Compositions of equi-dimensional fold maps

by

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Abstract. According to Ando's theorem, the oriented bordism group of fold maps of n-manifolds into n-space is isomorphic to the stable n-stem. Among such fold maps we define two geometric operations corresponding to the composition and to the Toda bracket in the stable stem through Ando's isomorphism. By using these operations we explicitly construct several fold maps with convenient properties, including a fold map which represents the generator of the stable 6-stem.

1. Introduction. A fold map, which is a smooth map between smooth manifolds with only fold singularities, can be considered as a simple extension of an immersion and also as a high-dimensional analogue of a Morse function. Many studies on fold maps have indicated that they are closely related to the geometry of manifolds (e.g. see [3, 4, 9, 16]). In this note we study equi-dimensional fold maps, rather in the light of their relation to algebraic topology.

A smooth map $f \colon N^n \to \mathbb{R}^n$ from an n-dimensional closed oriented manifold N^n into the n-dimensional Euclidean space is said to be a *fold map* if about each of its singular points it has the local form $f(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, \pm x_n^2)$ in suitable local coordinate systems in N^n and $\mathbb{R}^n$. We say that two such fold maps $f_i \colon N_i \to \mathbb{R}^n$ $(i = 0, 1)$ are *oriented bordant* if there exists a fold map F to $\mathbb{R}^n \times [0, 1]$ from an oriented cobordism (as a manifold) W^{n+1} between N_0 and N_1 such that $F|N_0 \times [0, \epsilon) = f_0 \times \mathrm{Id}_{[0,\epsilon)}$ and $F|N_1 \times (1 - \epsilon, 1] = f_1 \times \mathrm{Id}_{(-\epsilon,0]}$ (with ϵ being a small positive real number). This gives an equivalence relation on the set of all fold maps from closed oriented n-manifolds into $\mathbb{R}^n$, and the quotient space forms an abelian group called the *oriented fold bordism group*, which we denote by $\mathrm{SFold}(n, 0)$. Note that for more general singular maps the notion of the bordism group has

[119]

been introduced [11, 18] and intensively studied in recent years (see e.g.
[13, 17, 19]).

The fold bordism groups have been studied by many authors (see e.g.
[3–8, 14]). In particular, Ando [3, 5, 6] has proven that $\mathrm{SFold}(n, 0)$ is isomorphic to the stable homotopy group π_n^S of spheres. Under Ando's isomorphism, we introduce two geometric operations for (bordism classes of)
fold maps corresponding to the composition and to the Toda bracket in
the stable homotopy groups of spheres (in §3). In fact, Koschorke [10] has
formulated similar operations for the immersion bordism group $\mathrm{SI}(n, 1)$ of
immersions of oriented n-manifolds into $\mathbb{R}^{n+1}$, which is also isomorphic to
π_n^S [22]. Therefore in practice, we first establish in a geometric manner an
isomorphism between $\mathrm{SFold}(n, 0)$ and $\mathrm{SI}(n, 1)$ in §2, and then just interpret
Koschorke's composition and the Toda bracket through this isomorphism.
This attempt is natural and useful since, for codimension one immersions,
a kind of the Pontryagin–Thom construction gives a good understanding of
geometric counterparts to many algebraic operations in the stable stems (see
e. g. [1, 2] for recent studies). We detail many low-dimensional examples. As
an application, in §4 we describe a construction of a fold map $S^3 \times S^3 \to \mathbb{R}^6$
which represents the generator of the stable 6-stem $\pi_6^S \approx \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

2. An isomorphism between fold and immersion bordism groups.
Wells [22] studied the bordism groups of immersions and reduced the problem to the study of embeddings with appropriate vector fields, by lifting
immersions into a higher dimensional space. In particular, since a codimension one immersion $f\colon N^n \looparrowright \mathbb{R}^{n+1}$ of an oriented n-manifold N^n naturally
has a homotopically unique normal framing, by suspending and slightly perturbing it in a Euclidian space of sufficiently high dimension, we can obtain
a normally framed embedding. The isomorphism $\mathrm{SI}(n+1) \approx \pi_n^S$ is given by
applying the usual Pontryagin–Thom construction to the resulting normally
framed embedding.

2.1. The isomorphism $\mathfrak{m}$. In this section, we construct a natural isomorphism between the oriented fold bordism group $\mathrm{SFold}(n, 0)$ and the oriented immersion bordism group $\mathrm{SI}(n, 1)$, each of which we already know is
isomorphic to the stable homotopy group π_n^S of spheres.

Let $f\colon N^n \to \mathbb{R}^n$ be a fold map from an oriented n-manifold N^n to $\mathbb{R}^n$
($n \geq 1$). Then the *fold set* $S(f)$ of f is an $(n-1)$-dimensional orientable
submanifold of N^n and the restriction $f|S(f)$ is an immersion in $\mathbb{R}^n$ with
trivial normal line bundle (see e.g. [15, Lemma 2.2]). For each component
S_i of the fold set $S(f)$, we can take a tubular neighbourhood $S_i \times \mathbb{R} \subset N^n$
such that f immerses $S_i \times [0, \infty)$ orientation-preservingly into $\mathbb{R}^n$ and that
f immerses $S_i \times (-\infty, 0]$ orientation-reversingly into $\mathbb{R}^n$. This determines

an orientation of the normal bundle of $S_i \subset N^n$, which further induces an orientation of S_i from the given orientation of N^n. Thus, $S(f)$ becomes an oriented $(n-1)$-manifold.
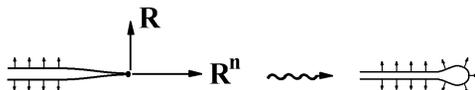

Fig. 1. Desingularisation of folds

Let $j \colon \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion and consider the composition $j \circ f \colon N^n \to \mathbb{R}^{n+1}$. Then $j \circ f$ is an immersion on $N^n \setminus S(f)$ and hence we can take a normal vector field ν on $N \setminus S(f)$ with respect to the orientations of N^n and of $\mathbb{R}^{n+1}$. The situation inside each fibre D_p^2 (at $p \in S(f)$) of the 2-dimensional normal disk bundle of $(j \circ f)(S(f))$ in $\mathbb{R}^{n+1}$, which is the trivial bundle, is as on the left of Figure 1 (the two curves in the left figure, depicted slightly away from each other, are in fact in $\mathbb{R}^n$, and the arrows attached to them indicate the normal vector field ν). Therefore, we can "desingularise" $j \circ f$ by modifying it inside each D_p^2 as in Figure 1. Since the normal bundle of $(j \circ f)(S(f))$ in $\mathbb{R}^{n+1}$ is trivial, this process can be done globally on each component of $(j \circ f)(S(f))$. Thus we obtain an immersion of N^n in $\mathbb{R}^{n+1}$, which we denote by $\bar{f}$. Furthermore, this gives rise to a homomorphism between the bordism groups,

$$\mathfrak{m} \colon \mathrm{SFold}(n,0) \to \mathrm{SI}(n,1), \quad [f] \mapsto [\bar{f}],$$

since we can perform the same operation for a fold bordism between two bordant fold maps, so that we can obtain an immersion bordism between the corresponding immersions.

EXAMPLE 2.1. The fold map $S^1 \to \mathbb{R}^1$ shown in Figure 2 (which we call the $\bowtie$-*fold map*) generates $\mathrm{SFold}(1,0) \approx \pi_1^S \approx \mathbb{Z}_2$. This is easily seen from [6, Theorem 1.3] and Figure 3, which depicts a stable map from D^2 to the half-plane $\mathbb{R}_+^2$ with one cusp point extending the fold map.
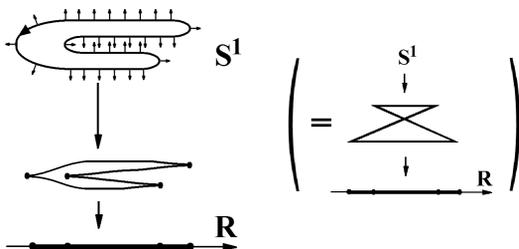

Fig. 2. The $\bowtie$-fold map generating $\mathrm{SFold}(1,0)$
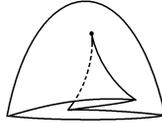
Fig. 3. An extension with a cusp of the ⋈-fold map

Then, Figure 4 describes the image of the ⋈-fold map shown in Figure 2 under the homomorphism $\mathfrak{m}\colon \mathrm{SFold}(1,0) \to \mathrm{SI}(1,1)$. In fact, the immersion with one crossing represents the generator of $\mathrm{SI}(1,1)$, since for a self-transverse immersion $S^1 \looparrowright \mathbb{R}^2$ the number modulo 2 of its double points gives the isomorphism $\mathrm{SI}(1,1) \to \mathbb{Z}_2$. Thus, we see that $\mathfrak{m}\colon \mathrm{SFold}(1,0) \to \mathrm{SI}(1,1)$ is an isomorphism.
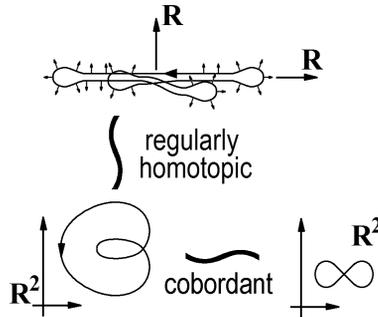
Fig. 4. The "figure 8" immersion generating $\mathrm{SI}(1,1)$

THEOREM 2.2. *The above homomorphism* $\mathfrak{m}\colon \mathrm{SFold}(n,0) \to \mathrm{SI}(n,1)$ *is an isomorphism for* $n \geq 1$.

Proof. It suffices to show that $\mathfrak{m}$ is surjective, since the groups on both sides are known to be isomorphic to π_n^S.

Let $F\colon N^n \looparrowright \mathbb{R}^{n+1}$ be an immersion of an oriented n-manifold N^n in $\mathbb{R}^{n+1}$. Then the regular homotopy class of F corresponds to the homotopy class of the induced stable framing of N^n. Therefore, due to Ando's h-principle [3, Corollary 2], we can deform F *by regular homotopy* into an immersion that becomes a fold map when followed by the projection $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$, $(y_1, \ldots, y_{n+1}) \mapsto (y_1, \ldots, y_n)$. Denote this resulting immersion by F'.

At each point x of the fold set $S(p \circ F')$, we can choose a normal vector $n(x)$ of $S(p \circ F') \subset N^n$ so that $dF_x(n(x))$ coincides with $(\partial/\partial y_{n+1})_{F'(x)}$. This defines a normal vector field n on each component of $S(p \circ F') \subset N^n$ since the normal bundle of $S(p \circ F') \subset N^n$ is orientable [15]. Thus we can choose an induced orientation σ_i of each component S_i of $S(p \circ F')$ such that (n, σ_i) agrees with the orientation of N^n.

Let ν be the normal vector field of $F(N^n) \subset \mathbb{R}^{n+1}$. Then $(dp(\nu), d(p \circ F')(\sigma_i))$ agrees with the orientation of $\mathbb{R}^n$ or with the opposite one, for each component S_i of $S(p \circ F')$. Denote by S_- the set of components of $S(p \circ F')$ on which $(dp(\nu), d(p \circ F')(\sigma_i))$ accords with the orientation of $\mathbb{R}^n$, and put $S_+ := S(p \circ F') \setminus S_-$. The left and right pictures of Figure 5 describe the situation of the normal (2-dimensional) disk at fold points of $F'(S_+) \subset \mathbb{R}^{n+1}$ and of $F'(S_-) \subset \mathbb{R}^{n+1}$, respectively.
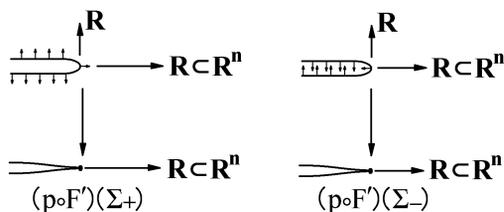

Fig. 5. S_+ and S_-

Now, we modify F' near S_- *by bordism*. Inside the 2-dimensional normal disk of $S_- \subset \mathbb{R}^{n+1}$ at a point $x \in S_-$, the modification is described as in Figure 6. This process changes F' by bordism and consequently we have an
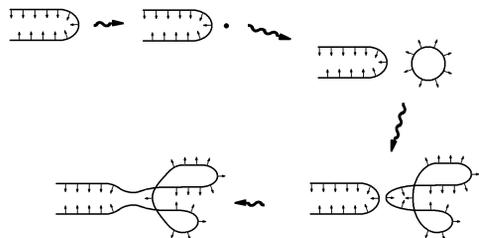

Fig. 6. A bordism around $S_- \subset \mathbb{R}^{n+1}$, in the 2-dimensional normal disk

immersion of $(S_- \times S^1) \sharp N^n$ into $\mathbb{R}^{n+1}$, bordant to F', which we denote by F''. Clearly, the composition $p \circ F''$ is a fold map into $\mathbb{R}^n$ and $\mathfrak{m}(p \circ F'')$ agrees with F'' bordant to F. ∎

REMARK 2.3. In the above proof, in order to obtain the inverse operation of $\mathfrak{m}$, we first deform F by regular homotopy into F' and then deform F' by bordism into F'' that has no S_- part. The regular homotopy alone is not enough here. This can also be seen from the following. If we regard the S^1 of the top of Figure 2 as an immersed (embedded) circle in $\mathbb{R}^2$, then the immersion belongs to the trivial regular homotopy class, but its projection represents the generator of SFold$(1,0)$ as a fold map. To obtain the correct inverse of the generator of SFold$(1,0)$, we need to eliminate the S_- part by bordism as shown in Figure 6.

3. Compositions of fold maps. In the case of codimension one immersions of closed oriented manifolds the isomorphism $\mathrm{SI}(n,1) \to \pi_n^S$ is given through the Pontryagin–Thom construction (see §2), which enables us to understand various algebraic operations geometrically. In Koschorke [10], for such codimension one immersions, the operations corresponding to the composition and to the Toda bracket in π_n^S are introduced. In this section, we interpret Koschorke's operations in terms of equi-dimensional fold maps via the isomorphism $\mathfrak{m} \colon \mathrm{SFold}(n,0) \to \mathrm{SI}(n,1)$, explained in the previous section.

3.1. The composition. Let $f \colon N^n \hookrightarrow \mathbb{R}^{n+1}$ be an immersion of a closed oriented manifold N^n. We can extend f to an immersion from the total space of its normal line bundle, diffeomorphic to $N^n \times \mathbb{R}$. Denote this extension by $f' \colon N^n \times \mathbb{R} \hookrightarrow \mathbb{R}^{n+1}$. For the symbol "$^-$" used below, see §2.1.

Let $\alpha \in \pi_a^S$ and $\beta \in \pi_b^S$. Let $i \colon A^a \to \mathbb{R}^a$ and $j \colon B^b \to \mathbb{R}^b$ be the respective fold maps. Suppose that $b \geq 1$. Then, in view of Koschorke [10, §1], we see that the fold map corresponding to the composition $\alpha \circ \beta \in \pi_{a+b}^S$ is defined to be

$$i * j \colon A^a \times B^b \xrightarrow{(\mathrm{Id}, j)} A^a \times \mathbb{R}^b = (A^a \times \mathbb{R}) \times \mathbb{R}^{b-1} \xrightarrow{((\bar{i})', \mathrm{Id})} \mathbb{R}^{a+1} \times \mathbb{R}^{b-1} = \mathbb{R}^{a+b}.$$

If $b = 0$, then $\beta \in \pi_0^S = \mathbb{Z}$ so j is represented by some integer s. Then we consider the fold map $i * j$ to be the union of s copies of i (each shifted in the last coordinate of $\mathbb{R}^a$ for convenience).

REMARK 3.1. We easily see from the above construction that the fold set $S(i * j)$ equals $A^a \times S(j) \subset A^a \times B^b$. Furthermore, the immersion $(i * j)|S(i * j)$ equals $\bar{i} \,\bar{*}\, (j|S(j)) \colon A^a \times S(j) \hookrightarrow \mathbb{R}^{a+b}$, where $\bar{*}$ stands for Koschorke's $*$-product [10] of codimension one immersions that represents the composition of the corresponding stable homotopy classes under the isomorphism $\mathrm{SI}(n,1) \to \pi_n^S$. Thus, we can see that $\overline{i * j}$ equals $\bar{i} \,\bar{*}\, \bar{j}$, from which we can easily deduce the associativity of the operation $*$ for fold maps.

EXAMPLE 3.2. (1) The fold map $T^2 \to \mathbb{R}^2$ in Figure 7, obtained by putting the $\bowtie$-fold map $S^1 \to \mathbb{R}^1$ (Figure 2) in each fibre of the normal line bundle of the "figure 8" immersion $S^1 \hookrightarrow \mathbb{R}^2$, represents the generator $\pi_2^S \approx \mathbb{Z}_2$ (cf. [12]), since $\eta \circ \eta$ generates $\pi_2^S \approx \mathbb{Z}_2$ (cf. [21, p. 189]).
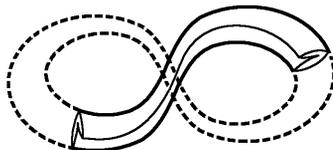

Fig. 7. A fold map $T^2 \to \mathbb{R}^2$ generating $\pi_2^S \approx \mathbb{Z}_2$

(2) The fold map $T^3 \to \mathbb{R}^3$ obtained by putting the $\bowtie$-fold map in each fibre of the normal line bundle of the "8 by 8" immersion $T^2 \looparrowright \mathbb{R}^3$ in Figure 8, represents $\eta \circ \eta \circ \eta$, which is known to be equal to 4ν for a generator ν of order 8 in $\pi_3^S \approx \mathbb{Z}_8 \oplus \mathbb{Z}_3$ [21, (5.5)].
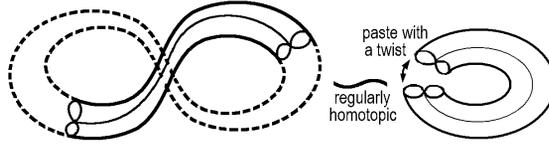
Fig. 8. The "8 by 8" immersion $T^2 \looparrowright \mathbb{R}^3$, that generates SI$(2,1)$

(3) We can repeat a similar construction in higher dimensions. However, the fold map $T^4 \to \mathbb{R}^4$ obtained by putting the $\bowtie$-fold map in each fibre of the normal line bundle of the "8 by 8 by 8" immersion $T^3 \looparrowright \mathbb{R}^4$ is null bordant, since $\eta \circ \eta \circ \eta \circ \eta \in \pi_4^S = 0$.

3.2. The Toda bracket. Let $\alpha \in \pi_a^S$, $\beta \in \pi_b^S$ and $\gamma \in \pi_c^S$. Let $i\colon A^a \to \mathbb{R}^a$, $j\colon B^b \to \mathbb{R}^b$ and $k\colon C^c \to \mathbb{R}^c$ be the respective fold maps. Suppose $\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$. Then, again in view of [10, §1], the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ is understood in terms of the fold maps i, j and k, as follows.

It follows from $\alpha \circ \beta = 0$ that $i * j\colon A^a \times B^b \to \mathbb{R}^{a+b}$ is null-bordant. Thus, we can take a null bordism, that is, a fold map from an $(a+b+1)$-dimensional manifold X^{a+b+1} with $\partial X^{a+b+1} = A^a \times B^b$,

$$\ell_+\colon X^{a+b+1} \to \mathbb{R}^{a+b} \times [0, \infty),$$

such that ℓ_+ coincides with $(i * j) \times \mathrm{Id}$ on a collar $\partial X^{a+b+1} \times [0, \epsilon) \subset X^{a+b+1}$. Similarly, as $\beta \circ \gamma = 0$ we can take a null bordism

$$\ell_-\colon Y^{b+c+1} \to \mathbb{R}^{b+c} \times (-\infty, 0]$$

of $j * k$. Thus, we have two null bordisms of $i * j * k$:

$$\ell_+ * k\colon X^{a+b+1} \times C^c \to \mathbb{R}^{a+b+c} \times [0, \infty)$$

and

$$i * \ell_-\colon A^a \times Y^{b+c+1} \to \mathbb{R}^{a+b+c} \times (-\infty, 0].$$

By pasting them along the common boundaries, we have a fold map from the closed manifold $(X^{a+b+1} \times C^c) \cup_\partial (A^a \times Y^{b+c+1})$ to $\mathbb{R}^{a+b+c+1}$. All fold maps constructed in this way form the Toda bracket $\langle \alpha, \beta, \gamma \rangle \subset \pi_{a+b+c+1}^S$.

EXAMPLE 3.3. Choose a a generator $\iota \in \pi_0^S \approx \mathbb{Z}$. Then the corresponding fold map is {one point} $\to \mathbb{R}^0$. We can check that the above construction for $\langle 2\iota, \eta, 2\iota \rangle$ provides the immersion of Figure 9, which is the same as the fold map (representing $\eta \circ \eta$) in Figure 7. Thus, we can show the relation $\langle 2\iota, \eta, 2\iota \rangle = \eta \circ \eta \in \pi_2^S$ [21, Corollary 3.7] by this example.
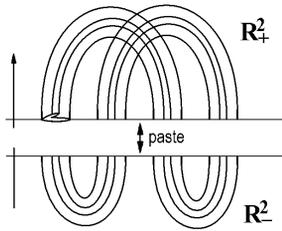
Fig. 9. $\langle 2\iota, \eta, 2\iota \rangle \colon T^2 \to \mathbb{R}^2$

4. A fold map which generates the stable 6-stem. Here, by using the composition in §3, we construct a fold map $S^3 \times S^3 \to \mathbb{R}^6$ which represents the generator of the stable 6-stem isomorphic to $\mathbb{Z}_2$.

First we need the following observation.

PROPOSITION 4.1. *Let $F\colon S^3 \looparrowright \mathbb{R}^6$ be a self-transverse immersion with an odd number of double points. Note that F has trivial normal bundle. Then an immersion $f\colon S^3 \looparrowright \mathbb{R}^4$ obtained by compressing F into $\mathbb{R}^4$ represents a generator of order 8 in $\mathrm{SI}(3,1) \approx \pi_3^S \approx \mathbb{Z}_8 \oplus \mathbb{Z}_3$.*

Proof. Assume that f represents an even element in $\mathrm{SI}(3,1) \approx \mathbb{Z}_{24}$. Then, by [20, Proposition 4.5], f is bordant to the compression of an embedding $S^3 \hookrightarrow \mathbb{R}^6$, which implies that F is bordant (as an immersion) to an embedding. This, however, is impossible since the parity of the number of double points of a self-transverse immersion $S^3 \looparrowright \mathbb{R}^6$ is invariant up to bordism. ∎

Let X be two copies of the 3-disk D^3 in 6-space intersecting each other in exactly one point, whose "boundary" consists of two copies of the 2-sphere. Consider two barycentric 3-spheres each of which is standardly embedded in $\mathbb{R}^6 = \mathbb{R}_1^3 \times \mathbb{R}_2^3$. Then, by removing from each 3-sphere a small 3-disk (centred at the point intersecting "the $\mathbb{R}_1^3$ axis", see Figure 10) and gluing X instead,
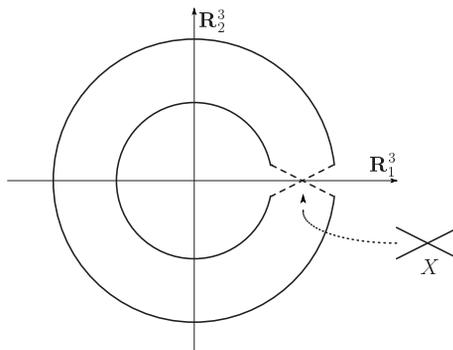

Fig. 10. The fold map $F\colon S^3 \looparrowright \mathbb{R}^6$ with one double point

(after suitably smoothing it) we obtain an immersion $F\colon S^3 \looparrowright \mathbb{R}^6$ with one double point as in Figure 10.

By Proposition 4.1, the composition of F and the projection $\mathbb{R}^6 \to \mathbb{R}^3$ in an appropriate direction (e.g. the projection $\mathbb{R}^6 = \mathbb{R}_1^3 \times \mathbb{R}_2^3 \to \mathbb{R}_1^3$ in Figure 10) becomes a fold map $f\colon S^3 \to \mathbb{R}^3$ which represents a generator ν of order 8 in $\mathrm{SFold}(3,0) \approx \mathbb{Z}_8 \oplus \mathbb{Z}_3$ (this is very similar to Example 2.1, see §2).

Thus, by mapping a copy of S^3 via f into each normal disk of the (trivial) normal bundle of the immersion $F\colon S^3 \looparrowright \mathbb{R}^6$, we obtain a fold map $G\colon S^3 \times S^3 \to \mathbb{R}^6$ which represents $\nu \circ \nu \in \pi_6^S$ (see §3.1). Since $\nu \circ \nu$ generates $\pi_6^S \approx \mathbb{Z}_2$ by [21, p. 189], we see that G represents a generator of π_6^S.

REMARK 4.2. Ando [6, §6] gives another explicit construction of a fold map $S^3 \times S^3 \to \mathbb{R}^6$ which represents the generator of π_6^S.

REMARK 4.3. Let $F'\colon S^3 \looparrowright \mathbb{R}^4$ be an immersion obtained by compressing the above F into $\mathbb{R}^4$ (see Proposition 4.1) and $j\colon \mathbb{R}^6 \to \mathbb{R}^7$ be the inclusion. If we immerse a copy of S^3 via F' into each normal 4-disk of the (trivial) normal bundle of the immersion $j \circ F\colon S^3 \looparrowright \mathbb{R}^7$, then we obtain an immersion $S^3 \times S^3 \looparrowright \mathbb{R}^7$ which represents the generator under $\mathrm{SI}(6,1) \approx \pi_6^S \approx \mathbb{Z}_2$.

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