overline{C})$ and either $C \subset G_n$, $C \subset Z_n$, or $C \subset \varphi_m^{-1}(Y_n)$ for some $n, m \in \mathbb{N}$.

As in the preceding case we can show that the last two inclusions are impossible. Thus, $C \subset G_n$ for some $n \geq 1$. Since C is dense in $\overline{C}$, we find that $\overline{C}\cap G_n$ is a dense G_δ-set in $\overline{C}$. By Lemma 6, $\overline{C}\cap\mathcal{N}_{q+1}$ is a dense G_δ-set in $\overline{C}$ as well. Then by the Baire Theorem, $\overline{C}\cap\mathcal{N}_{q+1}\cap G_n$ is dense in $\overline{C}$. But $G_n \cap \mathcal{N}_{q+1} = \emptyset$ by construction, a contradiction.

(3) Suppose X is sigma hereditarily disconnected and X^ω is $\mathcal{A}_2$-universal. Let Y be any completion of X. By Lemma 8, there is a map $\varphi : Q^\omega \to Y^\omega$ such that $\varphi^{-1}(X^\omega) = W(Q, s)$.

For $q_0, \ldots, q_n \in Q$ let $Q(q_0, \ldots, q_n) = \{(q_0, \ldots, q_n)\} \times \prod_{i>n} Q \subset Q^\omega$ and $s(q_0, \ldots, q_n) = \{(q_0, \ldots, q_n)\} \times \prod_{i>n} s \subset Q^\omega$. For $n \geq 0$ let $\varphi_n = \mathrm{pr}_n \circ\varphi : Q^\omega \to Y$.

By induction, for every $n \geq 0$ we will construct points $x_n \in X$, $q_n \in Q$ and a closed subset $A_n \subset Q(q_0, \ldots, q_n)$ such that

(1) $q_n \notin s$;
(2) $A_n \supset A_{n+1}$;
(3) A_n is an irreducible barrier in $Q(q_0, \ldots, q_n)$;
(4) $\varphi_n(A_n) = \{x_n\} \subset X$.

To get a contradiction, observe that the point $q = (q_n)_{n\geq 0} \in Q^\omega$, being the intersection of A_n's, belongs to $\varphi^{-1}(X^\omega) = W(Q, s)$ by (4). On the other hand, (1) implies $q \notin W(Q, s)$.

Inductive step. Let $A_{-1} = Q^\omega$. Suppose that for some $n \geq -1$ the points $q_0, \ldots, q_n \in Q$ and the irreducible barrier $A_n \subset Q(q_0, \ldots, q_n)$ have been constructed. Write $X = \bigcup_{i=1}^\infty X_i$, where X_i are hereditarily disconnected. Observe that $s(q_0, \ldots, q_n) \subset W(Q, s)$ is a ∞-dense G_δ-set in $Q(q_0, \ldots, q_n)$. Since the collection $\{A_n \cap \varphi_{n+1}^{-1}(X_i)\}_{i\in\mathbb{N}}$ covers $A_n \cap s(q_0, \ldots, q_n)$, we may apply Lemma 7 to find an $i \in \mathbb{N}$ and a connected set $C \subset A_n \cap \varphi_{n+1}^{-1}(X_i)$ such that $\overline{C}$ is an irreducible barrier in $Q(q_0, \ldots, q_n)$. Since $\varphi_{n+1}(C)$ is a connected subset of the hereditarily disconnected space X_i, we have $\varphi_{n+1}(C) = \{x_{n+1}\}$ for some $x_{n+1} \in X_i \subset X$. Then $\varphi_{n+1}(\overline{C}) = \{x_{n+1}\}$ as well. By Lemma 4, there is a $q_{n+1} \in Q \setminus s$ such that $\overline{C} \cap Q(q_0, \ldots, q_{n+1})$ contains an irreducible barrier A_{n+1} in $Q(q_0, \ldots, q_{n+1})$. Evidently, the points x_{n+1}, q_{n+1}, and the set A_{n+1} satisfy the conditions (1)–(4). ■

Proof of Theorem 3. Let Y be a subspace of a Polish space X.

(1) Suppose Y has a sigma hereditarily disconnected completion $\widehat{Y}$ and the weak product $W(X, Y)$ is $\mathcal{M}_2$-universal. By Lemma 8, there is an embedding $\varphi : Q^\omega \to X^\omega$ such that $\varphi^{-1}(W(X, Y)) = Q^\omega \setminus W(Q, s)$. For $n \geq 0$ let $\varphi_n : Q^\omega \to X$ be the composition of φ and the coordinate projection $\mathrm{pr}_n : X^\omega \to X$.

By induction, for every $n \geq 0$ we will construct a point $q_n \in Q$ and a closed subset $A_n \subset Q(q_0, \ldots, q_n)$ such that

(1) $A_n \supset A_{n+1}$;
(2) A_n is an irreducible barrier in $Q(q_0, \ldots, q_n)$;
(3) either $\varphi_n(A_n) \subset Y$ or $\varphi_n(A_n) \subset X \setminus Y$;
(4) $q_n \in s$ if and only if $\varphi_n(A_n) \subset Y$.

To get a contradiction, observe that the point $q = (q_n)_{n\geq 0} \in Q^\omega$ is the intersection of the sets A_n. Let $x = (x_n)_{n\geq 0} = \varphi(q) \in X^\omega$. By (3) and (4), $x_n \in Y$ if and only if $q_n \in s$. This yields $\varphi(q) = (x_n) \in W(X, Y)$ if and only if $q = (q_n) \in W(Q, s)$, contrary to $\varphi^{-1}(W(X, Y)) = Q^\omega \setminus W(Q, s)$.

Inductive step. Let $A_{-1} = Q^\omega$. Suppose that for some $n \geq -1$ the points $q_0, \ldots, q_n \in Q$ and the irreducible barrier $A_n \subset Q(q_0, \ldots, q_n)$ have been constructed. According to the Lavrent'ev Theorem, we may assume $\widehat{Y}$ to be a subspace of X. Write $\widehat{Y} = \bigcup_{i=1}^{\infty} Y_i$ and $X \setminus \widehat{Y} = \bigcup_{i=1}^{\infty} F_i$, where for every $i \geq 1$, Y_i is a hereditarily disconnected set and F_i is closed in X. Because the countable collection $\{\varphi_{n+1}^{-1}(Y_i), \varphi_{n+1}^{-1}(F_i) : i \in \mathbb{N}\}$ covers the irreducible barrier A_n, we may apply the Main Lemma to find a connected set $C \subset A_n$ such that $\overline{C}$ is an irreducible barrier in $Q(q_0, \ldots, q_n)$ and either $C \subset \varphi_{n+1}^{-1}(F_i)$ or $C \subset \varphi_{n+1}^{-1}(Y_i)$ for some i.

We claim that either $\varphi_{n+1}(\overline{C}) \subset X \setminus Y$ or $\varphi_{n+1}(\overline{C}) \subset Y$. Indeed, if $C \subset \varphi_{n+1}^{-1}(F_i)$, then $\varphi_{n+1}(\overline{C}) \subset F_i \subset X \setminus Y$ (because F_i is closed in X). If $C \subset \varphi_{n+1}^{-1}(Y_i)$, then because C is connected and Y_i is hereditarily disconnected, we deduce that $\varphi_{n+1}(C)$ consists of a unique point $y \in Y_i$. Then $\varphi_{n+1}(\overline{C}) = \{y\}$ and hence $\varphi_{n+1}(\overline{C}) \subset Y$ if $y \in Y$ and $\varphi_{n+1}(\overline{C}) \subset X \setminus Y$ otherwise.

By Lemma 4, there is a point $q_{n+1} \in Q$ such that $\overline{C} \cap Q(q_0, \ldots, q_{n+1})$ contains an irreducible barrier A_{n+1} in $Q(q_0, \ldots, q_{n+1})$. Moreover, since s and $Q \setminus s$ are ∞-dense in Q the point q_{n+1} can be chosen so that $q_{n+1} \in s$ if and only if $\varphi_{n+1}(\overline{C}) \subset Y$. Evidently, the point q_{n+1} and the set A_{n+1} satisfy the conditions (1)–(4).

(2) Suppose Y embeds into a σ-complete sigma hereditarily disconnected space $\widehat{Y}$ and the weak product $W(X,Y)$ is $\mathcal{M}_3$-universal. By Lemma 8, there is an embedding $\varphi : Q^\omega \to X^\omega$ such that $\varphi^{-1}(W(X,Y)) = Q^\omega \setminus W(Q,\sigma)$. For $n \geq 0$ let $\varphi_n = \mathrm{pr}_n \circ \varphi : Q^\omega \to X$. Let also $\pi_n : Q^\omega \to Q$ be the projection onto the nth coordinate.

According to the Lavrent'ev Theorem, we may assume $\widehat{Y}$ to be a subspace of X. Write $\widehat{Y} = \bigcup_{k=1}^{\infty} Y_k$, where each Y_k is an absolute G_δ-set closed in $\widehat{Y}$. Denote by $\overline{Y_k}$ the closure of Y_k in X. Write also $\sigma = \bigcup_{k=1}^{\infty} I_k$, where I_k are compact subsets of Q. By induction for every $k \geq 0$ we will construct a partition of $\{0, \ldots, k\}$ into three subsets $H_i(k)$, $i = 1, 2, 3$, so that

(1) for $i = 1, 2$, $H_i(k) \subset H_i(k')$ if $k \leq k'$.

For every $r \in \bigcup_{k \geq 0} H_1(k) \cup H_2(k)$ we will construct a point $q_r \in Q$ and for every $k \geq 0$ we let

$$P_k = \bigcap_{r \in H_1(k) \cup H_2(k)} \pi_r^{-1}(q_r) \subset Q^\omega$$

and $P_{-1} = Q^\omega$. By (1) we have $P_k \supset P_{k+1}$ for every k. We shall also construct a subcube R_k of P_k and an irreducible barrier A_k in R_k such that the following conditions are satisfied for every k:

(2) $A_k \supset A_{k+1}$;
(3) if $r \in H_1(k)$, then $q_r \in \sigma$ and $\varphi_r(A_k) \subset Y$;
(4) if $r \in H_2(k)$, then $q_r \in Q \setminus \sigma$ and $\varphi_r(A_k) \subset X \setminus Y$;
(5) if $r \in H_3(k)$, then $R_k \cap \pi_r^{-1}(I_k) = \emptyset$ and $\varphi_r(A_k) \cap \overline{Y_k} = \emptyset$.

To get a contradiction, observe that by (2) there exists a point $z = (z_r) \in \bigcap_{k\geq 0} A_k$. By (3)–(5), $z_r \in \sigma$ if and only if $\varphi_r(z) \in Y$. Thus $z \in W(Q,\sigma)$ if and only if $\varphi(z) \in W(X,Y)$, contrary to $\varphi^{-1}(W(X,Y)) = Q^\omega \setminus W(Q,\sigma)$.

Inductive construction. Let $R_{-1} = A_{-1} = Q^\omega$. Suppose $k = 0$ or $k \geq 1$ and our objects are constructed up to order $k-1$. Let $k = r_0, r_1, \ldots, r_l$ be the elements of the set $\{k\} \cup H_3(k-1)$. We shall construct two finite decreasing sequences

$$R_{k-1} = U_{-1} \supset U_0 \supset \ldots \supset U_l, \quad A_{k-1} = B_{-1} \supset B_0 \supset \ldots \supset B_l,$$

where U_j is a subcube in R_{k-1} and B_j is an irreducible barrier in U_j for $j \leq l$. From the construction of these sets we will see to which of the sets $H_i(k)$ an element r_j should be assigned (elements of $\{0,\ldots,k\} \setminus \{r_0,\ldots,r_l\}$ belong to $H_1(k)$ or $H_2(k)$ according to (1)).

Suppose for $j \geq 0$ the sets U_{j-1} and B_{j-1} are constructed. We distinguish two cases:

(a) $\varphi_{r_j}^{-1}(X \setminus \overline{Y}_k) \cap B_{j-1} \neq \emptyset$. Then we can find a subcube U_j in U_{j-1} whose interior in U_{j-1} meets the barrier B_{j-1} and $U_j \subset \varphi_{r_j}^{-1}(X \setminus \overline{Y}_k)$. By Lemmas 1 and 3, $B_{j-1} \cap U_j$ contains an irreducible barrier B_j in U_j. We assign r_j to $H_3(k)$.

(b) $\varphi_{r_j}(B_{j-1}) \subset \overline{Y}_k$. Since Y_k is closed in $\widehat{Y}$ we get $\overline{Y}_k \cap \widehat{Y} = Y_k$. Recalling that Y_k is a sigma hereditarily disconnected absolute G_δ-set, write $Y_k = \bigcup_{i=1}^\infty D_i$ and $\overline{Y}_k \setminus \widehat{Y} = \overline{Y}_k \setminus Y_k = \bigcup_{i=1}^\infty F_i$, where the sets D_i are hereditarily disconnected and F_i are closed in X. Then the countable collection $\{\varphi_{r_j}^{-1}(D_i), \varphi_{r_j}^{-1}(F_i) : i \in \mathbb{N}\}$ covers the irreducible barrier B_{j-1}. By the Main Lemma, there is a connected subset $C \subset B_{j-1}$ such that $\overline{C}$ is an irreducible barrier in B_{j-1} and either $C \subset \varphi_{r_j}^{-1}(F_i)$ or $C \subset \varphi_{r_j}^{-1}(D_i)$ for some i. As in the preceding proof, we have either $\varphi_{r_j}(\overline{C}) \subset Y$ or $\varphi_{r_j}(\overline{C}) \subset X \setminus Y$. Let $U_j = U_{j-1}$, $B_j = \overline{C}$, and assign z_j to $H_1(k)$ if $\varphi_{r_j}(B_j) \subset Y$ and to $H_2(k)$ if $\varphi_{r_j}(B_j) \subset X \setminus Y$.

Thus we constructed the sets $H_i(k)$, $i = 1,2,3$. Since the complement of the closed set $\bigcup_{r\in H_3(k)} \pi_r^{-1}(I_k)$ is ∞-dense in U_l, we may find a subcube $K \subset U_l$ whose interior relative to U_l meets the barrier B_l and such that $K \cap \bigcup_{r\in H_3(k)} \pi_r^{-1}(I_k) = \emptyset$ (see Lemma 6). By Lemma 1, the set $B_l \cap K$ contains an irreducible barrier B in K.

Applying Lemma 4 find for every $r \in H_1(k) \setminus H_1(k-1)$ a point $q_r \in \sigma$ and for every $r \in H_2(k) \setminus H_2(k-1)$ a point $q_r \in Q \setminus s$ such that $B \cap P_k$ contains an irreducible barrier A_k in the subcube $R_k = P_k \cap K$ of P_k. Clearly, the constructed objects satisfy the conditions (1)–(5). ∎

Proof of Theorem 2. First we recall some definitions. Let $0 \leq n \leq \infty$. A subset A of a space X is called a Z_n-*set* in X if A is closed in X and every map $f: I^n \to X$ of the n-dimensional cube can be uniformly approximated by maps into $X \setminus A$. A space X is called a *σZ_n-space* if X can be written as a countable union $X = \bigcup_{i=1}^{\infty} X_i$ of Z_n-sets X_i in X. Note that each σZ_n-space is a σZ_m-space for every $m \le n$. Observe also that a space X is of the first Baire category if and only if X is a σZ_0-space.

The following fact is proven in [BT].

LEMMA 9. *If an absolute retract X is a σZ_0-space, then for every $n \in \mathbb{N}$ its nth power X^n is a σZ_{n-1}-space.*

In Lemma 5.4 of [DMM] T. Dobrowolski, W. Marciszewski, and J. Mogilski have proven that if an absolute retract X is a σZ_∞-space, then for every σ-compact space A there is a proper map $f: A \to X$. Modifying their arguments and using results of [To] one may prove

LEMMA 10. *If for some $n \ge 0$ an absolute retract X is a σZ_n-space, then for every n-dimensional σ-compact space A there exists a proper map $f: A \to X$.*

For a class $\mathcal{C}$ of spaces and $n \ge 0$ let $\mathcal{C}[n] = \{C \in \mathcal{C} : \dim(C) \le n\}$. Let us recall that a map $f: A \to X$ is *proper* provided the preimage $f^{-1}(K)$ of any compact subset $K \subset X$ is compact.

LEMMA 11. *If X is an absolute retract of the first Baire category, then for every $n \in \mathbb{N}$ its power X^{3n+2} is $\mathcal{A}_1[n]$-universal.*

Proof. Fix $n \in \mathbb{N}$ and a σ-compact space A with $\dim(A) \le n$. By Lemmas 9 and 10 there exists a proper map $f: A \to X^{n+1}$. Since X, being an absolute retract, contains a topological copy of the interval I, we can apply the classical Menger–Nöbeling–Lefschetz Theorem [En, 1.11.4] to find an embedding $g: A \to X^{2n+1}$. Then $e = (f, g): A \to X^{n+1} \times X^{2n+1} = X^{3n+2}$ is a closed embedding. ∎

Proof of Theorem 2. By [To, 4.1, 2.4] the space X embeds into a complete-metrizable absolute retract $\widetilde{X}$ so that X is homotopy dense in $\widetilde{X}$. The latter means that there is a homotopy $h: \widetilde{X} \times [0,1] \to \widetilde{X}$ such that $h(\widetilde{X} \times (0,1]) \subset X$ and $h(x, 0) = x$ for every $x \in \widetilde{X}$.

Let $A \in \mathcal{A}_1(\text{s.c.d.c.})$, i.e., A is a σ-compact space having a strongly countable-dimensional completion C. By the Compactification Theorem [En, 5.3.5] the space C has a strongly countable-dimensional metrizable compactification K. Write $K = \bigcup_{i=0}^{\infty} K_i$, where each $K_i \subset K_{i+1}$ is a compact finite-dimensional subspace of K.

By Lemma 11, the countable power X^ω is $\mathcal{A}_1[n]$-universal for all $n \in \mathbb{N}$. Then Theorem 3.1.1 of [BRZ] implies that for every i there exists an embedding $f_i: K_n \to \widetilde{X}^\omega$ with $f_i^{-1}(X^\omega) = K_n \cap A$. Since X^ω is homotopy dense in the absolute retract $\widetilde{X}^\omega$, the map f_i can be extended to a map $\overline{f}_i: K \to \widetilde{X}^\omega$

such that $\overline{f}_i(K \setminus K_i) \subset X^\omega$. Consider the map $f = (\overline{f}_i)_{i=0}^\infty : K \to (\widetilde{X}^\omega)^\omega$ and notice that it is an embedding with $f^{-1}((X^\omega)^\omega) = A$. Thus the restriction $f|A : A \to (X^\omega)^\omega$ is a closed embedding, i.e., the space X^ω, being homeomorphic to $(X^\omega)^\omega$, is $\mathcal{A}_1$(s.c.d.c.)-universal. ∎

Some questions and comments. The exponent $3n+2$ in Lemma 11 is not optimal. In fact, for every locally path-connected space X of the first Baire category the power X^{2n+1} is $\mathcal{A}_1[n]$-universal for every $n \geq 0$. The proof of this statement requires more involved arguments and will be given in another paper.

QUESTION 1. *For which Borel classes $\mathcal{C}$ is there an absolute retract $A \in \mathcal{C}[1]$ whose power A^{n+1} is $\mathcal{C}[n]$-universal for every $n \in \mathbb{N}$?*

QUESTION 2. *Suppose that X, Y are finite-dimensional σ-compact absolute retracts of the first Baire category. Are their countable powers X^ω and Y^ω homeomorphic?*

Note that by Theorem 2 each of the spaces X^ω, Y^ω embeds as a closed subset into the other. By Lemma 9 these spaces are σZ_n-spaces for every $n \in \mathbb{N}$. By Theorem 1 and Lemma 5.4 of [DMM], they are not σZ-spaces, so that the standard technique of absorbing spaces (see [BRZ]) cannot be applied to answer Question 2.

Let us remark that the second assertion of Theorem 3 generalizes [Ca$_3$], the first assertion of Theorem 1 generalizes a result of [BR], and the third one generalizes [Ca$_1$]. As mentioned in the introduction, the Nagata space $\mathcal{N}$ admits no embedding into a sigma hereditarily disconnected absolute $F_{\sigma\delta}$-space. In this context it would be interesting to know answers to the following questions.

QUESTION 3. *Suppose $F \supset \mathcal{N}$ is an $F_{\sigma\delta}$-subset in Q containing the Nagata space $\mathcal{N}$.*

(a) *Does F contain a Hilbert cube* (cf. [En, 5.3.6])*?*

(b) *Is F strongly infinite-dimensional?*

(c) *Does $F \setminus \mathcal{N}$ contain an arc?* Note that $F \setminus \mathcal{N}$ is connected, moreover, $A \cap (F \setminus \mathcal{N})$ is connected for every irreducible barrier A in Q.

(d) *Does F contain a copy I of* $[0,1]$ *such that $I \cap \mathcal{N}$ is a countable dense subset of I?* Note that F always contains a copy K of the Cantor set such that $K \cap \mathcal{N}$ is countable and dense in K.

QUESTION 4. *Does there exist a countable-dimensional absolute $F_{\sigma\delta}$-space containing a copy of each countable-dimensional compactum?*

According to [En, 5.3.11 and 7.1.33], the Smirnov space σ contains a copy of all Smirnov cubes. This shows that there are σ-compact strongly countable-dimensional spaces containing compacta of arbitrary high transfinite dimension ind.

The Main Lemma implies that irreducible barriers in Q are not sigma hereditarily disconnected. In fact, every sigma hereditarily disconnected compactum is weakly infinite-dimensional [Kr, §6]. It is not clear if the converse is also true.

QUESTION 5. *Is every weakly infinite-dimensional compactum sigma hereditarily disconnected?*

It was remarked by R. Pol that this question is connected with the known open problem on existence of a weakly infinite-dimensional compactum whose square is strongly infinite-dimensional: such a compactum cannot be sigma hereditarily disconnected. Observe that the example of an uncountable-dimensional weakly infinite-dimensional compactum constructed by R. Pol [Po] is sigma hereditarily disconnected.

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