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Nonnormality points of $\beta X \setminus X$

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Abstract. Let X be a crowded metric space of weight κ that is either κ^ω-like or locally compact. Let $y \in \beta X \setminus X$ and assume GCH. Then a space of nonuniform ultrafilters embeds as a closed subspace of $(\beta X \setminus X) \setminus \{y\}$ with y as the unique limit point. If, in addition, y is a regular z-ultrafilter, then the space of nonuniform ultrafilters is not normal, and hence $(\beta X \setminus X) \setminus \{y\}$ is not normal.

1. Introduction. An important theorem about the structure of $\beta X \setminus X$ when X is discrete is due to Bešlagić and van Douwen [1].

THEOREM 1.1 (Bešlagić and van Douwen [1]). *Assume GCH. Let κ be an infinite cardinal, and let X be the discrete space of cardinality κ. Let y be any point of $\beta X \setminus X$. Then the space of nonuniform ultrafilters on κ^+ embeds in $(\beta X \setminus X) \setminus \{y\}$ as a closed subset. Hence neither $(\beta X \setminus X) \setminus \{y\}$ nor $\beta X \setminus \{y\}$ is normal.*

Recent research has extended nonnormality point results to nondiscrete spaces. For example:

THEOREM 1.2 (Logunov [9] and Terasawa [12], independently). *If X is a crowded metrizable space space, then $\beta X \setminus \{y\}$ is not normal for all $y \in \beta X \setminus X$.*

THEOREM 1.3 (Logunov [10]). *If X is a crowded realcompact locally compact metrizable space space, and y is not a P-point, then $(\beta X \setminus X) \setminus \{y\}$ is not normal for all $y \in \beta X \setminus X$.*

Logunov and Terasawa prove their results without extra axioms of set theory. They prove that $\beta X \setminus \{y\}$ or $(\beta X \setminus X) \setminus \{y\}$ is not normal, but do not embed closed subspaces of nonuniform ultrafilters. Our results are closer to those of Bešlagić and van Douwen.

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THEOREM 1.4. *Let X be a metric space of weight κ without isolated points that is either κ^ω-like or locally compact. Let $y \in \beta X \setminus X$. Assume GCH. Then a space of nonuniform ultrafilters embeds as a closed subspace of $(\beta X \setminus X) \setminus \{y\}$ with y as the unique limit point. If y is a regular z-ultrafilter, then neither $(\beta X \setminus X) \setminus \{y\}$ nor $\beta X \setminus \{y\}$ is normal.*

2. Topological spaces. All spaces X are Tikhonov, and hence have a Stone–Čech compactification βX. We consider a point of βX to be a z-ultrafilter on X. We identify a point x of X with the z-ultrafilter $\hat{x}$, the collection of all zero sets of X of which x is an element, so that X is embedded as a subspace of βX. When f is a bounded, continuous function from X to $\mathbb{R}$, we denote the unique extension of f by βf.

A space is called *crowded* if it has no isolated points. The topology, weight, and Lindelöf number of a space X are denoted $\tau(X)$, $w(X)$, and $L(X)$. We use the letters κ, λ, θ, etc. to denote infinite cardinals and the discrete spaces of that cardinality. We say that a space X is *κ^ω-like* if X is metrizable, nowhere locally compact, and every nonempty open subset of X has weight κ.

LEMMA 2.1. *Let X be a κ^ω-like metrizable space and let Z be a subset of X with $w(Z) = \lambda < \kappa$. There is a λ^ω-like closed subset Y of X containing Z.*

Proof. Set $Z_1 = Z$. Given Z_n with $L(Z_n) = \lambda$, choose $\mathcal{V}_n \in [\tau(X)]^\lambda$ such that $Z_n \subset \bigcup \mathcal{V}_n$ and $\operatorname{diam} V < 1/n$ for all $V \in \mathcal{V}$. Choose Z_{n+1} such that $Z_n \subseteq Z_{n+1}$, $|Z_{n+1} \setminus Z_n| \le \lambda$ (hence $L(Z_{n+1}) = \lambda$), and for all $V \in \mathcal{V}_n$ there is $E \in [V \cap Z_{n+1}]^\lambda$ which is closed discrete (hence $w(V \cap Z_{n+1}) = \lambda$). Set $Y_0 = \bigcup_{n\in\mathbb{N}} Z_n$; note that $w(Y) \le \lambda$ because $\{V \cap Y_0 : (\exists n)\ V \in \mathcal{V}_n\}$ is a base for Y_0.

Let $y \in W$ be open in Y_0. There are $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ such that $y \in V \cap Z^{n+1} \subseteq W$. Then $w(W) \ge w(V \cap Z_{n+1}) = \lambda$. Finally, set $Y = \operatorname{cl} Y_0$. ∎

3. Regular z-ultrafilters. The next result tells us for which cardinals θ the space of nonuniform ultrafilters is not normal.

LEMMA 3.1. *Let* $\mathrm{NU}(\theta)$ *denote the subspace of $\beta\theta$ of nonuniform ultrafilters. That is,* $\mathrm{NU}(\theta) = \{y \in \beta\theta : (\exists Z \in y)\ |Z| < \theta\}$.

(1) ([11]) *If θ is regular and not a strong limit cardinal (in particular, if $\theta = \kappa^+$), then* $\mathrm{NU}(\theta)$ *is not normal.*
(2) ([11]) *If θ is singular, then* $\mathrm{NU}(\theta)$ *is not normal.*
(3) ([8]) *The space* $\mathrm{NU}(\theta)$ *is normal if and only if θ is weakly compact.*

In the proof of Theorem 1.1, the reaping number $\mathfrak{r}(\kappa)$ of κ is defined, and the space $\mathrm{NU}(\mathfrak{r}(\kappa))$ is embedded in $(\beta\kappa \setminus \kappa) \setminus \{y\}$. The inequalities

$\kappa < \mathfrak{r}(\kappa) \leq 2^\kappa$ hold in ZFC, so GCH gives $\mathfrak{r}(\kappa) = \kappa^+$, and the embedded space is not normal.

In the proof of Theorem 1.4, we consider a point y of $\beta X \setminus X$. The analog of $\mathfrak{r}(\kappa)$ is θ_y, a cardinal which depends on the point y (not just the space X). The upper bound $\theta_y \leq 2^\kappa$ is proved as in [1], but the lower bound $\kappa < \theta_y$ requires assuming that y is a regular z-ultrafilter.

DEFINITION 3.2. Let y be a z-ultrafilter on a space X. We say that y is *κ-regular* if there is a subset $\mathcal{Z}$ of y such that $\mathcal{Z}$ is locally finite and $|\mathcal{Z}| = \kappa$. We say that y is *regular* if y is $w(X)$-regular.

If X is a discrete space of cardinality κ, then any ultrafilter on X is a z-ultrafilter (because every subset of X is an open set, a closed set, and a z-set). In this case, a κ-regular ultrafilter is exactly an (ω, κ)-regular ultrafilter as defined in [2]. The notion of regular ultrafilter appears implicitly in papers from the mid-1950's, for example [5].

THEOREM 3.3 ([7, Section 12.7]). *Let κ be an infinite cardinal. There is a maximal ideal M in $C(\kappa)$ such that $|C(\kappa)/M| > \kappa$. In fact, no set of power at most κ is cofinal in the ordered field $C(\kappa)/M$. If $2^\kappa = \kappa^+$, then $\mathrm{cf}(C(\kappa)/M) = |C(\kappa)/M| = 2^\kappa$.*

Proof. Because κ is infinite, there is a bijection $\alpha \mapsto s_\alpha$ from κ to $[\kappa]^{<\omega}$. For each $\alpha \in \kappa$, set $Z_\alpha = \{\gamma \in \kappa : \alpha \in s_\gamma\}$. By construction, $\{Z_\alpha : \alpha \in \kappa\}$ has the finite intersection property: if $s = s_\gamma \in [\kappa]^{<\omega}$, then $\gamma \in \bigcap\{Z_\alpha : \alpha \in s\}$. Extend $\{Z_\alpha : \alpha \in \kappa\}$ to a z-ultrafilter y, and set $M = \{f \in C(\kappa) : f^{\leftarrow}\{0\} \in y\}$.

Given $B = \{g_\alpha : \alpha < \kappa\} \subset C(\kappa)$, define

$$f(\gamma) = 1 + \max\{g_\alpha(\gamma) : \alpha \in s_\gamma\}.$$

The maximum exists because s_γ is finite, and f is continuous because κ is discrete. Let $g_\alpha \in B$ be arbitrary. For every $\gamma \in Z_\alpha$,

$$g_\alpha(\gamma) \leq \max\{g_{\alpha'} : \alpha' \in s_\gamma\} < f(\gamma). \blacksquare$$

We can generalize the previous theorem to show that if X is a paracompact space, and y is a regular z-ultrafilter on X, then $C(X)/M_y$ has cofinality greater than κ, where M_y is the maximal ideal of functions f such that $\{x \in X : f(x) = 0\} \in y$. We have also generalized the notion of "κ^+-good" to z-ultrafilters and proved the analogous theorem. If y is a κ^+-good z-ultrafilter on a paracompact space X, then $C(X)/M_y$ is an η_α-set, where $\kappa^+ = \aleph_\alpha$.

DEFINITION 3.4. Let $\mathrm{UR}(\kappa)$ be the assertion that every uniform ultrafilter on a set of cardinality κ is κ-regular. Let UR assert that $\mathrm{UR}(\kappa)$ holds for every infinite κ. Informally, we read UR as "every uniform ultrafilter is regular".

The most familiar example of a nonregular ultrafilter is a countably complete free ultrafilter on a measurable cardinal. Hence UR implies that there are no measurable cardinals. Like the assumption that there are no measurable cardinals, UR is safe. The assumption of Theorem 1.4, GCH + UR, is a consequence of V = L. Hence UR does not imply that ZFC is consistent. On the other hand, it has been shown that ¬UR does imply that ZFC is consistent. In fact, it is plausible to conjecture that ¬UR is equiconsistent with "there exists a measurable cardinal". See [3].

LEMMA 3.5. *Assume* UR(κ). *That is, every uniform ultrafilter p on a set of cardinality κ is κ-regular. Let X be a metrizable space of weight κ which is locally compact. Then every uniform z-ultrafilter y on X is κ-regular.*

Proof. Let $\mathcal{C}$ be the collection of open subsets of X that have compact closure. Because X is locally compact, $\mathcal{C}$ covers X. Let $\mathcal{R}$ be a locally finite open refinement of $\mathcal{C}$.

We claim that $|R| = \kappa$. Since y is free, X is not compact and therefore $\mathcal{R}$ cannot be finite. Hence if $\kappa = \omega$ then $|\mathcal{R}| = \kappa = \omega$. Suppose that $\kappa > \omega$. Let $\mathcal{B}$ be a base for X of cardinality κ. Because $\mathcal{R}$ is locally finite, $|\mathcal{R}| \le |\mathcal{B}| = \kappa$. In the other direction, if $R \in \mathcal{R}$, then $L(R) = \omega$. Hence $\kappa = L(\bigcup \mathcal{R}) \le |\mathcal{R}| \cdot \omega$. By the same argument, for all $\mathcal{S} \in [\mathcal{R}]^{<\kappa}$ and $Z \in y$, we have $Z \not\subseteq \bigcup \mathcal{S}$ because y is a uniform z-ultrafilter.

For each $Z \in y$, set $\mathcal{U}(Z) = \{U \in \mathcal{R} : U \cap Z \neq \emptyset\}$. Observe that $p^0 = \{\mathcal{U}(Z) : Z \in y\} \cup \{X \setminus \mathcal{S} : \mathcal{S} \in [\mathcal{R}]^{<\kappa}\}$ has the finite intersection property, and extend it to a uniform ultrafilter p on $\mathcal{R}$.

Because p is κ-regular, there is a point finite collection $\{\mathcal{U}_\alpha : \alpha \in \kappa\} \subset p$. For each α, set $Z_\alpha = \operatorname{cl} \bigcup \mathcal{U}_\alpha$. We now show that $Z_\alpha \in y$. Let $Z \in y$ be arbitrary. The collections $\mathcal{U}_\alpha$ and $\mathcal{U}(Z)$ are both members of p, so $\mathcal{U}_\alpha \cap \mathcal{U}(Z) \neq \emptyset$. Let $U \in \mathcal{U}_\alpha \cap \mathcal{U}(Z)$. Then $U \cap Z \neq \emptyset$ and therefore $\bigcup \mathcal{U}_\alpha \cap Z \neq \emptyset$. Hence $Z_\alpha \cap Z \neq \emptyset$, so $Z_\alpha \in y$.

We have shown that $\{Z_\alpha : \alpha \in \kappa\}$ is a subset of y; we must show that it is locally finite. Because $\mathcal{R}$ is locally finite, for each $x \in X$ there is an open set V such that $x \in V$ and $\{U \in \mathcal{R} : V \cap U \neq \emptyset\}$ is finite. Then $\{\alpha \in \kappa : (\exists U \in \mathcal{U}_\alpha)\ V \cap U \neq \emptyset\}$ is finite, and we are done. ∎

In the result above, the hypothesis "X is locally compact" can be replaced with the cumbersome "Let X have a cover $\mathcal{C}$ of open sets of weight less than λ, for some regular cardinal λ less than or equal to κ".

4. Pi-bases. In our constructions we will use locally finite pairwise disjoint collections ξ of open sets. The collections will come from an appropriate π-base. Following Terasawa we use ξ^* to denote $\bigcup \xi$. Observe that such a collection ξ is locally finite and maximal disjoint if and only if ξ^* is dense in X.

Proposition 4.1 (Terasawa). *Let X be a crowded metrizable space. Then X has a π-base*

$$\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$$

such that

(1) *$\mathcal{B}_n$ is a locally finite, maximal disjoint family of nonempty open sets;*
(2) *$\mathcal{B}_n$ refines $\mathcal{B}_{n-1}$;*
(3) *for each $B \in \mathcal{B}_{n-1}$, there are three sets $B^{(i)} \in \mathcal{B}_n$, $i = 0, 1, 2$, such that $\operatorname{cl} B^{(i)} \subset B$ and $\operatorname{cl} B^{(i)} \cap \operatorname{cl} B^{(i)} = \emptyset$ for $i \neq j$;*
(4) *every open cover of X is refined by a locally finite, maximal disjoint subfamily of $\mathcal{B}$.*

Suppose $y \in \beta X \setminus X$. Terasawa remarks that the π-base in Proposition 4.1 can be easily modified so that

$$(\#) \qquad y \notin \operatorname{cl}_{\beta X} B \quad \text{for all } B \in \mathcal{B}.$$

This property of $\mathcal{B}$ was not, however, necessary in his proof that $\beta X \setminus \{y\}$ is not normal; the butterfly sets did not need to be subsets of $\beta X \setminus X$. To show that $(\beta X \setminus X) \setminus \{y\}$ is not normal, our construction will require closed subsets of $\beta X \setminus X$. The following propositions define a π-base $\mathcal{B}$ for two types of metric spaces. For X locally compact, (#) is true for $\mathcal{B}$ for any $y \in \beta X \setminus X$. For X κ^ω-like, given $y \in \beta X \setminus X$, we construct $\mathcal{B}$ so that (#) is satisfied.

We say that a π-base $\mathcal{B}$ for a crowded metric space is *nice* if it satisfies (1), (2) and (4) in Proposition 4.1. In Section 5 we use the properties of a nice π-base to construct locally finite collections. In the sections after 5 we use a nice π-base with the additional properties (3) and (#).

The proofs of the next two results are omitted because they follow easily from Proposition 4.1.

Proposition 4.2. *Let X be a locally compact crowded metrizable space. Then X has a π-base*

$$\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$$

such that

(1) *$\mathcal{B}_n$ is a locally finite, maximal disjoint family such that $\operatorname{cl}_X B$ is compact for each $B \in \mathcal{B}$;*
(2) *$\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$ and $|\{B' \in \mathcal{B}_{n+1} : B' \subset B\}| = 4$ for all $B \in \mathcal{B}_n$;*
(3) *for $B \in \mathcal{B}_n$ there are $B^0, B^1 \in \mathcal{B}_{n+1}$ such that $\operatorname{cl} B^0 \cap \operatorname{cl} B^1 = \emptyset$ and $\operatorname{cl} B^0, \operatorname{cl} B^1 \subset B$;*
(4) *every open cover of X is refined by a locally finite, maximal disjoint subfamily of $\mathcal{B}$.*

PROPOSITION 4.3. *Let κ be an infinite cardinal and let X be a κ^ω-like metric space. Let y be a free z-ultrafilter on X. Then X has a π-base*

$$\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$$

such that

(1) *$\mathcal{B}_n$ is a locally finite, maximal disjoint family of nonempty open sets;*
(2) *$\mathcal{B}_n$ refines $\mathcal{B}_{n-1}$;*
(3) *$|\mathcal{B}_0| = \kappa$ and for each $B \in \mathcal{B}_{n-1}$, there are sets $B^{(\eta)} \in \mathcal{B}_n$, $\eta \in \kappa$, such that $\operatorname{cl} B^{(\eta)} \subset B$ and $\operatorname{cl} B^{(\eta)} \cap \operatorname{cl} B^{(\eta')} = \emptyset$ for $\eta \neq \eta'$;*
(4) *every open cover of X is refined by a locally finite, maximal disjoint subfamily of $\mathcal{B}$;*
(5) *$\operatorname{cl} B \notin y$ for all $B \in \mathcal{B}$.*

5. Locally finite collections and cofinalities. Let X be a crowded metrizable space with a nice π-base $\mathcal{B}$. Let Ξ be the collection of maximal pairwise disjoint, locally finite collections $\xi \subset \mathcal{B}$.

REMARK 1. For each $B, B' \in \mathcal{B}$, if $B \cap B' \neq \emptyset$ then either $B = B'$, $B \subsetneq B'$ or $B' \subsetneq B$.

REMARK 2. If $\xi, \eta \in \Xi$ and $B \in \xi$, then since both ξ^* and η^* are dense in X, because of Remark 1, there is $B' \in \eta$ such that either $B = B'$, $B \subsetneq B'$ or $B' \subsetneq B$.

Fix a free z-ultrafilter y on X and let τ_y be the collection of open neighborhoods of y in βX. Let $\mathcal{N}_y = \{X \cap O : y \in O, O \in \tau(\beta X)\}$. The collection $\mathcal{N}_y$ is a free open filter on X. We write $\hat{\mathcal{N}_y}$ for the collection of open subsets U of X that are dense in some $N \in \mathcal{N}_y$, that is, $N \subset \operatorname{cl} U$. Using $\hat{\mathcal{N}_y}$, we define a strict partial order $<_y$ on Ξ. For $\xi, \eta \in \Xi$ let $L(\xi, \eta) = \{B \in \xi : B' \subsetneq B \text{ for some } B' \in \eta\}$. Define $\xi <_y \eta$ if $L(\xi, \eta)^* \in \hat{\mathcal{N}_y}$.

The lemma below is analogous to Theorem 3.3.

LEMMA 5.1. *Let $\kappa = w(X)$. Suppose $y \in \beta X \setminus X$ is a regular z-ultrafilter. Any subset $\{\xi_\gamma : \gamma \in \lambda\}$ of Ξ where $\lambda \leq \kappa$ is bounded.*

Proof. Let $\{\xi_\gamma : \gamma \in \kappa\} \subset \Xi$. We construct $\xi \in \Xi$ such that $\xi_\gamma <_y \xi$ for all $\gamma \in \kappa$. Let $\{Z_\gamma : \gamma \in \kappa\} \subset y$ be a locally finite subcollection of y. Since X is paracompact, there is a locally finite collection $\mathcal{W} = \{W_\gamma : \gamma \in \kappa\}$ of open subsets of X such that $Z_\gamma \subset W_\gamma$ for all $\alpha \in \kappa$ (see [4, Remark 5.1.19]). Note that $W_\gamma \in \mathcal{N}_y$. For each $x \in X$ let $F_x = \{\gamma : x \in \operatorname{cl} W_\gamma\}$ and set $U_x^0 = X \setminus \bigcup\{\operatorname{cl} W_\gamma : \gamma \notin F_x\} = X \setminus \operatorname{cl}(\bigcup\{W_\gamma : \gamma \notin F_x\})$. For $\gamma \in F_x$ let $\mathcal{C}(x, \gamma) = \{B \in \xi_\gamma : x \in \operatorname{cl} B\}$ and set $\mathcal{C}_x = \bigcup\{\mathcal{C}(x, \gamma) : \gamma \in F_x\}$. Define $U_x = U_x^0 \setminus \bigcup\{\operatorname{cl} B : B \in \xi_\gamma \setminus \mathcal{C}_x, \gamma \in F_x\}$. Since ξ_γ is locally finite, U_x is an open neighborhood of x. Choose a finite set $E_x \subset X \setminus \{x\}$ such that $|E_x \cap B| \geq 1$ for each $B \in \mathcal{C}_x$. Let $V_x = U_x \setminus E_x$. For $B, B' \in \mathcal{B}$, observe that if $B \subset V_x$, $\gamma \in F_x$, $B' \in \xi_\gamma$ and $B \cap B' \neq \emptyset$ then $B \subsetneq B'$. The collection $\mathcal{V} = \{V_x : x \in X\}$ is an open cover of X. Let $\xi \in \Xi$ be a maximal locally finite collection refining $\mathcal{V}$. Suppose $\gamma \in \kappa$. We will show that $L(\xi_\gamma, \xi)^*$ contains $W_\gamma \cap \xi^* \cap \xi_\gamma^*$, and is therefore dense in W_γ, and hence $\xi_\gamma <_y \xi$.

Let $x' \in W_\gamma \cap \xi^* \cap \xi_\gamma^*$. So, there are $x \in X$, $B \in \xi$, and $B' \in \xi_\gamma$ such that $B \subset V_x$ and $x' \in B \cap B'$. Since $V_x \cap W_\gamma \neq \emptyset$ it must be that $\gamma \in F_x$. Following a previous observation, $B \subsetneq B'$. Hence $x' \in L(\xi_\gamma, \xi)^*$. ∎

If we assume that $2^\kappa = \kappa^+$, we may write Ξ as $\{\zeta_\gamma : \gamma \in \kappa^+\}$. We define $\{\xi_\gamma : \gamma \in \kappa^+\}$ by induction, using Lemma 5.1 to define ξ_γ greater than $\{\xi_\alpha : \alpha < \gamma\} \cup \{\zeta_\gamma\}$. The result is a $<_y$-increasing sequence $\{\xi_\gamma : \gamma \in \kappa^+\}$ cofinal in Ξ.

If y is a remote point, then the partial order $(\Xi, <_y)$ is a total order. We can show, without using axioms beyond ZFC, that if y is a remote point, then the cofinality of $(\Xi, <_y)$ is equal to the cofinality of $C(X)/M_y$.

6. H's and $\mathcal{L}$'s. Suppose y is a z-ultrafilter on a crowded metric space X with weight κ. Following Logunov [9] and Terasawa [12], in this section we use a cofinal sequence from Ξ to define a sequence of closed sets intersecting to y.

Suppose $\{\xi_\gamma : \gamma \in \theta_y\}$ is a cofinal $<_y$-increasing sequence in Ξ. We note now that $\theta_y \leq 2^\kappa$ and make extra assumptions on θ_y later. Without loss of generality we may assume that $\xi_\gamma \cap \mathcal{B}_0 = \emptyset$. If $\xi_\gamma \cap \mathcal{B}_0 \neq \emptyset$, replace ξ_γ with $(\xi_\gamma \setminus \mathcal{B}_0) \cup \{B \in \mathcal{B}_1 : (\exists B' \in \xi_\gamma \cap \mathcal{B}_0)\ B \subset B'\}$. Let $\mathcal{N}_\gamma = \{\mathcal{U} \subset \xi_\gamma : \mathcal{U}^* \in \hat{\mathcal{N}}_y\}$ and let

$$H_\gamma = \bigcap \{\mathrm{cl}_{\beta X}\, \mathcal{U}^* : \mathcal{U} \in \mathcal{N}_\gamma\}.$$

Claim. *For each $\gamma \in \theta_y$, $y \in H_\gamma$.*

Proof. If $\mathcal{U}^*$ and $\mathcal{V}^*$ are dense in N and N' from $\mathcal{N}_y$, then $\mathcal{U}^* \cap \mathcal{V}^*$ is dense in $N \cap N'$, which is also in $\mathcal{N}_y$. Hence, $\mathcal{N}_\gamma$ is a filter on ξ_γ. Every $U \in \hat{\mathcal{N}}_y$ is dense in some $N \in \mathcal{N}_y$, the trace of a neighborhood of y on X. Therefore, $y \in \mathrm{cl}_{\beta X}\, U$ for all $U \in \hat{\mathcal{N}}_y$. ∎

Claim. *For each $\gamma \in \theta_y$, $H_\gamma \subset \beta X \setminus X$.*

Proof. By Proposition 4.3(5), for any $B \in \xi_\gamma$, since $y \notin \mathrm{cl}_{\beta X}\, B$ it must be that $\xi_\gamma \setminus \{B\} \in \mathcal{N}_\gamma$. Fix $x \in X$. Since ξ_γ is locally finite, $\mathcal{U} = \{B \in \xi_\gamma : x \in \mathrm{cl}_{\beta X}\, B\}$ is finite and hence $\xi_\gamma \setminus \mathcal{U} \in \mathcal{N}_\gamma$. Also, $x \notin \mathrm{cl}_{\beta X}(\xi_\gamma \setminus \mathcal{U})^*$ and therefore $x \notin H_\gamma$. ∎

Claim. *If $\gamma' < \gamma$ then $H_\gamma \subset H_{\gamma'}$.*

Proof. Let $\gamma' < \gamma$ and let $\mathcal{U} \in \mathcal{N}_{\gamma'}$. We will show that $H_\gamma \subset \mathrm{cl}_{\beta X}\,\mathcal{U}^*$. Since $\gamma' < \gamma$, $\xi_{\gamma'} <_y \xi_\gamma$ and therefore $L(\xi_{\gamma'}, \xi_\gamma) \in \mathcal{N}_{\gamma'}$. Hence $\mathcal{U} \cap L(\xi_{\gamma'}, \xi_\gamma) \in \mathcal{N}_{\gamma'}$. Since $\mathcal{U}, L(\xi_{\gamma'}, \xi_\gamma) \subset \xi_{\gamma'}$ we find that $\mathcal{U}^* \cap L(\xi_{\gamma'}, \xi_\gamma)^* = (\mathcal{U} \cap L(\xi_{\gamma'}, \xi_\gamma))^*$. Let $\mathcal{W} = \mathcal{U} \cap L(\xi_{\gamma'}, \xi_\gamma)$ and $\mathcal{V} = \{V \in \xi_\gamma : V \cap U \neq \emptyset$ for some $U \in \mathcal{W}\}$. Since ξ_γ^* is dense in X, $\mathrm{cl}_X\,\mathcal{V}^* \supset \mathcal{W}^*$. Furthermore, $V \in \xi_\gamma$, $U \in L(\xi_{\gamma'}, \xi_\gamma)$ and $V \cap U \neq \emptyset$ imply that $V \subset U$. Therefore $\mathcal{V}^* \subset \mathcal{W}^*$ and hence $\mathrm{cl}_X\,\mathcal{V}^* = \mathrm{cl}_X\,\mathcal{W}^*$. Since $\mathcal{W}^* \in \hat{\mathcal{N}_y}$ and $\mathcal{V}^*$ is dense in $\mathcal{W}^*$ we deduce that $\mathcal{V} \in \mathcal{N}_\gamma$. Therefore, $H_\gamma \subset \mathrm{cl}_{\beta X}\,\mathcal{V}^* = \mathrm{cl}_{\beta X}\,\mathcal{W}^* \subset \mathrm{cl}_{\beta X}\,\mathcal{U}^*$. ∎

Claim. $\bigcap\{H_\gamma : \gamma \in \theta_y\} = \{y\}$.

Proof. We have seen that $y \in \bigcap\{H_\gamma : \gamma \in \theta_y\}$. Let $O' \in \tau_y$. We will find $\gamma \in \theta_y$ such that $H_\gamma \subset O'$. Let $W', U' \in \tau_y$ be such that

$$\mathrm{cl}_{\beta X}\,W' \subset W' \subset \mathrm{cl}_{\beta X}\,U \subset O.$$

Let $O = O' \cap X$, $U = U' \cap X$ and $W = W' \cap X$. So, $\mathrm{cl}_X\,W \subset U \subset \mathrm{cl}_X\,U \subset O$. Let $V = X \setminus \mathrm{cl}_X\,W$. Then $\{U, V\}$ is an open cover of X. By Proposition 4.2 there is $\xi \in \varXi$ that refines $\{U, V\}$. Let $\gamma \in \theta_y$ be such that $\xi <_y \xi_\gamma$. Note that $W \in \mathcal{N}_y$. Since $\xi <_y \xi_\gamma$ we have $L(\xi, \xi_\gamma)^* \in \hat{\mathcal{N}_y}$ and $W \cap L(\xi, \xi_\gamma)^* \in \hat{\mathcal{N}_y}$. Let $\hat{W} = W \cap L(\xi, \xi_\gamma)^*$ and let $\mathcal{V} = \{B \in \xi_\gamma : B \cap \hat{W} \neq \emptyset\}$. Since ξ_γ^* is dense in X and $\hat{W}$ is open, $\mathrm{cl}_X\,\mathcal{V}^* \supset \hat{W}$. Hence $\mathcal{V}^* \in \hat{\mathcal{N}_y}$. On the other hand, if $B \in \mathcal{V}$ then $B \cap L(\xi, \xi_\gamma)^* \neq \emptyset$ and therefore $B \subset B'$ for some $B' \in \xi$. Since ξ refines $\{U, V\}$, either $B \subset B' \subset U$ or $B \subset B' \subset V$. Since $B \cap W \neq \emptyset$, it cannot be the case that $B \subset V$. Therefore $B \subset U$ and hence $\mathcal{V}^* \subset U$ and $\mathrm{cl}_X\,\mathcal{V}^* \subset \mathrm{cl}_X\,U \subset O$. Then, since X is normal, $\mathrm{cl}_{\beta X}\,\mathcal{V}^* \subset O'$. Since $\mathcal{V} \subset \xi_\gamma$ and $\mathcal{V}^* \in \hat{\mathcal{N}_y}$ we have $H_\gamma \subset \mathrm{cl}_{\beta X}\,\mathcal{V}^* \subset O'$ as desired. ∎

Next, we will use the cofinal sequence to inductively define a pair of locally finite collections, $\mathcal{L}_\gamma^0$ and $\mathcal{L}_\gamma^1$, from $\mathcal{B}$ such that $\mathrm{cl}(\mathcal{L}_\gamma^0)^* \cap \mathrm{cl}(\mathcal{L}_\gamma^0)^* = \emptyset$. In this induction, we must do θ_y many tasks, and each step of the induction can have at most κ predecessors. Therefore, we assume $2^\kappa = \kappa^+$ to get $\theta_y \leq \kappa^+$. The constructions of the $\mathcal{L}$'s for the two types of spaces are not the same. However, in either case, the pairs will be used for the same purpose: to "split" the H_γ's.

6.1. X is locally compact. We are able to arrange the cofinal sequence of collections $\{\xi_\gamma : \gamma \in \theta_y\}$ as "step functions", which makes the definition of the $\mathcal{L}$'s easier than in the κ^ω-like case. List $\mathcal{B}_0 = \{B_{\alpha,\emptyset} : \alpha \in \kappa\}$ and $\mathcal{B}_i = \{B_{\alpha,\sigma} : \alpha \in \kappa,\ \sigma \in {}^i4\}$ in such a way that $B_{\alpha,\sigma} \subset B_{\alpha,\sigma'}$ if σ extends σ'. We may assume that for $\alpha \in \kappa$ and $\sigma \in {}^i4$, $\mathrm{cl}_X\,B_{\alpha,\sigma^\frown 0} \cap \mathrm{cl}_X\,B_{\alpha,\sigma^\frown 1} = \emptyset$ and $\mathrm{cl}_X\,B_{\alpha,\sigma^\frown 0}, \mathrm{cl}_X\,B_{\alpha,\sigma^\frown 1} \subset B_{\alpha,\sigma}$. Notice that the collections ξ from $\varXi$ that have the property that $B_{\alpha,\sigma}, B_{\alpha,\sigma'} \in \xi$ implies $|\sigma| = |\sigma'|$ form an unbounded set in $\varXi$. To see this, let $\xi' \in \varXi$ and let $n(\alpha) = \max\{|\sigma| : B_{\alpha,\sigma} \in \xi'\} + 1$.

Then the collection $\xi = \{B_{\alpha,\sigma} : \alpha \in \kappa, \sigma \in {}^{n(\alpha)}4\}$ has the property that $\xi >_y \xi'$ since $L(\xi', \xi) = \xi'$.

Therefore, we may assume that $\{\xi_\gamma : \gamma \in \theta_y\}$ is a sequence of collections that have the property that for each $\gamma \in \theta_y$ and $\alpha \in \kappa$ if $B_{\alpha,\sigma}, B_{\alpha,\sigma'} \in \xi_\gamma$ then $|\sigma| = |\sigma'|$. For each $\gamma \in \theta_y$ define the function $n(\gamma, \cdot) : \kappa \to \omega$ such that $\xi_\gamma = \{B_{\alpha,\sigma} : \alpha \in \kappa, \sigma \in {}^{n(\gamma,\alpha)}4\}$. Notice that for any $\gamma' < \gamma < \theta_y$ the set $L(\xi_\gamma, \xi_{\gamma'})^*$ is dense in $\{B_\alpha : \alpha \in S\}^*$ for any nonempty set $S \subset \kappa$.

Defining the $\mathcal{L}^i_\gamma$*'s.* For $\gamma \in \theta_y$ and $i = 0, 1$ define $\mathcal{L}^i_\gamma = \{B_{\alpha,\sigma^\frown i} : \alpha \in \kappa, \sigma \in {}^{n(\gamma,\alpha)}4\}$.

Claim. *For all* $\gamma \in \theta_y$, $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\gamma) \cap \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^1_\gamma) = \emptyset$.

Proof. For each $\alpha \in \kappa$ and $\sigma \in {}^i4$, $\mathrm{cl}_X B_{\alpha,\sigma^\frown 0} \cap \mathrm{cl}_X B_{\alpha,\sigma^\frown 1} = \emptyset$. Also, $B_{\alpha,\sigma} \cap B_{\alpha,\beta} = \emptyset$ for $\sigma \neq \beta \in {}^{n(\gamma,\alpha)}4$, and for $i = 0, 1$ we have $\mathrm{cl}_X B_{\alpha,\sigma^\frown i} \subset B_{\alpha,\sigma}$ and $\mathrm{cl}_X B_{\alpha,\beta^\frown i} \subset B_{\alpha,\beta}$. Therefore

$$\mathrm{cl}_X B_{\alpha,\sigma^\frown i} \cap \mathrm{cl}_X B_{\alpha,\beta^\frown j} = \emptyset$$

for $i, j = 0, 1$. So,

$$\bigcup\{\mathrm{cl}_X B_{\alpha,\sigma^\frown 0} : \sigma \in {}^{n(\gamma,\alpha)}4\} \cap \bigcup\{\mathrm{cl}_X B_{\alpha,\sigma^\frown 0} : \sigma \in {}^{n(\gamma,\alpha)}4\} = \emptyset.$$

Now, since $\{B_{\alpha,\emptyset} : \alpha \in \kappa\}$ is a locally finite family and since $\mathrm{cl}_X B_{\alpha,\sigma^\frown i} \subset B_{\alpha,\emptyset}$ for each $\sigma \in \bigcup_{n\in\omega} {}^n4$ and $i = 0, 1$, we have

$$\begin{aligned}\mathrm{cl}_X\Big(\bigcup \mathcal{L}^0_\gamma\Big) \cap \mathrm{cl}_X\Big(\bigcup \mathcal{L}^1_\gamma\Big) &= \bigcup\{\mathrm{cl}_X B_{\alpha,\sigma^\frown 0} : \sigma \in {}^{n(\gamma,\alpha)}4,\ \alpha \in \kappa\} \\ &\quad \cap \bigcup\{\mathrm{cl}_X B_{\alpha,\sigma^\frown 1} : \sigma \in {}^{n(\gamma,\alpha)}4,\ \alpha \in \kappa\} = \emptyset.\end{aligned}$$

Finally, since $\mathrm{cl}_X(\bigcup \mathcal{L}^0_\gamma) \cap \mathrm{cl}_X(\bigcup \mathcal{L}^1_\gamma) = \emptyset$ we conclude that $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\gamma) \cap \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^1_\gamma) = \emptyset$. ∎

Since $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\gamma) \cap \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^1_\gamma) = \emptyset$, y can be in at most one of $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\gamma)$ or $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^1_\gamma)$. Without loss of generality, assume $y \notin \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\gamma)$ for each $\gamma \in \theta_y$.

Consider a finite collection $\{\xi_{\gamma_i} : i \in m\} \subset \{\xi_\gamma : \gamma \in \theta_y\}$ such that $\gamma_i < \gamma_j$ for $i < j \leq m$ and let $U(i,j) = L(\xi_{\gamma_i}, \xi_{\gamma_j})$. It is the case that $U(i,j)^* \in \hat{\mathcal{N}}_y$ for each $i < j$ and hence $U = \bigcap\{U(i,j)^* : i < j \leq m\} \in \hat{\mathcal{N}}_y$. For any $B \in \xi_{\gamma_0}$ such that $B \cap U \neq \emptyset$ we observe that $\{B' \in \gamma_i : B' \subset B\}$ refines $\{B' \in \gamma_j : B' \subset B\}$ whenever $0 < j < i \leq m$.

A special case of the following claim, in particular when Φ is constant, is proven in [12, Lemma 3] and in [9, Proposition 6].

Claim 6.1. *For any* $\rho < \theta_y$ *and* $\Phi : D \subset [\rho, \theta_y) \to 2$, *the collection* $\{H_\rho\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^{\Phi(\gamma)}_\gamma) : \gamma \in D\}$ *has nonempty intersection.*

Proof. Let $\rho < \theta_y$ and $\Phi : D \to 2$ for some $D \subset [\rho, \theta_y)$. We will show that $\{\mathrm{cl}_{\beta X} \mathcal{U}^* : \mathcal{U} \in \mathcal{N}_\rho\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}_\gamma^{\Phi(\gamma)}) : \gamma \geq \rho\}$ has the finite intersection property. Let $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \mathcal{N}_\rho$ and let $\gamma_1, \ldots, \gamma_m \in D$ be such that $\gamma_m \geq \cdots \geq \gamma_1 \geq \rho$. Since $\mathcal{N}_\rho$ is a filter, $\mathcal{U} = \bigcap\{\mathcal{U}_i : 1 \leq i \leq n\} \in \mathcal{N}_\rho$ and therefore $V = \mathcal{U}^* \in \hat{\mathcal{N}_y}$. For $i < j \leq m$, let $U(i,j)^* = L(\xi_{\gamma_i}, \xi_{\gamma_j})$ and notice that $U = \bigcap\{U(i,j)^* : i < j \leq m\} \in \hat{\mathcal{N}_y}$. Let $B_{\alpha,\sigma} \in \xi_\rho$ be such that $B_{\alpha,\sigma} \subset V$ and $B_{\alpha,\sigma} \cap U \neq \emptyset$. As noted before, $\{B \in \gamma_i : B \subset B_{\alpha,\sigma}\}$ refines $\{B \in \gamma_j : B \subset B_{\alpha,\sigma}\}$ whenever $0 < j < i \leq m$. Define $\sigma' \in {}^{n(\gamma_m,\alpha)+1}4$ as follows: $\sigma'|_{n(\rho,\alpha)} = \sigma$, $\sigma'(n(\gamma_i,\alpha)+1) = \Phi(\gamma_i)$ for each $1 \leq i \leq m$ and $\sigma'(k) = 0$ otherwise. Then $B_{\alpha,\sigma'} \subset B_{\alpha,\sigma}$, since σ' extends σ and hence $B_{\alpha,\sigma'} \subset \mathcal{U}^*$. Furthermore, $B_{\alpha,\sigma} \subset \bigcup \mathcal{L}_{\gamma_i}^{\Phi(\gamma_i)}$ since σ' extends $\sigma'|_{n(\gamma_i,\alpha)+1} = \sigma'|_{n(\gamma_i,\alpha)} {}^\frown \Phi(\gamma_i)$ and $B_{\alpha,\sigma'|_{n(\gamma_i,\alpha)} {}^\frown \Phi(\gamma_i)} \in \mathcal{L}_{\gamma_i}^{\Phi(\gamma_i)}$. ∎

6.2. X is κ^ω-like. Consider a finite collection $\{\xi_{\gamma_i} : i \in n\} \subset \{\xi_\gamma : \gamma \in \theta_y\}$ such that $\gamma_i < \gamma_j$ for $i < j \leq n$ and let $U(i,j) = L(\xi_{\gamma_i}, \xi_{\gamma_j})$. It is the case that $U(i,j)^* \in \hat{\mathcal{N}_y}$ for each $i < j$ and hence $U = \bigcap\{U(i,j)^* : i < j \leq n\} \in \hat{\mathcal{N}_y}$. It is tempting to assume that, as in the locally compact case, $\{B \in \xi_{\gamma_0} : B \subset \mathrm{cl}\, U\} \neq \emptyset$. However, there may not exist $B \in \xi_{\gamma_0}$ such that $\{B' \in \gamma_i : B' \subset B\}$ refines $\{B' \in \gamma_j : B' \subset B\}$ whenever $0 < j < i \leq n$.

Defining the $\mathcal{L}_\gamma^i$'s. We define $\{\mathcal{L}_\gamma^i : i \in 2,\ \gamma \in \theta_y\}$ by induction on $\gamma \in \theta_y$.

Let $P = \{p : \mathrm{dom}(p) \in [\theta_y]^{<\omega}, \mathrm{ran}(p) \subset 2\}$. Let $\gamma_p = \max(\mathrm{dom}(p))$ and $n(p) = |p|$. Define $p|_i$ to be the function p restricted to the first i elements of $\mathrm{dom}(p)$. We say $B \in \mathcal{B}$ and $p \in P$ are *aligned* if for each $\gamma \in \mathrm{dom}(p)$ and $B' \in \xi_\gamma$ such that $B' \cap B \neq \emptyset$, we have $B' \subsetneq B$. We will define $\mathcal{L}(B,p)$ for each B and p and set

$$\mathcal{L}_\gamma^i = \bigcup\{\mathcal{L}(B,p) : \gamma_p = \gamma \text{ and } p(\gamma) = i\}.$$

If B and p are not aligned, set $\mathcal{L}(B,p) = \emptyset$.

Stage $\gamma = 0$. There are two $p \in P$ with $\mathrm{dom}(p) = \{0\}$, namely $p^0 = \{(0,0)\}$ and $p^1 = \{(0,1)\}$. Notice that $B \in \mathcal{B}$ is aligned with p^0 or p^1 if there exists $B' \in \xi_0$ such that $B' \subsetneq B$, and that there are κ such B. List as $\{(B_\nu, p_\nu) : \nu \in \kappa\}$ all pairs (B,p) such that $p = p^0$ or $p = p^1$ and B is aligned with p, so that each (B,p) appears in the list κ times. We will define a sequence $\{L(\nu) : \nu \in \kappa\}$ and for each p and B aligned with p, we will set $\mathcal{L}(B,p) = \{L(\nu) : (B,p) = (B_\nu, p_\nu)\}$.

Suppose we have defined $L(\mu) \in \mathcal{B}$ for each $\mu < \nu$ such that $L(\mu) \subsetneq V_\mu \subsetneq B_\mu$ where V_μ is some element of ξ_0. Also assume that if $L(\mu), L(\mu') \subset V \in \xi_0$, then $\mu = \mu'$. We now define $L(\nu)$. For each $V \in \xi_0$ such that $V \cap B_\nu \neq \emptyset$ there is $\eta \in \kappa$ such that $V \subset B_\nu^\eta$. Furthermore, since ξ_0^* is dense in X, for each $\eta \in \kappa$ there is $V \in \xi_0$ such that $V \subset B_\nu^\eta$. For each $\mu < \nu$,

$L(\mu)$ is contained in an element V of ξ_0 and $|\nu| < \kappa$. Therefore, there are κ many $\eta \in \kappa$ such that for all $\mu < \nu$, $B^\eta_\nu \cap L(\mu) = \emptyset$. So, let η_0 be one such η and choose $L(\nu) \in \mathcal{B}$ so that $L(\nu) \subsetneq V_\nu \subset B^{\eta_0}_\nu \subsetneq B_\nu$ for some $V_\nu \in \xi_0$.

For $p = p^0$ or p^1 and each B aligned with p, set

$$\mathcal{L}(B,p) = \{L(\nu) : (B,p) = (B_\nu, p_\nu)\}.$$

Let

$$\mathcal{L}^i_0 = \bigcup \{\mathcal{L}(B,p) : p = p^i \text{ and } B \text{ is aligned with } p\}.$$

Notice that if $L(\nu), L(\mu) \subset B' \in \xi_0$ then $\nu = \mu$. So, since ξ_0 is locally finite, $\mathrm{cl}(\bigcup \mathcal{L}^0_0)$ is disjoint from $\mathrm{cl}(\bigcup \mathcal{L}^1_0)$. Since each (B,p) is listed κ times, $|\{\nu : L(B_\nu, p_\nu) \subsetneq B\}| = \kappa$. Consequently, $|\{\eta \in \kappa :$ there is $L \in \mathcal{L}(B,p)$, $L \subset B^\eta\}| = \kappa$.

Induction hypothesis. Let B and p be aligned such that $\gamma_p \le \gamma$ and $n(p) > 1$. Then, for κ many $\eta \in \kappa$, there is a sequence $\{L_i : 0 \le i < n(p)$, $L_i \in \mathcal{L}(B, p|_i)\}$ such that

$$L_{n(p)-1} \subset L_{n(p)-2} \subset \cdots \subset L_0 \subset B^\eta \subset B.$$

Also, for each $\gamma' < \gamma$, $\mathrm{cl}(\bigcup \mathcal{L}^0_{\gamma'})$ is disjoint from $\mathrm{cl}(\bigcup \mathcal{L}^1_\gamma)$.

Stage γ. Consider all (B,p) such that $\gamma_p = \gamma$ and B is aligned with p. We have assumed $2^\kappa = \kappa^+$. So, $\gamma < \kappa^+$ and hence there are $\le\kappa$ many p with $\gamma_p = \gamma$. Therefore, we can list the collection of such (B,p) as $\{(B_\nu, p_\nu) : \nu \in \kappa\}$ in such a way that each (B,p) appears κ times. Assume we have defined $L(\mu) \in \mathcal{B}$ for each $\mu < \nu$ so that $L(\mu) \subsetneq V_\mu \subsetneq B_\mu$ where V_μ is some element of ξ_γ. Also assume that if $L(\mu), L(\mu') \subset V \in \xi_\gamma$, then $\mu = \mu'$. Let $\eta \in \kappa$ be such that there is $\{L_i : 0 \le i < n(p_\nu),\ L_i \in \mathcal{L}(B_\nu, p_\nu|_i)\}$ with $L_{n(p_\nu)-1} \subset L_{n(p_\nu)-2} \subset \cdots \subset L_0 \subset B^\eta_\nu \subset B_\nu$. Since we have defined $L(\mu)$ for $|\nu| < \kappa$ many μ, by the inductive hypothesis we may also assume that η satisfies $B^\eta_\nu \cap L(\mu) = \emptyset$ for all $\mu < \nu$.

Let $V \in \xi_\gamma$ be such that $L_{n(p_\nu)-1} \cap V \ne \emptyset$. Let $L(\nu)$ be an element of $\mathcal{B}$ such that

$$L(\nu) \subsetneq (V \cap L_{n(p_\nu)-1}) \subset L_{n(p_\nu)-2} \subset \cdots \subset L_0 \subset B^\eta_\nu \subset B_\nu.$$

Set $\mathcal{L}(B,p) = \{L(\nu) : (B_\nu, p_\nu) = (B,p)\}$ and observe that

$$\Big(\bigcup \mathcal{L}(B,p)\Big) \cap \bigcap \Big\{ \bigcup \mathcal{L}(B, p|_i) : i < n(p)\Big\} \ne \emptyset.$$

Now, set $\mathcal{L}^i_\gamma = \bigcup\{\mathcal{L}(B,p) : \gamma_p = \gamma \text{ and } p(\gamma) = i\}$. This concludes stage γ.

For each p and B aligned with p, we have

$$\Big(\bigcup \mathcal{L}(B,p)\Big) \cap \bigcap \Big\{ \bigcup \mathcal{L}(B, p|_i) : i < n(p)\Big\} \ne \emptyset.$$

Therefore, if $\mathrm{dom}(p) \setminus \{\gamma_p\} = \{\gamma_i : 1 \le i < n(p)\}$, we deduce that $\bigcap\{\mathcal{L}^{p(\gamma_i)}_{\gamma_i} : i < n(p)\} \cap B \ne \emptyset$.

Claim 6.2. *For any $\rho < \theta_y$ and $\Phi : D \subset [\rho, \theta_y) \to 2$, the collection $\{H_\rho\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}_\gamma^{\Phi(\gamma)}) : \gamma \in D\}$ has nonempty intersection.*

Proof. Let $\rho < \theta_y$ and $\Phi : D \to 2$ for some $D \subset [\rho, \theta_y)$. We will show that $\{\mathrm{cl}_{\beta X} \mathcal{U}^* : \mathcal{U} \in \mathcal{N}_\rho\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}_\gamma^{\Phi(\gamma)}) : \gamma \geq \rho\}$ has the finite intersection property. Let $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \mathcal{N}_\rho$ and let $\gamma_1, \ldots, \gamma_m \in D$ be such that $\gamma_m > \cdots > \gamma_1 > \rho$. For each $i \leq m$, $L(\xi_\rho, \xi_{\gamma_i}) \in \mathcal{N}_\rho$ since $\xi_{\gamma_i} > \xi_\rho$. Hence, $\mathcal{U} = \bigcap\{\mathcal{U}_i : 1 \leq i \leq n\} \cap \bigcap\{L(\xi_\rho, \xi_{\gamma_i}) : 1 \leq i \leq m\} \in \mathcal{N}_\rho$. Let p be the function Φ restricted to $\{\gamma_i : 1 \leq i \leq m\}$. Note that if $B \in \mathcal{U}$ then B is aligned with p. From the previous construction we conclude that $\bigcap\{\bigcup \mathcal{L}_{\gamma_i}^{p(\gamma_i)} : i \leq m\} \cap B \neq \emptyset$. ∎

7. Theorems

Theorem 7.1. *Let X be a crowded metrizable space of weight κ that is either κ^ω-like or locally compact. Let $y \in \beta X \setminus X$. Suppose that $2^\kappa = \kappa^+$ and $\theta_y^{<\theta_y} = \theta_y$. Then there is a closed copy of $\mathrm{NU}(\theta_y)$ in $(\beta X \setminus X) \setminus \{y\}$.*

Proof. We follow the argument found in [1] to embed $\mathrm{NU}(\theta_y)$ into $(\beta X \setminus X) \setminus \{y\}$, using the $\mathcal{L}_\gamma$'s to play the role of the reaping sets.

The induction. Denote by θ_y the discrete space of size θ_y. We define a 1-1 function g from θ_y into a compact subset of $\beta X \setminus X$ such that

(1) $y \in \mathrm{cl}_{\beta X}\, g[A]$ if and only if $|A| = \theta_y$.
(2) If $A, B \in [\theta_y]^{<\theta_y}$ and $A \cap B = \emptyset$ then $\mathrm{cl}_{\beta X}\, g[A] \cap \mathrm{cl}_{\beta X}\, g[B] = \emptyset$.

By assumption, we have $\theta_y^{<\theta_y} = \theta_y$. List $\theta_y \cup \{(A, B) : A, B \in [\theta_y]^{<\theta_y}$ and $A \cap B = \emptyset\}$ as $\{T_\eta : \eta \in \theta_y\}$ in such a way that if $T_\eta = (A, B)$, then $\eta \geq \sup(A \cup B)$, and if $T_\eta \in \theta_y$, then $\eta \geq T_\eta$. For $\rho \in \theta_y$ let $D_\rho = \{\eta : T_\eta = (A, B)$ and $\rho \in A \cup B\} \cup \{\eta : \rho \in T_\eta\}$. Note that $D_\rho \subset [\rho, \theta_y)$.

For each $\rho \in \theta_y$ we define $\Phi_\rho : D_\rho \to 2$ and choose $g(\rho)$ to be any element of $K_\rho := \bigcap(\{H_\rho\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}_\gamma^{\Phi_\rho(\gamma)}) : \gamma \in D_\rho\})$. We define Φ_ρ by induction.

Let $\eta \in \theta_y$ and assume we have defined $\Phi_\rho|_{\eta \cap D_\rho}$. If $T_\eta \in \theta_y$, let $\Phi_\beta(\eta) = 0$ for all $\beta < T_\eta$. If $T_\eta = (A, B)$, let $\Phi_\beta(\eta) = 0$ for all $\beta \in A$ and let $\Phi_\beta(\eta) = 1$ for all $\beta \in B$. By Claims 6.1 and 6.2, $K_\rho \neq \emptyset$ for each $\rho \in \theta_y$, so we may choose $g(\rho) \in K_\rho$.

To show (1), let $A \subset \theta_y$ be such that $|A| < \theta_y$. There is $\gamma \in \theta_y$ with $A \subset [0, \gamma)$. Let η satisfy $T_\eta = \gamma$. Note that $\eta \geq \gamma$. For any $\rho < \gamma = T_\eta$, $\Phi_\rho(\eta) = 0$. So, for $\rho \in A$, $K_\rho \subset \mathcal{L}_\eta^0$. But $y \notin \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}_\eta^0)$. Hence, $y \notin \mathrm{cl}_{\beta X}\, g[A]$. For the other direction, let $A \subset \theta_y$ be such that $|A| = \theta_y$. Since θ_y is regular, A is unbounded in θ_y. Let $U \in \mathcal{N}$. There is $\gamma \in \theta_y$ such that $H_\gamma \subset U$. For $\rho \geq \gamma$, $g(\rho) \in H_\rho \subset H_\gamma \subset U$. Hence $y \in \mathrm{cl}_{\beta X}\, g[A]$.

To show (2), let $A, B \in [\theta_y]^{<\theta_y}$ be such that $A \cap B = \emptyset$. Let η be such that $T_\eta = (A, B)$. Then, for each $\rho \in A$, $\Phi_\rho(\eta) = 0$, and for each $\rho \in B$, $\Phi_\rho(\eta) = 1$.

Hence $g(\rho) \in K_\rho \subset \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\eta)$ for $\rho \in A$ and $g(\rho) \in K_\rho \subset \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^1_\eta)$ for $\rho \in B$. But $\mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^0_\eta) \cap \mathrm{cl}_{\beta X}(\bigcup \mathcal{L}^1_\eta) = \emptyset$. Hence $\mathrm{cl}_{\beta X}\, g[A] \cap \mathrm{cl}_{\beta X}\, g[B] = \emptyset$. Note (2) implies g is one-to-one.

Since θ_y is discrete, g is continuous. Extend g to $\beta g : \beta\theta_y \to \beta X \setminus X$. It follows from Bešlagić and van Douwen's [1, Lemma 2.2] that the image of βg is a closed subset of $(\beta X \setminus X) \setminus \{y\}$ which is homeomorphic to $\mathrm{NU}(\theta_y)$. ∎

Theorem 7.2. ($2^\kappa = \kappa^+$) *Let X be a metric space of weight κ that is either crowded locally compact or κ^ω-like. Any regular z-ultrafilter is a nonnormality point of $\beta X \setminus X$.*

Proof. Since y is regular, by Lemma 5.1, $\theta_y > \kappa$. By the hypothesis, $\theta_y = \kappa^+ = 2^\kappa$ and hence θ_y is regular and not a strong limit. By Lemma 3.1, $\mathrm{NU}(\theta_y)$ is not normal. Hence, by Theorem 7.1, y is a nonnormality point of $\beta X \setminus X$. ∎

Corollary 7.3. *Suppose GCH+UR. Let X be a crowded locally compact metric space. Then each $y \in \beta X \setminus X$ is a nonnormality point of $\beta X \setminus X$.*

Proof. We have seen that if $y \in \beta X \setminus X$ is uniform then it is a nonnormality point of $\beta X \setminus X$. Suppose that $y \in \beta X \setminus X$ is not uniform. That is, there exists $Z \in y$ for which $w(Z) < w(X)$. Let $Z \in y$ be such that $\lambda = w(Z)$ is minimum. Then y is a uniform z-ultrafilter on the set Z, and by UR, it is regular. However, it may be the case that Z has isolated points. We aim to find a crowded locally compact closed subset Y of X with weight λ such that $Z \subset Y$. There is a cover of Z consisting of sets $\mathrm{cl}\, B$ from a subcollection $\mathscr{Z}$ of $\mathcal{B}_0$ of size λ. Let $Y = \bigcup\{\mathrm{cl}\, B : B \in \mathscr{Z}\}$. Since $\mathcal{B}_0$ is locally finite, Y is closed. Each $B \in \mathscr{Z}$ is crowded and has compact closure, so Y is crowded locally compact.

So, $y \in \mathrm{cl}_{\beta X} Y$. Since X is normal and Y is closed, Y is C^*-embedded in X. Therefore, $\beta Y = \mathrm{cl}_{\beta X} Y$ and $y|_Y$ is uniform on Y. So, by the theorem, y is a nonnormality point of the set $(\mathrm{cl}_{\beta X} Y) \setminus Y$ and hence a nonnormality point of $\beta X \setminus X$. ∎

8. Questions. Gillman's question [6], which started research in this area, is still not completely answered.

Problem 8.1. *Let X be $\mathbb{N}$. Let y be any point of $\beta X \setminus X$. Without extra axioms of set theory, is $(\beta X \setminus X) \setminus \{y\}$ not normal? If yes, what if X is any discrete space? If yes, what if X is any metrizable space?*

There are many ways that our work can be extended. For example

Problem 8.2. *Assume GCH. For every crowded metrizable space X and every $y \in \beta X \setminus X$, is $(\beta X \setminus X) \setminus \{y\}$ not normal?*

Katětov (see [4, 5.5.10]) showed that if there is a nonrealcompact metrizable (more generally, paracompact) space, then there is a measurable cardinal. In other words, if there is a countably complete free z-ultrafilter on a metrizable (more generally, paracompact) space, then there is a countably complete free ultrafilter on a set. Is there an analogue for nonregular ultrafiters?

Problem 8.3. *If there is a nonregular ultrafilter on a metrizable* (*more generally, paracompact*) *space, is there a nonregular ultrafilter on a set?*

Problem 8.4. *What can be proved about θ_y and the normality of $(\beta X \setminus X) \setminus \{y\}$ when y is a nonregular z-ultrafilter?*

We do not know whether it is possible that θ_y is an uncountable weakly compact cardinal. It is possible that $\theta_y = \omega$. For example, let q be a κ-complete ultrafilter on a measurable cardinal κ. Let X be $\kappa \times \mathbb{R}$. Then X is crowded, locally compact, metrizable. (If a nowhere locally compact example is wanted, we can use $\mathbb{Q}$ in place of $\mathbb{R}$.) For $r \in \mathbb{R}$ let $e_r : \kappa \to X$ be defined by $e_r(\alpha) = (\alpha, r)$, and let $\beta e_r : \beta\kappa \to \beta X$ be the extension. Let y be $\beta e_0(q)$. Then $\theta_y = \omega$. In fact, $\{\beta e_{1/n}(q) : n \in \mathbb{N}\}$ is a sequence converging to y. We can show that $(\beta X \setminus X) \setminus \{y\}$ is not normal. Observe that neither Theorem 1.3 (X is not realcompact) nor Theorem 1.4 (y is nonregular) applies here.

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