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Affine Birman–Wenzl–Murakami algebras and tangles in the solid torus

by

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Dedicated to William G. Bade on his 80th birthday

Abstract. The affine Birman–Wenzl–Murakami algebras can be defined algebraically, via generators and relations, or geometrically as algebras of tangles in the solid torus, modulo Kauffman skein relations. We prove that the two versions are isomorphic, and we show that these algebras are free over any ground ring, with a basis similar to a well known basis of the affine Hecke algebra.

Contents

1. INTRODUCTION

The purpose of this paper is to establish an isomorphism between the affine Birman–Wenzl–Murakami algebras (defined by generators and relations) and the algebras of (n, n)-tangles in the solid torus, modulo Kauffman skein relations.

The Birman–Wenzl–Murakami (or BMW) algebras were conceived in [2] and [11] as an algebraic framework for the Kauffman link invariant of [8]. The definition of these algebras by generators and relations was motivated

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by certain canonical elements, and relations satisfied by these elements, in Kauffman tangle algebras in $D^2 \times I$, that is, algebras of framed (n, n)-tangles in $D^2 \times I$, modulo Kauffman skein relations. It is therefore not surprising, and is sometimes taken as evident, that the BMW algebras are isomorphic to the Kauffman tangle algebras. A proof of the isomorphism was given by Morton and Wassermann [10] in a paper from 1989 that unfortunately was never published. (A related result is given by Kauffman [8, Theorem 4.4]. See Remark 3.2 below.) Some aspects of the BMW algebras can be understood more clearly in the Kauffman tangle picture; for example, the existence of the Markov trace and conditional expectations is evident from the tangle picture, and an argument of Morton and Traczyk [9] gives the freeness of the Kauffman tangle algebras over an arbitrary ring.

The affine BMW algebras are related to the ordinary BMW algebras as the affine Hecke algebras of type A are related to the ordinary Hecke algebras of type A. The affine algebras have an "extra" generator x_1 which satisfies the braid relation $x_1 g_1 x_1 g_1 = g_1 x_1 g_1 x_1$ with the first "ordinary" braid generator g_1. The extra generator of the affine algebras can be imagined geometrically as a strand looping around the hole in $A \times I$, where A denotes the annulus $S^1 \times I$. The full set of relations for the affine BMW algebras (due to Häring-Oldenburg [5]) are modeled after relations which hold in the Kauffman tangle algebras in $A \times I$. So the affine BMW algebras "should" be isomorphic to the Kauffman tangle algebras in $A \times I$, assuming that a sufficient list of relations has been discovered. This isomorphism is our main result. On the way to proving the isomorphism, we also show that the affine BMW algebras, over any ring, are free with a basis generalizing a well known basis of the affine Hecke algebras.

As for the ordinary BMW algebras, certain properties of the affine BMW algebras are most easily obtained from the Kauffman tangle picture—in particular the existence of the Markov trace and conditional expectations. Moreover, we require the tangle picture even to describe our basis of the affine BMW algebras.

In proving our main theorems, we rely on the results of Morton and Wassermann for the ordinary BMW and Kauffman tangle algebras, as well as on techniques from their paper. Since their paper is not generally available, we have reproduced their results in Section 5 of the present paper.

The outline of the paper is as follows. In Sections 2 and 3 we introduce the ordinary and affine Birman–Wenzl–Murakami algebras and the Kauffman tangle algebra and derive basic properties of these algebras. In Section 4, we introduce an affine version of the Brauer centralizer algebra, which is a homomorphic image of the Kauffman tangle algebra in $A \times I$. Section 5 is devoted to the Morton–Wassermann proof of the isomorphism

of the ordinary BMW and Kauffman tangle algebras. Finally, in Section 6 we obtain the main results of the paper.

In a future paper, we intend to study the cyclotomic BMW algebras, which are quotients of the affine BMW algebras in which the affine generator x_1 satisfies a polynomial relation. Cyclotomic BMW algebras have previously appeared in [4], [5], [15].

Let us mention some additional antecedents for the algebras studied here. The affine and cyclotomic Brauer algebras, which we use in Section 4, were introduced by Häring-Oldenburg [5], and the cyclotomic case was studied in [18]. Formally similar algebras determined by weighted Brauer diagrams are used in the approach of Olshanski and Okounkov to the representation theory of the infinite symmetric group; see, for example, [14], [13]. Nazarov introduced a "degenerate" affine BMW algebra, under the name "degenerate affine Wenzl algebra", in [12]. The cyclotomic version of this algebra was recently studied in [1]. Ram and Orellana [15] have studied certain representations of the affine BMW algebras.

We would like to thank Hans Wenzl for finding a significant error in a previous version of this paper.

2. THE AFFINE KAUFFMAN TANGLE ALGEBRA

In this section we introduce the geometric versions of the ordinary and affine Birman–Wenzl–Murakami algebras.

2.1. Framed tangles. The objects considered in this subsection, namely framed tangles in $S \times I$, where S is an oriented surface, do not enter directly into the definition of the Kauffman tangle algebra and its affine counterpart, but they provide important motivation for the definition.

Let S be a smooth oriented surface (with boundary). Let I denote the unit interval $[0, 1]$. Choose once and for all a countable family of mutually disjoint oriented intervals $\{J_i : i \in \mathbb{N}\}$ in S.

DEFINITION 2.1. Fix integers $k, n \geq 0$. A *framed* (k, n)*-tangle* in $S \times I$ is a family of non-intersecting embedded piecewise smooth rectangles (ribbons) in $S \times I$ such that:

(1) with F denoting the union of the ribbons,

$$\begin{aligned} F \cap \partial(S \times I) &= F \cap (S \times \{0, 1\}) \\ &= J_1 \times \{0, 1\} \cup \cdots \cup J_k \times \{0, 1\}, \end{aligned}$$

(2) any ribbon in the family intersects the boundary of $S \times I$ transversally.

REMARK 2.2. Note that $k = 0$, or $n = 0$, or both, are allowed in the definition of tangles. If one of k or n is positive, then a (k, n)-tangle must be a non-empty family of ribbons, but the empty tangle, with no ribbons, is an allowable $(0, 0)$-tangle.

We will identify framed tangles which are isotopic via an isotopy fixing the intersections of the curves with the boundary of $S \times I$; so strictly speaking, a tangle is an isotopy class of families of ribbons.

One can compose (k, n)-tangles and (n, m)-tangles by "stacking". Namely, the first tangle (the (k, n)-tangle) is placed above the second tangle (the (n, m)-tangle), the n oriented intervals at which the ribbons intersect the lower boundary of the first tangle are identified with the n oriented intervals at which the ribbons intersect the upper boundary of the second tangle, and the resulting family of ribbons is compressed into $S \times I$.

Since composition of tangles is evidently associative, framed tangles in $S \times I$ may be regarded as morphisms in a certain category. Namely, the objects of the category are $\{0, 1, \dots\}$, and the morphisms from k to n are the (k, n)-tangles in $S \times I$. In particular, the set of (n, n)-tangles forms a monoid, whose identity is the tangle $J_1 \times I \cup \cdots \cup J_n \times I$.

We are interested here in two possibilities for S, namely the plane $\mathbb{R}^2$ and the annulus $A^2 = \{(x, y) \in \mathbb{R}^2 : 1/4 \le x^2 + y^2\}$. We refer to tangles in $\mathbb{R}^2 \times I$ as "ordinary" framed tangles and tangles in $A^2 \times I$ as "affine" framed tangles. For both cases we take our intervals J_i to lie on the positive x-axis and to be ordered by their order on the x-axis. For convenience, we can take $J_i = [i, i + 1/2] \times \{0\}$.

2.2. Tangle diagrams. Fix points a_i in $\mathbb{R}$, for $i \ge 0$, with $0 = a_0 < a_1 < a_2 \cdots$.

DEFINITION 2.3. An *ordinary* (k, n)*-tangle diagram* is a piece of a knot diagram in $R = \mathbb{R} \times I$ such that:

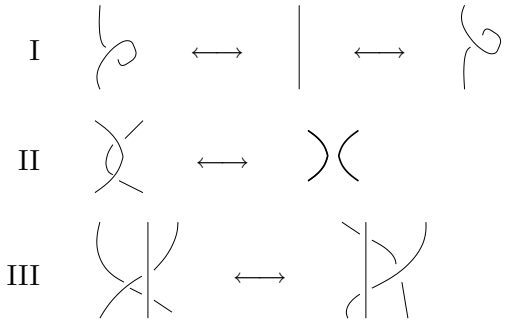
(1) with F denoting the union of the curves comprising the knot diagram,

$$\begin{aligned} F \cap \partial(R) &= F \cap (I \times \{0, 1\}) \\ &= \{(a_0, 1), \dots, (a_{k-1}, 1)\} \cup \{(a_0, 0), \dots, (a_{n-1}, 0)\}, \end{aligned}$$

(2) any curve in the family intersects the boundary of R transversally.

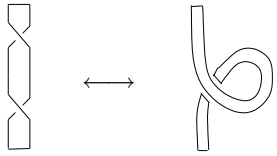
Recall that a knot diagram means a collection of piecewise smooth curves which may have intersections and self-intersections, but only simple transverse intersections. At each intersection or crossing, one of the two strands (pieces of curves) which intersect is indicated as crossing over the other.

Definition 2.4. Two ordinary tangle diagrams are said to be *ambient isotopic* if they are related by a sequence of *Reidemeister moves* of types I, II, and III, followed by an isotopy of R fixing the boundary. Two ordinary tangle diagrams are said to be *regularly isotopic* if they are related by a sequence of Reidemeister moves of types II and III only, followed by an isotopy of R fixing the boundary. See the following figure for the Reidemeister moves of types I, II, and III.



Tangle diagrams can be composed, similarly to tangles, and composition respects ambient isotopy or regular isotopy. Thus one obtains a product on ambient isotopy classes of tangle diagrams, respectively on regular isotopy classes of tangle diagrams.

As is well known, isotopy classes of ordinary framed tangles correspond bijectively to regular isotopy classes of ordinary tangle diagrams, by a framed version of Reidemeister's theorem. The correspondence is as follows: A representative of an isotopy class of framed tangles (in (x, y, z)-space) can always be chosen that lies close to and almost parallel with the xz plane; twists in ribbons are converted to "kinks" (or almost planar loops) in such a representative:

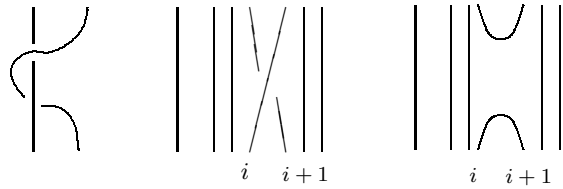


Replace each ribbon in such a representative by its "core", a curve running lengthwise along the center of the ribbon, and project the collection of these curves in the xz plane. The result is a tangle diagram. Any two such projections are regularly isotopic tangle diagrams. Thus one has a well defined map from isotopy classes of ordinary framed tangles to regular isotopy classes of tangle diagrams. This map is bijective.

It follows that the monoid of (isotopy classes of) ordinary (n, n)-framed tangles is isomorphic to the monoid of regular isotopy classes of (n, n)-tangle diagrams.

DEFINITION 2.5. An *affine* (k, n)*-tangle diagram* is an ordinary $(k+1, n+1)$-tangle diagram which includes a distinguished curve P connecting $(a_0, 1)$ and $(a_0, 0)$ such that the height coordinate (the second coordinate) varies monotonically along the curve P.

Note that an affine tangle diagram is always equivalent by an isotopy of R fixing the boundary to a diagram including the curve $\{a_0\} \times I$. We will draw affine tangle diagrams with the distinguished curve drawn as a thickened vertical segment. We refer to the distinguished curve as the "flagpole". The figure below shows some fundamental affine tangle diagrams from which more complex affine tangle diagrams can be built by composition.



Two affine (k, n)-tangle diagrams are ambient (respectively regularly) isotopic if they are ambient (resp. regularly) isotopic as ordinary $(k+1, n+1)$-tangle diagrams. Note that in an ambient isotopy, no Reidemeister move of type I is ever applied to the flagpole, as the flagpole is required to be represented by a monotonic path.

The set of affine tangle diagrams, regarded as a subset of ordinary tangle diagrams, is evidently closed under the composition of tangle diagrams. Thus one obtains a product on ambient isotopy classes of affine tangle diagrams, respectively on regular isotopy classes of affine tangle diagrams.

We can extend the correspondence between isotopy classes of ordinary framed tangles and regular isotopy classes of ordinary tangle diagrams to a correspondence between isotopy classes of affine framed tangles and regular isotopy classes of affine tangle diagrams, as follows. First for each affine (k, n)-framed tangle T in $A^2 \times I$, we can consider $T \cup (\{0\} \times I)$ in $\mathbb{R}^2 \times I$. The latter projects to an ordinary $(k+1, n+1)$-tangle with distinguished curve $P = \{0\} \times I$.

It follows that the monoid of (isotopy classes of) affine (n, n)-tangles is isomorphic to the monoid of regular isotopy classes of affine (n, n)-tangle diagrams.

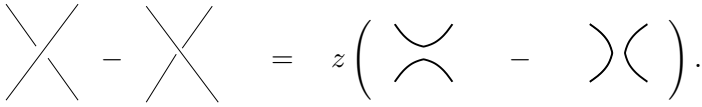
2.3. The ordinary Kauffman tangle algebras. Let $\mathcal{U}(k, n)$ denote the family of ordinary (k, n)-tangle diagrams modulo regular isotopy. Likewise, let $\widehat{\mathcal{U}}(k, n)$ denote the family of affine (k, n)-tangle diagrams modulo regular isotopy. Then $\mathcal{U}(n, n)$ and $\widehat{\mathcal{U}}(n, n)$ are monoids for each n.

DEFINITION 2.6. Let R be any (commutative, unital) ring with distinguished elements λ, z and δ, with λ and δ invertible, satisfying the relation

$$\lambda^{-1} - \lambda = z(\delta - 1).$$

The (ordinary) *Kauffman tangle algebra* $\mathrm{KT}_{n,R}$ over R is the monoid algebra $R\,\mathcal{U}(n,n)$ modulo the following relations:

(1) (Kauffman skein relation)



Here, the figures indicate tangle diagrams which differ only in the region shown and are identical outside this region.

(2) (Untwisting relation)

(3) (Free loop relation)

$$T \cup \bigcirc = \delta T,$$

where $T \cup \bigcirc$ is a tangle diagram consisting of the union of T and an additional closed loop having no crossing with T and no self-crossings.

The *generic Kauffman tangle algebra* (as considered by Morton and Traczyk [9] and Morton and Wassermann [10]) is the Kauffman tangle algebra $\mathrm{KT}_{n,\Lambda}$ over the ring

$$\Lambda = \mathbb{Z}[\boldsymbol{\lambda}^{\pm 1}, \boldsymbol{z}, \boldsymbol{\delta}^{\pm 1}]/\langle \boldsymbol{\lambda}^{-1} - \boldsymbol{\lambda} = \boldsymbol{z}(\boldsymbol{\delta} - 1)\rangle.$$

Here $\boldsymbol{\lambda}$, $\boldsymbol{z}$, $\boldsymbol{\delta}$ are indeterminates. For any ring R as above, there is a homomorphism from Λ to R determined by $\boldsymbol{\lambda} \mapsto \lambda$, $\boldsymbol{z} \mapsto z$, and $\boldsymbol{\delta} \mapsto \delta$. Thus the specialization $\mathrm{KT}_{n,\Lambda} \otimes_\Lambda R$ makes sense. It follows from the freeness of Kauffman tangle algebras [10] that for any R,

$$\mathrm{KT}_{n,R} \cong \mathrm{KT}_{n,\Lambda} \otimes_\Lambda R;$$

see Corollary 5.9 below. This is actually a special case of a general universal coefficient theorem for skein modules due to J. Przytycki; see [17, Lemma 5].

REMARK 2.7. If R is a ring with distinguished elements λ, z, and δ as above, and $S \supseteq R$ is a ring containing R, then the inclusion

$$R\,\mathcal{U}(n,n) \to S\,\mathcal{U}(n,n)$$

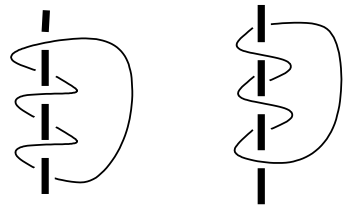
induces an R-algebra homomorphism

$$\mathrm{KT}_{n,R} \to \mathrm{KT}_{n,S}.$$

This map is always injective; see Corollary 5.10.

2.4. The affine Kauffman tangle algebras. Let R be a ring with distinguished elements λ, z and δ as above. Our *preliminary* definition of the affine Kauffman tangle algebra over R is the monoid algebra $R\,\widehat{\mathcal{U}}(n,n)$ modulo the Kauffman skein relation, the untwisting relation, and the free loop relation, as for the ordinary Kauffman tangle algebras. Here it is understood that none of the curves in the diagrams for these relations represent a part of the flagpole. Denote this algebra temporarily by $\widehat{K}_{n,R}$.

For $r \geq 1$, let Θ_r (resp. Θ_{-r}) denote the (regular isotopy class of) the closed curve with no self-crossings that winds r times around the flagpole in the positive sense (resp. in the negative sense).



Θ_3 $\qquad$ Θ_{-3}

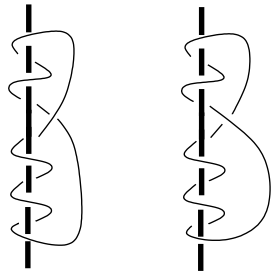
It is not difficult to see that the curves Θ_r generate $\widehat{K}_{0,R}$. A theorem of Turaev [19] says that $\{\Theta_r : r \geq 1\}$ is algebraically independent in $\widehat{K}_{0,\Lambda}$. Hence $\widehat{K}_{0,\Lambda}$ is the polynomial algebra over Λ in the infinitely many variables Θ_r for $r \geq 1$.

For any R, one has an algebra map from $\widehat{K}_{0,R}$ to $\widehat{K}_{0,\Lambda} \otimes_\Lambda R$; since the former is generated by $\{\Theta_r : r \geq 1\}$ and the latter is the polynomial algebra over R in the variables $\{\Theta_r : r \geq 1\}$, it follows that the map is an isomorphism. Thus, $\widehat{K}_{0,R}$ is the polynomial algebra over R in the variables Θ_r for $r \geq 1$.

We have $\Theta_1 = \Theta_{-1}$ in $\widehat{K}_{0,R}$, but Θ_r is not ambient isotopic to Θ_{-r} for $r \geq 2$. According to Turaev's theorem, Θ_{-r} is a polynomial in the variables Θ_k, $k \geq 1$,

$$\Theta_{-r} = f_r(\Theta_1, \Theta_2, \dots). \tag{2.1}$$

Later we will need a little more information about the polynomials f_r. For $a \geq 1$, $b \geq 0$, let $\Theta_{a,b}$ be the curve with a positive windings around the flagpole, and one positive crossing, and b negative windings, as in the first of the following figures. Let $\Theta^-_{a,b}$ be the curve with the crossing reversed.



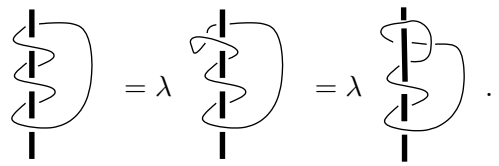
$\Theta_{2,3}$ $\Theta^{-}_{2,3}$

In particular, $\Theta_{r,0} = \lambda^{-1}\Theta_r$; we interpret Θ_0 as δ.

LEMMA 2.8. *Let $r, a, b \geq 1$.*

(1) $\Theta_{-r} = \lambda\Theta_{1,r-1}$.
(2) $\Theta_{a,b} = \lambda^2\Theta_{a+1,b-1} + z(\Theta_a\Theta_{-b} - \Theta_{a-b})$.
(3) $f_r(\Theta_1, \Theta_2, \ldots) = \lambda^{2r-2}\Theta_r + zf'_r(\Theta_1, \ldots, \Theta_{r-1})$, *where f'_r is a polynomial in $\Theta_1, \ldots, \Theta_{r-1}$.*

Proof. Point (1) follows from introducing a twist at the top of Θ_{-r}:



The tangle obtained by smoothing the crossing in $\Theta_{a,b}$ horizontally is $\Theta_a\Theta_{-b}$, and the tangle obtained by smoothing the crossing vertically is Θ_{a-b}, while $\Theta^{-}_{a,b} = \lambda^2\Theta_{a+1,b-1}$. Thus the Kauffman skein relation gives

$$\Theta_{a,b} = \lambda^2\Theta_{a+1,b-1} + z(\Theta_a\Theta_{-b} - \Theta_{a-b}). \tag{2.2}$$

An induction based on points (1) and (2) yields (3). ∎

We now return to the definition of the affine Kauffman tangle algebra. Because $\widehat{K}_{0,R}$ is a polynomial algebra over R and $\widehat{K}_{n,R}$ is a $\widehat{K}_{0,R}$ module, it makes sense to just absorb $\widehat{K}_{0,R}$ into the ground ring for the affine Kauffman tangle algebra, and this is what we do.

DEFINITION 2.9. Let S be a ring containing distinguished elements λ, z, δ, and q_r ($r \geq 1$), with λ and δ invertible, such that the relation $\lambda^{-1} - \lambda = z(\delta - 1)$ holds. The *affine Kauffman tangle algebra*

$$\widehat{\mathrm{KT}}_{n,S} = \widehat{\mathrm{KT}}_{n,S}(\lambda, z, \delta, q_1, q_2, \ldots)$$

is the monoid algebra $S\widehat{\mathcal{U}}(n,n)$ modulo the following relations:

(1) The Kauffman skein relation, the untwisting relations, and the free loop relation, as for the ordinary Kauffman tangle algebra.
(2) $T \cup \Theta_r = q_r T$, where $T \cup \Theta_r$ is the union of a tangle T and a copy of the curve Θ_r, such that there are no crossings between T and Θ_r.

REMARK 2.10. Note that $\widehat{\mathrm{KT}}_{0,S} \cong S$.

If R is a ring with distinguished elements λ, z, and δ as above, and $\boldsymbol{q}_r$, $r \geq 1$, are indeterminates, we denote by $\widehat{R}$ the polynomial ring $R[\boldsymbol{q}_1, \boldsymbol{q}_2, \dots]$.

REMARK 2.11. Let R be a ring with distinguished elements λ, z, and δ as above, and $S \supseteq R$ a ring with additional elements $q_1, q_2, \dots$. Adding a flagpole to ordinary (n,n)-tangle diagrams induces a homomorphism of R-algebras $i_n : \mathrm{KT}_{n,R} \to \widehat{\mathrm{KT}}_{n,S}$. This homomorphism is always injective. For the moment, we verify injectivity in two special cases.

(1) $i_n : \mathrm{KT}_{n,S} \to \widehat{\mathrm{KT}}_{n,S}$ is injective. In fact the image of i_n is the span of affine (n,n)-tangle diagrams having no intersection with the flagpole. We can define a map o_n of such diagrams to ordinary (n,n)-tangle diagrams by removing the flagpole. The map o_n induces an S-algebra homomorphism inverse to i_n; that is, $o_n \circ i_n$ is the identity on $\mathrm{KT}_{n,S}$.
(2) $i_n : \mathrm{KT}_{n,R} \to \widehat{\mathrm{KT}}_{n,\widehat{R}}$ is injective. In fact, removing the flagpole from affine (n,n)-tangle diagrams and mapping $\boldsymbol{q}_r \mapsto \delta$ determines a homomorphism of R-algebras $f_n : \widehat{\mathrm{KT}}_{n,\widehat{R}} \to \mathrm{KT}_{n,R}$, and $f_n \circ i_n$ is the identity on $\mathrm{KT}_{n,R}$.

Once we have verified that $\mathrm{KT}_{n,R}$ imbeds in $\mathrm{KT}_{n,S}$ (see Corollary 5.10), it will follow from (1) that $\mathrm{KT}_{n,R}$ imbeds in $\widehat{\mathrm{KT}}_{n,S}$ in general.

The *generic affine Kauffman tangle algebra* is the Kauffman tangle algebra

$$\widehat{\mathrm{KT}}_{n,\widehat{\Lambda}} = \widehat{\mathrm{KT}}_{n,\widehat{\Lambda}}(\boldsymbol{\lambda}, \boldsymbol{z}, \boldsymbol{\delta}, \boldsymbol{q}_1, \boldsymbol{q}_2, \dots)$$

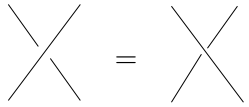
over the ring $\widehat{\Lambda} = \Lambda[\boldsymbol{q}_1, \boldsymbol{q}_2, \dots]$.

Since for any ring S as above, we have a homomorphism $\widehat{\Lambda}$ to S, determined by the assignments $\boldsymbol{\lambda} \mapsto \lambda$, $\boldsymbol{z} \mapsto z$, $\boldsymbol{\delta} \mapsto \delta$, and $\boldsymbol{q}_i \mapsto q_i$, the specialization $\widehat{\mathrm{KT}}_{n,\widehat{\Lambda}} \otimes_{\widehat{\Lambda}} S$ makes sense. We will eventually show that

$$\widehat{\mathrm{KT}}_{n,S} \cong \widehat{\mathrm{KT}}_{n,\widehat{\Lambda}} \otimes_{\widehat{\Lambda}} S;$$

see Corollary 6.14.

REMARK 2.12. We will use the following specialization below, in connection with an affine version of the Brauer algebra: Set $R = \mathbb{Z}[\delta^{\pm 1}]$. We have a homomorphism of $e : \Lambda \to R$ determined by $\lambda \mapsto 1$, $z \mapsto 0$, $\delta \mapsto \delta$. Then $\widehat{\mathrm{KT}}_{n,\widehat{R}}$ is the quotient of $\widehat{R}\,\widehat{\mathcal{U}}(n,n)$ by the relations:

(1) (Kauffman skein relation)  for a crossing of strands neither of which represents a part of the flagpole.

(2) (Untwisting relation)

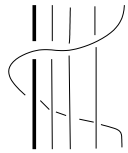
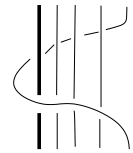
(3) (Removal of closed loops) $T \cup \bigcirc = \delta T$, where $T \cup \bigcirc$ is a tangle diagram consisting of the union of T and an additional closed loop having no crossings with the flagpole, and $T \cup \Theta_r = \boldsymbol{q}_r T$, for $r \geq 1$.

Since in this specialization, two tangle diagrams related by crossing changes are equivalent, for any $r \geq 1$ and for any closed curve c which loops r times around the flagpole, $c = \Theta_r = \boldsymbol{q}_r$.

2.5. The elements X_r. We define elements $X_r \in \widehat{\mathrm{KT}}_{n,S}$ for $1 \leq r \leq n$ by

$$X_r = (G_{r-1} \cdots G_2 G_1) X_1 (G_1 G_2 \cdots G_{r-1}).$$

For example,

$X_4 =$  and $X_4^{-1} =$ 

2.6. Symmetries. The map on ordinary (respectively, affine) tangle diagrams which flips diagrams top to bottom induces an anti-automorphism α of $\mathrm{KT}_{n,R}$ (respectively, of $\widehat{\mathrm{KT}}_{n,S}$). The anti-automorphism α fixes E_i, G_i, and X_1.

There is an isomorphism $\beta : \mathrm{KT}_{n,R}(\lambda, z, \delta) \to \mathrm{KT}_{n,R}(\lambda^{-1}, -z, \delta)$ determined by the map of ordinary tangle diagrams that reverses all crossings.

For $r \geq 1$, define $q_{-r} = f_r(q_1, \ldots, q_r)$, where f_r is the polynomial of equation (2.1). There is an isomorphism

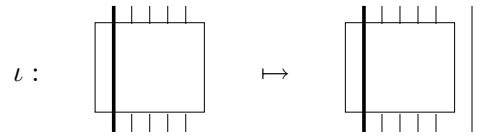
$$\beta : \widehat{\mathrm{KT}}_{n,S}(\lambda, z, \delta, q_1, q_2, \ldots) \to \widehat{\mathrm{KT}}_{n,S}(\lambda^{-1}, -z, \delta, q_{-1}, q_{-2}, \ldots)$$

determined by the map of affine tangle diagrams that reverses all crossings (also crossings of ordinary strands with the flagpole).

The ordinary Kauffman tangle algebra $\mathrm{KT}_{n,R}$ has an automorphism ϱ_n which flips ordinary tangle diagrams left to right. The automorphism ϱ_n satisfies $\varrho_n(E_i) = E_{n+1-i}$ and $\varrho_n(G_i) = G_{n+1-i}$.

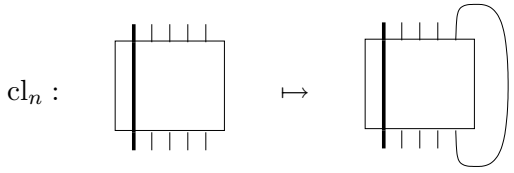
2.7. Inclusions, conditional expectations, and trace. The observations in this section apply equally to the ordinary and affine Kauffman tangle algebras. We fix a ring S containing distinguished elements λ, z, δ, and q_r $(r \geq 1)$ as above, and we let $\widehat{\mathrm{KT}}_n$ denote $\widehat{\mathrm{KT}}_{n,S}$.

For $n \geq 2$, the map ι from affine $(n-1, n-1)$-tangle diagrams to affine (n, n)-tangle diagrams that adds an additional strand on the right without adding any crossings:



respects regular isotopy, composition of tangle diagrams, and the relations of the affine Kauffman tangle algebras, so induces a homomorphism $\iota : \widehat{\mathrm{KT}}_{n-1} \to \widehat{\mathrm{KT}}_n$. Note that $\alpha \circ \iota = \iota \circ \alpha$, where α is the anti-automorphism of the affine Kauffman tangle algebras described in Section 2.6.

The map of affine (n, n)-tangle diagrams to affine $(n-1, n-1)$-tangle diagrams that "closes" the rightmost strand, without adding any crossings:



respects regular isotopy and the relations of the affine Kauffman tangle algebras, so induces an S-linear map $\widehat{\mathrm{KT}}_n \to \widehat{\mathrm{KT}}_{n-1}$. We define $\varepsilon_n : \widehat{\mathrm{KT}}_n \to \widehat{\mathrm{KT}}_{n-1}$ by

$$\varepsilon_n(T) = \delta^{-1}\mathrm{cl}_n(T).$$

Note that $\varepsilon_n \circ \alpha = \alpha \circ \varepsilon_n$, where α is the anti-automorphism of the affine Kauffman tangle algebras described above.

We have

$$\varepsilon_n \circ \iota(x) = x$$

for $x \in \widehat{\mathrm{KT}}_{n-1}$. In particular, the map ι of $\widehat{\mathrm{KT}}_{n-1}$ into $\widehat{\mathrm{KT}}_n$ is injective. Consequently, we will drop the notation ι and regard $\widehat{\mathrm{KT}}_{n-1}$ as a subalgebra of $\widehat{\mathrm{KT}}_n$.

More generally,

$$\varepsilon_n(xy) = x\varepsilon_n(y), \qquad \varepsilon_n(yx) = \varepsilon_n(y)x$$

for $x \in \widehat{\mathrm{KT}}_{n-1}$ and $y \in \widehat{\mathrm{KT}}_n$; that is, ε_n is a $\widehat{\mathrm{KT}}_{n-1}$-$\widehat{\mathrm{KT}}_{n-1}$-bimodule map, or *conditional expectation.*

REMARK 2.13. One has the formula

$$E_n x E_n = \mathrm{cl}_n(x) E_n$$

for x an (affine) (n,n)-tangle diagram. Equivalently, the idempotent E_n/δ implements the conditional expectation:

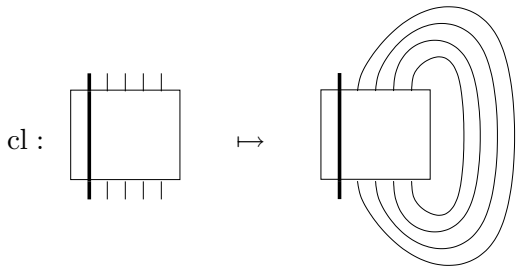
$$(E_n/\delta)x(E_n/\delta) = \varepsilon_n(x)(E_n/\delta)$$

for $x \in \widehat{\mathrm{KT}}_n$.

We define $\varepsilon : \widehat{\mathrm{KT}}_n \to \widehat{\mathrm{KT}}_0 \cong S$ by

$$\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_n.$$

Equivalently, if we define the closure of an affine (n,n)-tangle diagram to be the affine $(0,0)$-diagram obtained by closing all strands without introducing any new crossings:



then $\varepsilon(x) = \delta^{-n}\mathrm{cl}(x)$. Note that

$$\varepsilon \circ \iota = \varepsilon, \qquad \varepsilon \circ \varepsilon_n = \varepsilon.$$

We will show below that $\widehat{\mathrm{KT}}_n$ is generated as a unital algebra by $X_1^{\pm 1}$, $G_i^{\pm 1}$ and E_i $(1 \le i \le n-1)$. In particular $\widehat{\mathrm{KT}}_1$ is generated by $X_1^{\pm 1}$, so is commutative.

For $n \ge 2$, let $\varepsilon_{n,1}$ denote the map

$$\varepsilon_{n,1} = \varepsilon_2 \circ \cdots \circ \varepsilon_n : \widehat{\mathrm{KT}}_n \to \widehat{\mathrm{KT}}_1.$$

Then $\varepsilon_{n,1}$ is a conditional expectation, so for $x \in \widehat{\mathrm{KT}}_n$,

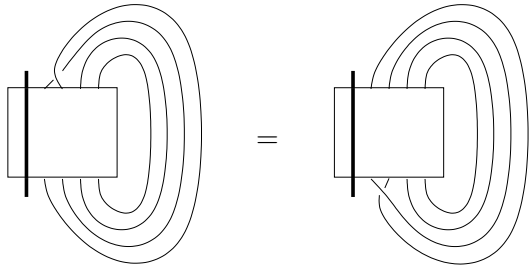
$$\varepsilon_{n,1}(X_1^{\pm 1}x) = X_1^{\pm 1}\varepsilon_{n,1}(x) = \varepsilon_{n,1}(x)X_1^{\pm 1} = \varepsilon_{n,1}(xX_1^{\pm 1}).$$

Since $\varepsilon = \varepsilon_1 \circ \varepsilon_{n,1}$, we have

$$\varepsilon(X_1^{\pm 1}x) = \varepsilon(xX_1^{\pm 1}).$$

For any affine (n,n)-tangle diagram T and any $r \le n-1$, the closure of the affine tangle diagram $E_r T$ is isotopic to the closure of the affine tangle

diagram TE_r, and similarly for E_r replaced by $G_r^{\pm 1}$. The following figure illustrates this for G_r^{-1}:



It follows that for $x \in \widehat{\mathrm{KT}}_n$ and $r \leq n-1$,

$$\varepsilon(E_r x) = \varepsilon(xE_r), \qquad \varepsilon(G_r^{\pm 1} x) = \varepsilon(xG_r^{\pm 1})$$

PROPOSITION 2.14. *$\varepsilon : \widehat{\mathrm{KT}}_n \to \widehat{\mathrm{KT}}_0 \cong S$ is a trace. That is, ε is S-linear and $\varepsilon(xy) = \varepsilon(yx)$ for all $x, y \in \widehat{\mathrm{KT}}_n$.*

Proof. We have shown that the trace property

$$\varepsilon(xy) = \varepsilon(yx)$$

holds when y is arbitrary and x is one of $X_1^{\pm 1}$, $G_i^{\pm 1}$ and E_i $(1 \leq i \leq n-1)$. It follows that the trace property also holds when x is any product of the elements $X_1^{\pm 1}$, $G_i^{\pm 1}$ and E_i. Since these elements generate $\widehat{\mathrm{KT}}_n$ as a unital algebra, the property holds for all x. ∎

REMARK 2.15. One easily checks by picture proofs that $\varepsilon_{r+1}(TG_r^{\pm 1}) = (\lambda^{\pm 1}/\delta)T$, and hence

$$\varepsilon(TG_r^{\pm 1}) = \varepsilon(T)\lambda^{\pm 1}/\delta$$

for $T \in \widehat{\mathrm{KT}}_r$. Likewise, $\varepsilon_{r+1}(TE_r) = (1/\delta)T$, and hence

$$\varepsilon(TE_r) = \varepsilon(T)/\delta$$

for $T \in \widehat{\mathrm{KT}}_r$. Moreover, if X_r' denotes

$$X_r' = G_{r-1} \cdots G_1 X_1 G_1^{-1} \cdots G_{r-1}^{-1},$$

then for $s \geq 1$ and $T \in \widehat{\mathrm{KT}}_r$, one has $\varepsilon_{r+1}(T(X_r')^s) = q_s T$. Consequently,

$$\varepsilon(T(X_r')^s) = q_s \varepsilon(T).$$

It follows from Lemma 2.8 that

$$\varepsilon(T(X_r')^{-s}) = f_s(q_1, \ldots, q_s)\varepsilon(T).$$

A trace with these properties is usually called a *Markov trace*. The terminology originated in [6].

3. THE AFFINE BIRMAN–WENZL–MURAKAMI ALGEBRA

3.1. The Birman–Wenzl–Murakami algebra. The Birman–Wenzl–Murakami algebras were introduced independently by Birman and Wenzl [2] and by Murakami [11] as an algebraic setting for the Kauffman link invariant [8]. These algebras are known to appear as centralizer algebras for the quantum universal enveloping algebras of $\mathrm{sp}(2n, \mathbb{C})$ or $\mathrm{so}(n, \mathbb{C})$ acting on tensor powers of the vector representation. They are deformations of Brauer's centralizer algebras (see [20]), and are extensions of the Hecke algebras of type A, as will be explained below.

The presentation which we give here follows Morton and Wassermann [10]. The parameters differ slightly from those used by Birman and Wenzl.

As before, let R be a commutative unital ring with invertible elements λ and δ and an element z satisfying

$$\lambda^{-1} - \lambda = z(\delta - 1).$$

Definition 3.1. The *Birman–Wenzl–Murakami* algebra $W_{n,R}$ is the R-algebra with generators $g_i^{\pm 1}$ and e_i $(1 \le i \le n-1)$ and relations:

(1) (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$.
(2) (Idempotent relation) $e_i^2 = \delta e_i$.
(3) (Braid relations) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i-j| \ge 2$.
(4) (Commutation relations) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i-j| \ge 2$.
(5) (Tangle relations) $e_i e_{i\pm1} e_i = e_i$, $g_i g_{i\pm1} e_i = e_{i\pm1} e_i$, and $e_i g_{i\pm1} g_i = e_i e_{i\pm1}$.
(6) (Kauffman skein relation) $g_i - g_i^{-1} = z(e_i - 1)$.
(7) (Untwisting relations) $g_i e_i = e_i g_i = \lambda^{-1} e_i$, and $e_i g_{i\pm1} e_i = \lambda e_i$.

Remark 3.2.

(1) If z is taken to be invertible, then

$$e_i = 1 + z^{-1}(g_i^{-1} - g_i).$$

In this case, several of the relations are redundant. Moreover, the algebra is a quotient of the braid group algebra.

(2) In the parametrization of [21], z is replaced by $q - q^{-1}$, where q is another indeterminate, and $q - q^{-1}$ is assumed to be invertible. The Kauffman skein relation then becomes a cubic relation for the braid generators g_i.

(3) The main result of [10] is that the Birman–Wenzl–Murakami algebra $W_{n,\Lambda}$ defined over Λ is isomorphic to the Kauffman tangle algebra $\mathrm{KT}_{n,\Lambda}$.

In [8], Theorem 4.4, Kauffman gives a presentation for the algebra of tangles generated by the E_i's, G_i's, and free loops. It is not clear to us how this result is related to the Morton–Wassermann theorem.

Remark 3.3.

(1) The assignment $e_i \mapsto e_i$, $g_i \mapsto g_i$ defines a homomorphism ι from $W_{n,R}$ to $W_{n+1,R}$, since the relations are preserved. It is not evident that ι is injective, but this will follow eventually from the isomorphism of $W_{n,R}$ with $\mathrm{KT}_{n,R}$; see Theorem 5.41.

(2) Define $S(x) : W_{n,R} \to W_{n+1,R}$ by

$$S(x) = \mathrm{Ad}(g_n g_{n-1} \cdots g_1)(x).$$

Then S is an injective homomorphism. It follows from the relations that $S(e_i) = e_{i+1}$ and $S(g_i) = g_{i+1}$ for $i \leq n-1$. The map S is called the *shift endomorphism*.

(3) The assignment $e_i \mapsto e_{n-i}$, $g_i \mapsto g_{n-i}$ determines an automorphism ϱ_n of $W_{n,R}$, as the relations are preserved by this assignment. The shift map $S : W_{n,R} \to W_{n+1,R}$ satisfies $S = \varrho_{n+1} \circ \varrho_n$.

(4) The assignment $g_i \mapsto g_i$ and $e_i \mapsto e_i$ determines an anti-automorphism α of $W_{n,R}$.

(5) The assignment $g_i \mapsto g_i^{-1}$, $e_i \mapsto e_i$ determines an isomorphism β from $W_{n,R}(\lambda, z, \delta)$ to $W_{n,R}(\lambda^{-1}, -z, \delta)$.

We have the following relation of the BMW algebras to the Hecke algebras of type A:

Proposition 3.4. *Let $J_{n,R}$ denote the ideal $W_{n,R}e_{n-1}W_{n,R}$ in $W_{n,R}$.*

(1) *$J_{n,R}$ is the ideal in $W_{n,R}$ generated by $\{e_1, \ldots, e_{n-1}\}$.*

(2) *$W_{n,R}/J_{n,R}$ is isomorphic to the Hecke algebra over Λ with generators $g_i^{\pm 1}$ $(1 \leq i \leq n-1)$ satisfying the braid relations and the quadratic relation $g_i^2 = 1 - zg_i$.*

Proof. For (1), it follows from the relation $e_j e_{j+1} e_j = e_j$ that all the e_i's are elements of $W_n e_{n-1} W_n$. Statement (2) is evident, since the e_i's are zero in the quotient. ∎

Remark 3.5. For $\omega \in \mathfrak{S}_n$, let $\omega = s_{i_1} \cdots s_{i_r}$ be a reduced expression for ω (in terms of the generators s_i of the symmetric group). Let g_ω be the corresponding word in the generators g_i of $W_{n,R}$. It follows from the proposition and well known facts concerning the Hecke algebra that g_ω is well defined modulo $J_{n,R}$. In fact, g_ω is well defined in $W_{n,R}$, not merely well defined modulo the ideal $J_{n,R}$; see Proposition 5.14.

Recall that Λ denotes the ring

$$\Lambda = \mathbb{Z}[\boldsymbol{\lambda}^{\pm 1}, \boldsymbol{z}, \boldsymbol{\delta}^{\pm 1}]/\langle \boldsymbol{\lambda}^{-1} - \boldsymbol{\lambda} = \boldsymbol{z}(\boldsymbol{\delta} - 1)\rangle,$$

where $\boldsymbol{\lambda}$, $\boldsymbol{z}$, $\boldsymbol{\delta}$ are indeterminates. The *generic Birman–Wenzl–Murakami algebra* is the BMW algebra $W_{n,\Lambda}$ over the ring Λ. Given a ring R as above, we have two possible specializations to an algebra over R, namely $W_{n,R}$

and $W_{n,\Lambda} \otimes_\Lambda R$. We will show later that these are in fact isomorphic; see Corollary 5.43.

3.2. The affine Birman–Wenzl–Murakami algebra. Let S be a ring with distinguished elements λ, z, and δ as above, and with additional elements $q_1, q_2, \ldots$.

DEFINITION 3.6. The *affine Birman–Wenzl–Murakami* algebra $\widehat{W}_{n,S}$ is the S-algebra with generators $x_1^{\pm 1}$, $g_i^{\pm 1}$ and e_i $(1 \le i \le n-1)$ and relations:

(1) (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$ and $x_1 x_1^{-1} = x_1^{-1} x_1 = 1$.
(2) (Idempotent relation) $e_i^2 = \delta e_i$.
(3) (Affine braid relations)
  (a) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i-j| \ge 2$.
  (b) $x_1 g_1 x_1 g_1 = g_1 x_1 g_1 x_1$ and $x_1 g_j = g_j x_1$ if $j \ge 2$.
(4) (Commutation relations)
  (a) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i-j| \ge 2$.
  (b) $x_1 e_j = e_j x_1$ if $j \ge 2$.
(5) (Affine tangle relations)
  (a) $e_i e_{i\pm 1} e_i = e_i$.
  (b) $g_i g_{i\pm 1} e_i = e_{i\pm 1} e_i$ and $e_i g_{i\pm 1} g_i = e_i e_{i\pm 1}$.
  (c) For $r \ge 1$, $e_1 x_1^r e_1 = q_r e_1$.
(6) (Kauffman skein relation) $g_i - g_i^{-1} = z(e_i - 1)$.
(7) (Untwisting relations) $g_i e_i = e_i g_i = \lambda^{-1} e_i$ and $e_i g_{i\pm 1} e_i = \lambda e_i$.
(8) (Unwrapping relations) $e_1 x_1 g_1 x_1 = \lambda^{-1} e_1 = x_1 g_1 x_1 e_1$.

REMARK 3.7. Let R be a ring with distinguished elements λ, z, and δ as above, and $S \supseteq R$ a ring with additional elements $q_1, q_2, \ldots$.

(1) The assignment $e_i \mapsto e_i$, $g_i \mapsto g_i$, $x_1 \mapsto x_1$ defines a homomorphism
$$\iota : \widehat{W}_{n,S} \to \widehat{W}_{n+1,S},$$
since the relations are preserved. It is not evident that ι is injective, but this will follow from the isomorphism $\widehat{W}_{n,S} \cong \widehat{\mathrm{KT}}_{n,S}$; see Corollary 6.15.
(2) The assignment $e_i \mapsto e_i$, $g_i \mapsto g_i$ defines an R-algebra homomorphism
$$i_n : W_{n,R} \to \widehat{W}_{n,S}.$$
This map is always injective, as we will verify later. (For the moment we verify that $i_n : W_{n,R} \to \widehat{W}_{n,\widehat{R}}$ is injective. In fact, the assignment $e_i \mapsto e_i, g_i \mapsto g_i, x_1 \mapsto 1, \boldsymbol{q}_i \mapsto \delta$ defines an R-algebra homomorphism from $f_n : \widehat{W}_{n,\widehat{R}} \to W_{n,R}$, and $f_n \circ i_n$ is the identity on $W_{n,R}$.)

Note that the following diagram commutes:

$$\begin{array}{ccc} W_{n,R} & \xrightarrow{i_n} & \widehat{W}_{n,S} \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ W_{n+1,R} & \xrightarrow{i_{n+1}} & \widehat{W}_{n+1,S} \end{array}$$

(3) The assignment $g_i \mapsto g_i$, $e_i \mapsto e_i$, $x_1 \mapsto x_1$ determines an anti-automorphism α of $\widehat{W}_{n,S}$. One has $\alpha \circ i_n = i_n \circ \alpha$, where $i_n : W_{n,R} \to \widehat{W}_{n,S}$ and α also denotes the anti-automorphism of $W_{n,R}$ of Remark 3.3.

(4) For $r \geq 1$, let $q_{-r} = f_r(q_1, \ldots, q_r)$, where f_r is the polynomial of equation (2.1). The assignment $g_i \mapsto g_i^{-1}$, $e_i \mapsto e_i$, $x_1 \mapsto x_1^{-1}$ determines an isomorphism

$$\beta : \widehat{W}_{n,S}(\lambda, z, \delta, , q_1, q_2, \ldots) \to \widehat{W}_{n,S}(\lambda^{-1}, -z, \delta, q_{-1}, q_{-2}, \ldots).$$

The following diagram commutes:

$$\begin{array}{ccc} W_{n,R}(\lambda, z, \delta) & \xrightarrow{i_n} & \widehat{W}_{n,S}(\lambda, z, \delta, q_1, q_2, \ldots) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\ W_{n,R}(\lambda^{-1}, -z, \delta) & \xrightarrow{i_n} & \widehat{W}_{n,S}(\lambda^{-1}, -z, \delta, q_{-1}, q_{-2}, \ldots) \end{array}$$

LEMMA 3.8. *The Kauffman skein relation implies*

$$g_i^2 = 1 + \lambda^{-1} z e_i - z g_i, \qquad g_i^{-2} = 1 - \lambda z e_i + z g_i.$$

Proof. Multiply by g_i or g_i^{-1} and simplify using the untwisting relation. ∎

LEMMA 3.9.

(1) $g_i^{\pm 1}(g_{i+1} g_i) = (g_{i+1} g_i) g_{i+1}^{\pm 1}$.
(2) $e_i(g_{i+1} g_i) = (g_{i+1} g_i) e_{i+1}$.
(3) $g_i g_{i+1} e_i e_{i+2} = g_{i+2} g_{i+1} e_i e_{i+2}$.
(4) *If* $i < m$, *then*

$$g_i^{\pm 1}(g_m g_{m-1} \cdots g_1) = (g_m g_{m-1} \cdots g_1) g_{i+1}^{\pm 1},$$
$$g_{i+1}^{\pm 1}(g_1 g_2 \cdots g_m) = (g_1 g_2 \cdots g_m) g_i^{\pm 1}.$$

(5) *If* $i < m$, *then*

$$e_i(g_m g_{m-1} \cdots g_1) = (g_m g_{m-1} \cdots g_1) e_{i+1},$$
$$e_{i+1}(g_1 g_2 \cdots g_m) = (g_1 g_2 \cdots g_m) e_i.$$

Proof. The first statement is just the braid relation. The second results from two applications of the tangle relation (5b) from Definition 3.6:

$$e_i(g_{i+1}g_i) = e_ie_{i+1} = (g_{i+1}g_i)e_{i+1}.$$

Statement (3) also results from two applications of the tangle relation (5b) from Definition 3.6:

$$g_ig_{i+1}e_ie_{i+2} = e_{i+1}e_ie_{i+2} = e_{i+1}e_{i+2}e_i = g_{i+2}g_{i+1}e_{i+2}e_i.$$

The first parts of statements (4) and (5) follow from statements (1) and (2) and the commutation relations (3a) and (4a) of Definition 3.6. The second parts of statements (4) and (5) follow by applying the anti-automorphism α. ∎

3.3. The elements x_i. For $2 \le r \le n$ define

$$x_r = (g_{r-1}\cdots g_1)x_1(g_1\cdots g_{r-1}).$$

Proposition 3.10.

(1) *For all r and $j \notin \{r, r-1\}$, $g_jx_r = x_rg_j$.*
(2) *For all r and $j \notin \{r, r-1\}$, $e_jx_r = x_re_j$.*
(3) *For all r and j, $x_jx_r = x_rx_j$.*

Proof. For $j < r-1$, relations (1) and (2) follow by applying Lemma 3.9(4, 5) and Definition 3.6(3b, 4b).

For $j > r$, relations (1) and (2) follow from Definition 3.6(3, 4).

For part (3), we first observe that x_1 commutes with x_r for $r \ge 2$, by the braid relations Definition 3.6(3b). Namely,

$$\begin{aligned} x_1x_r &= x_1(g_{r-1}\cdots g_2g_1)x_1(g_1g_2\cdots g_{r-1}) = (g_{r-1}\cdots g_2)x_1g_1x_1g_1(g_2\cdots g_{r-1}) \\ &= (g_{r-1}\cdots g_2)g_1x_1g_1x_1(g_2\cdots g_{r-1}) = (g_{r-1}\cdots g_2g_1)x_1(g_1g_2\cdots g_{r-1})x_1 \\ &= x_rx_1. \end{aligned}$$

Finally, if $j < r$, then

$$x_jx_r = (g_{j-1}\cdots g_1)x_1(g_1\cdots g_{j-1})x_r = x_r(g_{j-1}\cdots g_1)x_1(g_1\cdots g_{j-1}) = x_rx_j,$$

since x_r commutes with g_i for $i \le j-1$ and with x_1. ∎

Proposition 3.11.

(1) *For $r < n$, $g_rx_r = x_{r+1}g_r^{-1}$ and $g_r^{-1}x_{r+1} = x_rg_r$.*
(2) *For $r < n$,*

$$\begin{aligned} g_rx_{r+1} &= x_rg_r - zx_{r+1} + z\lambda e_rx_{r+1}, \\ g_r^{-1}x_r &= x_{r+1}g_r^{-1} + zx_r - ze_rx_r. \end{aligned}$$

(3) *For $r < n$, $e_rx_r = \lambda^{-2}e_rx_{r+1}^{-1}$ and $x_re_r = \lambda^{-2}x_{r+1}^{-1}e_r$.*
(4) *For $r < n$, $e_rx_r^{-1} = \lambda^2e_rx_{r+1}$ and $x_r^{-1}e_r = \lambda^2x_{r+1}e_r$.*

Proof. Statement (1) follows from the definition of x_r. Statement (2) uses the Kauffman skein relation, the quadratic relation of Lemma 3.8, and part (1). Statement (3) is equivalent to

$$e_r x_r x_{r+1} = \lambda^{-2} e_r.$$

For $r = 1$, this follows from Definition 3.6(7, 8). The proof is completed by induction on r. For $r \geq 2$, one has (using braid and tangle relations)

$$\begin{aligned}
e_r x_r x_{r+1} &= e_r(g_{r-1}\cdots g_1)x_1(g_1\cdots g_{r-2}(g_{r-1}g_r g_{r-1})g_{r-2}\cdots g_1)x_1(g_1\cdots g_r)\\
&= e_r(g_{r-1}\cdots g_1)x_1(g_1\cdots g_{r-2}(g_r g_{r-1} g_r)g_{r-2}\cdots g_1)x_1(g_1\cdots g_r)\\
&= (e_r g_{r-1} g_r)(g_{r-2}\cdots g_1)x_1(g_1\cdots g_{r-2}g_{r-1}g_{r-2}\cdots g_1)\\
&\quad \cdot x_1(g_1\cdots g_{r-2})(g_r g_{r-1} g_r)\\
&= (e_r e_{r-1})(g_{r-2}\cdots g_1)x_1(g_1\cdots g_{r-2}g_{r-1}g_{r-2}\cdots g_1)\\
&\quad \cdot x_1(g_1\cdots g_{r-2})(g_{r-1}g_r g_{r-1})\\
&= e_r e_{r-1} x_{r-1} x_r g_r g_{r-1}\\
&= \lambda^{-2} e_r e_{r-1} g_r g_{r-1} \quad \text{by the induction assumption}\\
&= \lambda^{-2} e_r e_{r-1} e_r = \lambda^{-2} e_r.
\end{aligned}$$

Statement (4) follows from (3) by applying the isomorphism β of Remark 3.7. ∎

The next proposition contains general unwrapping and affine braid relations:

PROPOSITION 3.12. *For all $n \geq 1$ and all $r \geq 1$,*

(1) $g_n x_n g_n x_n = x_n g_n x_n g_n$.
(2) $e_n x_n g_n x_n = \lambda^{-1} e_n$.
(3) $e_n x_n^r g_n x_n = \lambda^{-2} e_n x_n^{r-1} g_n^{-1}$.
(4) $e_n x_n^{-r} g_n^{-1} x_n^{-1} = \lambda^2 e_n x_n^{-r+1} g_n$.

Proof. Statement (1) is equivalent to

$$x_{n+1} x_n = x_n x_{n+1},$$

so follows from Proposition 3.10. Statement (2) follows from (3) and the untwisting relation of Definition 3.6. For statement (3), we have

$$\begin{aligned}
e_n x_n^r g_n x_n &= \lambda^{-2} e_n x_n^{r-1} x_{n+1}^{-1} g_n x_n\\
&= \lambda^{-2} e_n x_n^{r-1} g_n^{-1} x_n^{-1} x_n = \lambda^{-2} e_n x_n^{r-1} g_n^{-1}.
\end{aligned}$$

by Proposition 3.11. Statement (4) follows by applying the isomorphism β to (3). ∎

COROLLARY 3.13. *For $r \geq 1$, $e_1 x_1^{-r} e_1 = f_r(q_1, \ldots, q_r)e_1$, where f_r is the polynomial of equation* (2.1) *in Section* 2.4.

Proof. For $r \geq 1$, set $\psi_r = e_1 x_1^r e_1 = q_r e_1$, and $\psi_{-r} = e_1 x_1^{-r} e_1$. For $a \geq 1$ and $b \geq 0$, define $\psi_{a,b} = e_1 x_1^{-b} g_1 x_1^a e_1$, and $\psi^-_{a,b} = e_1 x_1^{-b} g_1^{-1} x_1^a e_1$. In particular, $\psi_{a,0} = e_1 g_1 x_1^a e_1 = \lambda^{-1} q_a e_1$. We have

$$\psi_{-r} = e_1 x_1^{-r} e_1 = \lambda^{-1} e_1 x_1^{-r} g_1^{-1} e_1 = \lambda \psi_{1,r-1}, \tag{3.1}$$

and

$$\psi^-_{a,b} = \lambda^2 \psi_{a+1,b-1}, \tag{3.2}$$

using Proposition 3.12(4). Moreover, for $a, b \geq 1$,

$$\begin{aligned} \psi_{a,b} &= e_1 x_1^{-b} g_1 x_1^a e_1 \\ &= e_1 x_1^{-b} g_1^{-1} x_1^a e_1 + z(e_1 x_1^{-b} e_1 x_1^a e_1 - e_1 x_1^{a-b} e_1) \\ &= \psi^-_{a,b} + z(\psi_{-b} q_a - \psi_{a-b}) \\ &= \psi_{a+1,b-1} + z(\psi_{-b} q_a - \psi_{a-b}), \end{aligned} \tag{3.3}$$

by the Kauffman skein relation, and equation 3.2. Comparing the recursion relations of equations 3.1 and 3.3 with those of Lemma 2.8 gives that $\psi_{-r} = f_r(q_1, \ldots, q_r) e_1$. ∎

3.4. $\widehat{W}_n$ as a $\widehat{W}_{n-1}$-bimodule. $W_{n,R}$ has a simple structure as a $W_{n-1,R}$-bimodule, described by the following proposition, which is Lemma 3.1 from [2].

PROPOSITION 3.14. *$W_{n,R}$ is the span of elements of the form $a\chi b$, where $a, b \in W_{n-1,R}$ and $\chi \in \{g_{n-1}^{\pm 1}, e_{n-1}, 1\}$.*

We generalize this result to the affine Birman–Wenzl–Murakami algebras in this section. Fix S and let $\widehat{W}_n$ denote $\widehat{W}_{n,S}$.

LEMMA 3.15.

(1) *For $n \geq 1$ and $s \in \mathbb{Z}$, there exists an element $b \in \widehat{W}_{n-1}$ such that*

$$e_n x_n^s e_n = b e_n.$$

(2) *For $n \geq 2$, let a be an element of $\widehat{W}_n$ of the form*

$$a = z_0 x_{n-1}^{r_1} z_1 x_{n-1}^{r_2} \cdots z_{k-1} x_{n-1}^{r_k} z_k,$$

where $z_i \in \{e_{n-1}, g_{n-1}\}$ for all i, and $r_i \geq 1$. There exists $b \in \widehat{W}_{n-1}$ such that

$$e_n a e_n = b e_n.$$

Proof. For $n = 1$, the first assertion follows from the relation (5c) of Definition 3.6 and Corollary 3.13. We take this as the base for an inductive proof of both assertions.

For $n \geq 2$ and $s \geq 1$, the first assertion follows from the second, since $x_n^s = (g_{n-1}x_{n-1}g_{n-1})^s$. The assertion for $s \leq -1$ follows by applying the symmetry β.

The proof of statement (2) is by induction first on n and then on the number k of factors $x_{n-1}^{r_j}z_j$ in the expression for a. For $n \geq 2$ and $k = 0$, the statement is immediate from the BMW relations. Fix $n \geq 2$, $k \geq 1$, and a of the form above with k factors; assume inductively that

(1) $e_m x_m^s e_m \in \widehat{W}_{m-1}e_m$ for $m < n$ and $s \geq 1$, and
(2) $e_n a' e_n \in \widehat{W}_{n-1}e_n$ if a' is of the same form with fewer than k factors.

We consider several cases:

(a) Two successive z_i are equal to e_{n-1}. Then

$$a = a'e_{n-1}x_{n-1}^s e_{n-1}a'' = a'be_{n-1}a'',$$

where $b \in \widehat{W}_{n-2}$, by the induction hypothesis. Thus

$$e_n a e_n = be_n a' e_{n-1}a''e_n \in \widehat{W}_{n-1}e_n,$$

since $a'e_{n-1}a''$ has fewer than k factors.

(b) For some i, $z_i = e_{n-1}$ and $z_{i+1} = g_{n-1}$. Then

$$\begin{aligned} e_n a e_n &= e_n a' e_{n-1}x_{n-1}^r g_{n-1}a''e_n \\ &= \lambda^{-2}e_n a' e_{n-1}x_{n-1}^{r-1}g_{n-1}^{-1}x_{n-1}^{-1}a''e_n \quad \text{(by Proposition 3.12)} \\ &= \lambda^{-2}e_n a' e_{n-1}x_{n-1}^{r-1}[g_{n-1} + ze_{n-1} - z]x_{n-1}^{-1}a''e_n \\ &\equiv \lambda^{-2}e_n a' e_{n-1}x_{n-1}^{r-1}g_{n-1}x_{n-1}^{-1}a''e_n \bmod \widehat{W}_{n-1}e_n, \end{aligned}$$

using case (a) and the induction hypothesis. If $a'' = 1$, then the final expression is equal to

$$\lambda^{-2}e_n a' e_{n-1}x_{n-1}^{r-1}g_{n-1}e_n x_{n-1}^{-1}.$$

Repeating this step r times in all, we get

$$\begin{aligned} e_n a e_n &\equiv \lambda^{-2r}e_n a' e_{n-1}g_{n-1}e_n x_{n-1}^{-r} \bmod \widehat{W}_{n-1}e_n \\ &= \lambda^{-2r-1}e_n a' e_{n-1}e_n x_{n-1}^{-r}. \end{aligned}$$

By the induction hypothesis, this is in $\widehat{W}_{n-1}e_n$. If $a'' \neq 1$, then $a'' = x_{n-1}^s z_{i+2}a'''$, so we get

$$e_n a e_n \equiv \lambda^{-2}e_n a' e_{n-1}x_{n-1}^{r-1}g_{n-1}x_{n-1}^{s-1}z_{i+2}a'''e_n \bmod \widehat{W}_{n-1}e_n.$$

If $r > s$, we repeat this step s times in all, obtaining finally

$$e_n a e_n \equiv \lambda^{-2s}e_n a' e_{n-1}x_{n-1}^{r-s}g_{n-1}z_{i+2}a'''e_n \bmod \widehat{W}_{n-1}e_n.$$

Now $g_{n-1}z_{i+2}$ is a linear combination of e_{n-1}, g_{n-1}, and 1, so by the induction hypothesis, the latter expression is in $\widehat{W}_{n-1}e_n$. The cases $r < s$ and $r = s$ are similar.

(c) For some i, $z_i = g_{n-1}$ and $z_{i+1} = e_{n-1}$. This is essentially the same as case (b).

(d) For all i, $z_i = g_{n-1}$. If $a = g_{n-1}x_{n-1}^r g_{n-1}$, then

$$\begin{aligned} e_n a e_n &= e_n g_{n-1} x_{n-1}^r g_{n-1} e_n \\ &\equiv e_n g_{n-1} x_{n-1}^r g_{n-1}^{-1} e_n \bmod \widehat{W}_{n-1}e_n \quad \text{(as in case (b))} \\ &= e_n g_{n-1} g_n x_{n-1}^r g_n^{-1} g_{n-1}^{-1} e_n \\ &= e_n e_{n-1} x_{n-1}^r e_{n-1} e_n \in \widehat{W}_{n-1}e_n, \end{aligned}$$

by case (a). Otherwise, $a = g_{n-1}x_{n-1}^r g_{n-1}x_{n-1}^s g_{n-1}a'$, and

$$\begin{aligned} e_n a e_n &= e_n g_{n-1} x_{n-1}^r g_{n-1} x_{n-1}^s g_{n-1} a' e_n \\ &\equiv e_n g_{n-1}^{-1} x_{n-1}^r g_{n-1} x_{n-1}^s g_{n-1} a' e_n \bmod \widehat{W}_{n-1}e_n \quad \text{(as in case (b))}. \end{aligned}$$

The affine braid relation of Proposition 3.12 implies

$$x_{n-1}^r g_{n-1} x_{n-1} g_{n-1} = g_{n-1} x_{n-1} g_{n-1} x_{n-1}^r,$$

or

$$g_{n-1}^{-1} x_{n-1}^r g_{n-1} x_{n-1} = x_{n-1} g_{n-1} x_{n-1}^r g_{n-1}^{-1}.$$

Applying this, we get

$$e_n a e_n \equiv e_n x_{n-1} g_{n-1} x_{n-1}^r g_{n-1}^{-1} x_{n-1}^{s-1} g_{n-1} a' e_n \bmod \widehat{W}_{n-1}e_n.$$

Now we can change the g_{n-1}^{-1} to g_{n-1} while maintaining congruence mod $\widehat{W}_{n-1}e_n$, as in case (b), so

$$e_n a e_n \equiv x_{n-1} e_n g_{n-1} x_{n-1}^r g_{n-1} x_{n-1}^{s-1} g_{n-1} a' e_n \bmod \widehat{W}_{n-1}e_n.$$

Repeating this step a total of s times, we get

$$e_n a e_n \equiv x_{n-1}^s e_n g_{n-1} x_{n-1}^r g_{n-1}^2 a' e_n \bmod \widehat{W}_{n-1}e_n.$$

Since g_{n-1}^2 is a linear combination of e_{n-1}, g_{n-1}, and 1, the last expression is in $\widehat{W}_{n-1}e_n$ by the induction assumption. ∎

For the remainder of this section, we maintain the following notation: A_1 denotes the linear span of x_1^r for $r \in \mathbb{Z}$. (Thus $A_1 = \widehat{W}_1$.) For each $n \geq 2$, A_n denotes the linear span of e_{n-1}, $g_{n-1}^{\pm 1}$, and x_n^r for $r \in \mathbb{Z}$.

The following proposition is the analogue for the $\widehat{W}_n$ of Lemma 3.1 in [2]. This result is due to Häring-Oldenburg [5].

PROPOSITION 3.16. *Every element of $\widehat{W}_n$ is a linear combination of elements of the form $a\chi b$, where $a, b \in \widehat{W}_{n-1}$ and $\chi \in \{e_{n-1}, g_{n-1}^{\pm 1}\} \cup \{x_n^r : r \in \mathbb{Z}\}$.*

Proof. We have to show that for all $n \geq 1$, $\widehat{W}_n = \widehat{W}_{n-1}A_n\widehat{W}_{n-1}$. Since $\widehat{W}_n$ is generated as an algebra by $\widehat{W}_{n-1}$ and A_n, it suffices to show that $A_n\widehat{W}_{n-1}A_n \subseteq \widehat{W}_{n-1}A_n\widehat{W}_{n-1}$.

The assertion is evident for $n = 1$. We assume it holds for a particular $n \geq 1$ and prove that $A_{n+1}\widehat{W}_nA_{n+1} \subseteq \widehat{W}_nA_{n+1}\widehat{W}_n$. By the induction assumption, $\widehat{W}_n = \widehat{W}_{n-1}A_n\widehat{W}_{n-1}$, so $A_{n+1}\widehat{W}_nA_{n+1} = A_{n+1}\widehat{W}_{n-1}A_n\widehat{W}_{n-1}A_{n+1}$ $= \widehat{W}_{n-1}A_{n+1}A_nA_{n+1}\widehat{W}_{n-1}$. Thus it suffices to show that

$$A_{n+1}A_nA_{n+1} \subseteq \widehat{W}_nA_{n+1}\widehat{W}_n.$$

We consider several cases:

(a) $\chi'_n\chi_{n-1}\chi''_n \in \widehat{W}_nA_{n+1}\widehat{W}_n$, where $\chi'_n, \chi''_n \in \{e_n, g_n^{\pm 1}\}$ and $\chi_{n-1} \in \{e_{n-1}, g_{n-1}^{\pm 1}\}$. This follows easily from the BMW relations.

(b) $e_nx_n^re_n \in \widehat{W}_{n-1}e_n \subseteq \widehat{W}_nA_{n+1}$. This follows from Lemma 3.15.

(c) $g_n^{\pm 1}x_n^re_n \in \widehat{W}_ne_n \subseteq \widehat{W}_nA_{n+1}$, and $e_nx_n^rg_n^{\pm 1} \in e_n\widehat{W}_n \subseteq A_{n+1}\widehat{W}_n$. The second statement follows from the first by applying the symmetry α. Note that

$$g_n^{-1}x_n^re_n \equiv g_nx_n^re_n \bmod \widehat{W}_ne_n,$$

by the Kauffman skein relation and case (b). Moreover, the assertion for $r \leq -1$ follows from the assertion for $r \geq 1$ by applying the symmetry β. For $r \geq 1$, we have

$$g_nx_n^re_n = \lambda^{-2}x_n^{-1}g_n^{-1}x_n^{r-1}e_n \equiv x_n^{-1}g_nx_n^{r-1}e_n \bmod \widehat{W}_ne_n,$$

by the unwrapping relation of Proposition 3.12. Repeating this step a total of r times gives

$$\begin{aligned} g_nx_n^re_n &\equiv \lambda^{-2r}x_n^{-r}g_ne_n \bmod \widehat{W}_ne_n \\ &= \lambda^{-2r-1}x_n^{-r}e_n. \end{aligned}$$

(d) $A_{n+1}A_nx_{n+1}^t \subseteq \widehat{W}_nA_{n+1}\widehat{W}_n$ and $x_{n+1}^tA_nA_{n+1} \subseteq \widehat{W}_nA_{n+1}\widehat{W}_n$. The second assertion follows from the first by applying the symmetry α. Since x_{n+1}^t commutes with A_n, it suffices to show $A_{n+1}x_{n+1}^t \subseteq \widehat{W}_nA_{n+1}\widehat{W}_n$. Since $x_{n+1}^sx_{n+1}^t = x_{n+1}^{s+t} \in \widehat{W}_nA_{n+1}\widehat{W}_n$, we only have to check that $e_nx_{n+1}^t \in \widehat{W}_nA_{n+1}\widehat{W}_n$, and $g_n^{\pm 1}x_{n+1}^t \in \widehat{W}_nA_{n+1}\widehat{W}_n$.

We have

$$e_nx_{n+1}^t = \lambda^{-2t}e_nx_n^{-t} \in \widehat{W}_nA_{n+1}\widehat{W}_n,$$

by Proposition 3.11. It follows from this and the Kauffman skein relation that

$$g_nx_{n+1}^t \equiv g_n^{-1}x_{n+1}^t \bmod \widehat{W}_nA_{n+1}\widehat{W}_n.$$

Moreover, for $t \geq 1$,

$$g_n^{-1}x_{n+1}^t = x_ng_nx_{n+1}^{t-1} \equiv x_ng_n^{-1}x_{n+1}^{t-1} \bmod \widehat{W}_nA_{n+1}\widehat{W}_n.$$

Repeating this step t times gives

$$g_n^{-1}x_{n+1}^t \equiv x_n^t g_n^{-1} \bmod \widehat{W}_n A_{n+1}\widehat{W}_n.$$

Thus for $t \geq 1$,

$$g_n^{\pm 1}x_{n+1}^t \in \widehat{W}_n A_{n+1}\widehat{W}_n,$$

and the same statement for $t \leq -1$ follows by applying the symmetry β. ∎

PROPOSITION 3.17. *For $n \geq 1$, $e_n\widehat{W}_n e_n = \widehat{W}_{n-1}e_n$.*

Proof. We only have to prove the containment $e_n\widehat{W}_n e_n \subseteq \widehat{W}_{n-1}e_n$. This is obvious for $n = 1$. For $n \geq 2$, we have

$$e_n\widehat{W}_n e_n = e_n\widehat{W}_{n-1}A_n\widehat{W}_{n-1}e_n = \widehat{W}_{n-1}e_n A_n e_n\widehat{W}_{n-1}.$$

But $e_n A_n e_n \subseteq \widehat{W}_{n-1}e_n$ by the BMW relations and Lemma 3.15. ∎

COROLLARY 3.18. *For $n \geq 1$, $e_n W_n e_n = W_{n-1}e_n$.*

LEMMA 3.19. *For $n \geq 1$, $g_n^{\pm 1}\widehat{W}_n e_n \subseteq \widehat{W}_n e_n$ and $x_{n+1}^r\widehat{W}_n e_n \subseteq \widehat{W}_n e_n$.*

Proof. Let χ be one of x_{n+1}^r or $g_n^{\pm 1}$. We have

$$\chi\widehat{W}_n e_n = \chi\widehat{W}_{n-1}A_n\widehat{W}_{n-1}e_n = \widehat{W}_{n-1}\chi A_n e_n\widehat{W}_{n-1}.$$

But $g_n^{\pm 1}A_n e_n \subseteq \widehat{W}_n e_n$ by the BMW relations and case (c) in the proof of Proposition 3.16. Likewise, $x_{n+1}^r A_n e_n = A_n x_{n+1}^r e_n = \lambda^{-2r}x_n^{-r}e_n \subseteq \widehat{W}_n e_n$. Thus $\chi\widehat{W}_n e_n \subseteq \widehat{W}_n e_n\widehat{W}_{n-1} = \widehat{W}_n e_n$. ∎

PROPOSITION 3.20. *For $n \geq 1$, $\widehat{W}_{n+1}e_n = \widehat{W}_n e_n$.*

Proof. We have $\widehat{W}_{n+1}e_n = \widehat{W}_n A_{n+1}\widehat{W}_n e_n \subseteq \widehat{W}_n e_n$, by Proposition 3.17 and Lemma 3.19. ∎

COROLLARY 3.21. *For $n \geq 1$, $W_{n+1}e_n = W_n e_n$.*

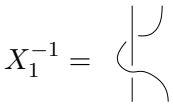
Note that Corollaries 3.18 and 3.21 also follow directly from Proposition 3.14 by similar reasoning.

3.5. A homomorphism from $\widehat{W}_n$ to $\widehat{\mathrm{KT}}_n$. Let R be a ring with distinguished elements λ, z, and δ as above, and $S \supseteq R$ a ring with additional elements $q_1, q_2, \ldots$.

Let X_1, G_i, and E_i denote the affine tangle diagrams shown in the figure following Definition 2.5, regarded as elements of the affine Kauffman tangle algebra $\widehat{\mathrm{KT}}_{n,S}$.

PROPOSITION 3.22. *The assignment $x_1 \mapsto X_1$, $g_i \mapsto G_i$, and $e_i \mapsto E_i$ determines an S-algebra homomorphism φ from $\widehat{W}_{n,S}$ to $\widehat{\mathrm{KT}}_{n,S}$.*

Proof. One needs to check that X_1, G_i, and E_i satisfy the relations of the affine BMW algebra (Definition 3.6), using regular isotopy of affine tangles and the relations of the affine Kauffman tangle algebra. The inverses of G_i and X_1 in $\widehat{\mathrm{KT}}_{n,S}$ are represented by the tangle diagrams with the crossings reversed:

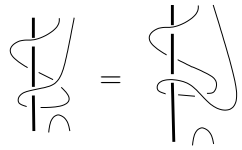
$X_1^{-1} =$  $G_i^{-1} =$ 

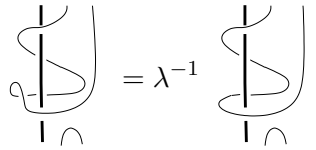
The verification that these elements are in fact inverses involves applications of Reidemeister II.

The Kauffman skein relation for G_i depends on point (1) of Definition 2.6. The relation $E_i^2 = \delta E_i$ results from point (3) of Definition 2.6. The relation $E_1 X_1^r E_1 = q_r E_1$ results from point (2) of Definition 2.9.

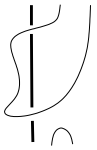
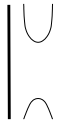
The affine braid relations result from applications of Reidemeister moves II and III. The remaining commutation relations and the tangle relation (6a) only use planar isotopy, while (6b) also uses Reidemeister move II. The untwisting relations (7) follow from the untwisting relation (2) for $\widehat{\mathrm{KT}}_{n,S}$.

Finally, the unwrapping relation follows from the following graphical "computation", which we give here for the sake of illustration. The first equality of pictures is just isotopy, the second a Reidemeister III move, the third an untwisting move, and the last two are Reidemeister II moves.

$E_1 X_1 G_1 X_1 =$ 

$=$ 

$= \lambda^{-1}$

$= \lambda^{-1}$  $= \lambda^{-1}$  $= \lambda^{-1} E_1$. ∎

Proposition 3.23. *The assignment $g_i \mapsto G_i$ and $e_i \mapsto E_i$ determines an R-algebra homomorphism φ from $W_{n,R}$ to $\mathrm{KT}_{n,R}$.*

Proof. One has to verify relations, as in the previous proof. ∎

REMARK 3.24.

(1) The homomorphism $\varphi : \widehat{W}_n \to \widehat{\mathrm{KT}}_n$ gives us a trace $\varepsilon : \widehat{W}_n \to S$ defined by $\varepsilon = \varepsilon \circ \varphi$.

(2) The following diagram commutes (see Propositions 3.22 and 3.23 and Remarks 2.11 and 3.7):

$$\begin{array}{ccc} \widehat{W}_n & \xrightarrow{\varphi} & \widehat{\mathrm{KT}}_n \\ {\scriptstyle i_n}\uparrow & & \uparrow{\scriptstyle i_n} \\ W_n & \xrightarrow{\varphi} & \mathrm{KT}_n \end{array}$$

(3) The following diagrams commute:

$$\begin{array}{ccc} W_n & \xrightarrow{\varphi} & \mathrm{KT}_n \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ W_{n+1} & \xrightarrow{\varphi} & \mathrm{KT}_{n+1} \end{array} \qquad \begin{array}{ccc} \widehat{W}_n & \xrightarrow{\varphi} & \widehat{\mathrm{KT}}_n \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ \widehat{W}_{n+1} & \xrightarrow{\varphi} & \widehat{\mathrm{KT}}_{n+1} \end{array}$$

(4) The following diagrams commute (see Section 2.6, Remark 3.3, and Remark 3.7):

$$\begin{array}{ccc} \widehat{W}_n & \xrightarrow{\varphi} & \widehat{\mathrm{KT}}_n \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \widehat{W}_n & \xrightarrow{\varphi} & \widehat{\mathrm{KT}}_n \end{array} \qquad \begin{array}{ccc} \widehat{W}_n & \xrightarrow{\varphi} & \widehat{\mathrm{KT}}_n \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ \widehat{W}_n & \xrightarrow{\varphi} & \widehat{\mathrm{KT}}_n \end{array} \qquad \begin{array}{ccc} W_n & \xrightarrow{\varphi} & \mathrm{KT}_n \\ {\scriptstyle \varrho_n}\downarrow & & \downarrow{\scriptstyle \varrho_n} \\ W_n & \xrightarrow{\varphi} & \mathrm{KT}_n \end{array}$$

(5) One has $\varphi(x_r) = X_r$ for all r.

We will eventually show $\varphi : \widehat{W}_{n,S} \to \widehat{\mathrm{KT}}_{n,S}$ is an isomorphism for any S.

4. THE AFFINE BRAUER ALGEBRA

4.1. The Brauer algebra. The Brauer algebra D_n is an algebra of planar (n,n)-tangle diagrams in which *crossings are ignored*. The precise definition follows.

Fix points a_i in I, for $i \geq 0$, as in the description of (n,n)-tangles. For convenience write $\boldsymbol{i} = (a_i, 1)$ and $\overline{\boldsymbol{i}} = (a_i, 0)$.

DEFINITION 4.1. An (n,n)-*Brauer diagram* (also called an n-*connector*) consists of a collection of n curves in the rectangle $R = I \times I$ such that

(1) The curves connect the points $\{\mathbf{1},\dots,\boldsymbol{n},\overline{\mathbf{1}},\dots,\overline{\boldsymbol{n}}\}$ in pairs.
(2) For each curve C in the collection, the intersection of C with $\partial(R)$ consists of the two endpoints of C.

Consider the free $\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]$-module D_n with basis the set of (n,n)-Brauer diagrams. The product of two Brauer diagrams is defined to be a certain multiple of another Brauer diagram. Namely, given two Brauer diagrams a, b, first "stack" b over a (as for tangle diagrams). Let r denote the number of closed curves in the interior of R in the resulting planar "tangle", and let c be the Brauer diagram obtained by removing all the closed curves. Then

$$ab = \boldsymbol{\delta}^r c.$$

Definition 4.2. The *Brauer algebra* D_n over $\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]$ is the free $\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]$-module with basis the set of (n,n)-Brauer diagrams, with the bilinear product determined by the multiplication of Brauer diagrams.

The Brauer algebras were introduced by R. Brauer [3] as a device for studying the invariant theory of orthogonal and symplectic groups. The generic structure of the Brauer algebras, and conditions for semisimplicity of $D_n \otimes_{\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]} k$, where k is a field, were determined by H. Wenzl [20].

Note that the Brauer diagrams with only vertical strands, that is, diagrams in which upper points are paired only with lower points, are in bijection with permutations of $\{1,\dots,n\}$, and that the multiplication of two such diagrams coincides with the multiplication of permutations. Thus the Brauer algebra contains the group algebra of the permutation group $\mathfrak{S}_n$ (over $\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]$).

4.2. The affine Brauer algebra. We will define the affine Brauer algebra as a sort of "wreath product" of $\mathbb{Z}$ with the Brauer algebra (containing the wreath product of $\mathbb{Z}$ with $\mathfrak{S}_n$).

Definition 4.3. A *colored* (n,n)*-Brauer diagram*, or *colored* n*-connector* is a Brauer diagram in which each strand is labeled by an integer.

We will define the affine Brauer algebra over the ring

$$\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^{\wedge} = \mathbb{Z}[\boldsymbol{\delta}^{\pm 1}, \boldsymbol{q}_1, \boldsymbol{q}_2, \dots],$$

where $\boldsymbol{q}_1, \boldsymbol{q}_2, \dots$ are indeterminates.

Order the points $\{\mathbf{1},\dots,\boldsymbol{n},\overline{\mathbf{1}},\dots,\overline{\boldsymbol{n}}\}$ by $\mathbf{1} < \cdots < \boldsymbol{n} < \overline{\boldsymbol{n}} < \cdots < \overline{\mathbf{1}}$. The colors on the strands of a colored Brauer diagram should be regarded as assigning integer values to *oriented strands* of the Brauer diagram. If a strand is colored by the integer r, then the strand endowed with the positive orientation (i.e. the orientation from a lower numbered vertex to a higher numbered vertex) takes the value r, but the same strand endowed with the negative orientation takes the value $-r$.

Consider the free $\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge$-module $\widehat{D}_n$ with basis the set of colored (n,n)-Brauer diagrams. The product of two colored Brauer diagrams is defined to be a certain multiple of another colored Brauer diagram, determined as follows.

Given two colored Brauer diagrams a, b, first "stack" b over a (as for tangle diagrams and ordinary Brauer diagrams). In the resulting "tangle" there are three types of curves:

(1) *Vertical strands.* These are concatenations of one vertical strand from b, an even number of horizontal strands from the bottom of b and the top of a, and finally one vertical strand from a. Traversing such a composite strand in its standard orientation (from lower numbered vertex to higher numbered vertex) sum the integer values of the *oriented* strands encountered. (We repeat for emphasis: if a strand colored by r is traversed in the negative direction, then it contributes $-r$ to the sum.) Color the strand by the resulting sum.

(2) *Horizontal strands.* These include horizontal strands remaining from the original diagrams, namely horizontal strands from the top of b, and from the bottom of a. These strands retain their original coloring from the original diagrams. The horizontal strands also include concatenations of a vertical strand from b or a followed by an odd number of horizontal strands from the bottom of b and the top of a, and finally a second vertical strand from the same diagram as the first vertical strand. The color of the composite strand is determined as for composite vertical strands.

(3) *Closed strands.* These are concatenations of an even number of horizontal strands from the bottom of b and the top of a. There is no preferred orientation on such a strand, so pick an orientation arbitrarily, and obtain a color by summing the integer values of the oriented strands encountered in traversing the curve, as for vertical strands. For each $i \in \{0, 1, \dots\}$, let m_i be the number of closed loops with color $\pm i$.

Let c be the colored Brauer diagram obtained by removing all the closed curves. Then

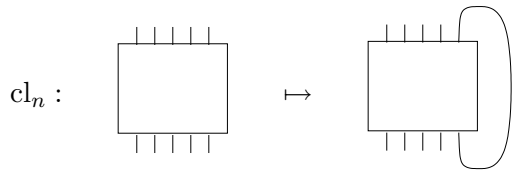
$$ab = (\boldsymbol{\delta}^{m_0}\boldsymbol{q}_1^{m_1}\boldsymbol{q}_2^{m_2}\cdots)c.$$

Definition 4.4. The *affine Brauer algebra* $\widehat{D}_n$ over $\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge$ is the free $\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge$-module with basis the set of colored (n,n)-Brauer diagrams, with the bilinear product determined by the multiplication of colored Brauer diagrams.

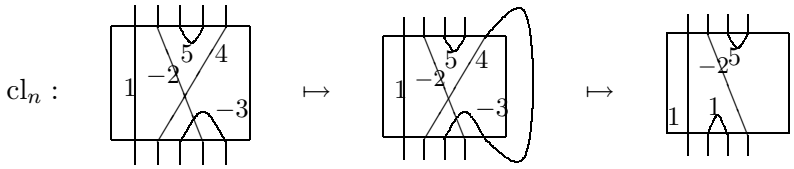
One can easily check that the multiplication is associative. Note that the subalgebra generated by colored Brauer diagrams with only vertical strands is isomorphic to the wreath product of $\mathbb{Z}$ with $\mathfrak{S}_n$.

4.3. Conditional expectation and trace for the affine Brauer algebras. Just as for affine Kauffman tangle algebras, one has a homomorphism ι of $\widehat{D}_{n-1}$ into $\widehat{D}_n$ by attaching an additional strand on the right of an affine Brauer diagram (colored by 0).

Moreover, one has a conditional expectation $\varepsilon_n : \widehat{D}_n \to \widehat{D}_{n-1}$ defined as follows: First define a map cl_n from colored n-connectors to colored $(n-1)$-connectors by joining the rightmost pair of vertices $\overline{\boldsymbol{n}}, \boldsymbol{n}$ of a colored n-connector d by a new strand, with color 0:



The new strand is part of a concatenated (vertical, horizontal or closed) strand. If the concatenated strand is vertical or horizontal, orient it positively (from lower numbered vertex to higher) and obtain its color by adding the colors of its (oriented) components. If the concatenated strand is closed (which happens precisely if d contains a strand connecting $\boldsymbol{n}$ and $\overline{\boldsymbol{n}}$ with some color r) then remove the closed loop and multiply the resulting colored $(n-1)$-connector by $\boldsymbol{q}_{|r|}$ if $r \neq 0$ or $\boldsymbol{\delta}$ if $r = 0$. For example:

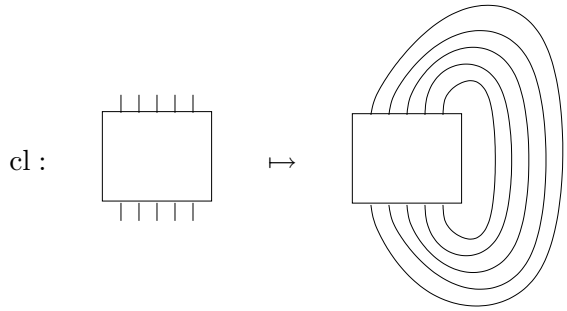


Define $\varepsilon_n : \widehat{D}_n \to \widehat{D}_{n-1}$ by

$$\varepsilon_n(d) = \boldsymbol{\delta}^{-1}\,\mathrm{cl}_n(d).$$

One can check that ε_n is a conditional expectation. Since $\varepsilon_n \circ \iota(x) = x$, the map ι is injective; therefore, we consider $\widehat{D}_{n-1}$ as a subalgebra of $\widehat{D}_n$.

Define $\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_n : \widehat{D}_n \to \widehat{D}_0 = \mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge$. Alternatively, define the closure cl of a colored n-connector by closing all the strands:

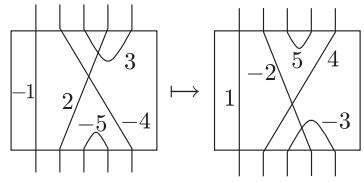


Compute the color of each (closed) strand in the resulting diagram as before, and replace each closed strand by the appropriate factor $\boldsymbol{q}_r$ or $\boldsymbol{\delta}$. Then $\varepsilon(d) = \boldsymbol{\delta}^{-n}\,\mathrm{cl}(d)$.

Using this picture for ε, one can check that ε is a trace.

We can define a symmetric $\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge$-bilinear form on $\widehat{D}_n$ by $(x, y) \mapsto \varepsilon(xy)$.

Define the reflection $\bar{d}$ of a colored n-connector d by reflecting the diagram vertically and changing the color of each strand to its opposite:



Note that for colored n-connectors d, d', the closure $\mathrm{cl}(dd')$ has at most n (closed) strands, and that it has n precisely when the underlying diagrams of d and d' are reflections of one another. Moreover, each of these closed loops has color 0 precisely when $d' = \bar{d}$. Consequently, we have:

Lemma 4.5.

(1) *For colored n-connectors d, d',*

$$\varepsilon(dd') = \boldsymbol{\delta}^{-n} m,$$

where m is a monomial in $\boldsymbol{\delta}, \boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ of total degree $\leq n$. The degree of $\varepsilon(dd')$ in $\boldsymbol{\delta}$ is strictly negative unless $d' = \bar{d}$, while $\varepsilon(d\bar{d}) = 1$.

(2) *Let S be a finite set of colored n-connectors that is closed under the reflection $d \mapsto \bar{d}$. Consider the matrix $A_S = (\varepsilon(dd'))_{d,d' \in S}$. The determinant of A_S is non-zero.*

Proof. The first statement follows from the preceding discussion. For the second statement, each row and column of A_S has exactly one entry

equal to 1. All other entries have strictly negative degree in $\boldsymbol{\delta}$. Therefore, $\det(A_S) = \pm 1 + \boldsymbol{\delta}^{-1}p$, where p is a polynomial in $\boldsymbol{\delta}^{-1}$ with coefficients in $\mathbb{Z}[\boldsymbol{q}_1, \boldsymbol{q}_2, \dots]$. ∎

4.4. A homomorphism from $\widehat{\mathrm{KT}}_{n,\widehat{\Lambda}}$ to $\widehat{D}_n$. We wish to define a map from affine tangle diagrams to colored Brauer diagrams which basically forgets the sense of crossings of ordinary strands, but remembers the sense of crossings of ordinary strands with the flagpole.

Define a map c (the connector map) from affine (n,n)-tangle diagrams to $\widehat{D}_n$ as follows. Number and order the $2n$ vertices of affine (n,n)-tangle diagrams by the same convention as for colored Brauer diagrams. For each strand of an affine (n,n)-tangle diagram a that connects two vertices, draw a curve connecting the corresponding vertices in $c(a)$. Determine the color r of the curve as follows: with the strand oriented from the lower numbered vertex to the higher numbered vertex, r is the number of clockwise rotations of the strand around the flagpole, viewed from above ([1]).

Let d be the resulting colored n-connector. Give each closed strand of a an arbitrary orientation, and determine the color of the strand as above; for each $r \geq 1$, let m_r be the number of closed loops in a with color $\pm r$. Finally, define

$$c(a) = (\boldsymbol{\delta}^{m_0}\boldsymbol{q}_1^{m_1}\boldsymbol{q}_2^{m_2}\cdots)d.$$

The map c respects regular isotopy (in fact, ambient isotopy), so induces a map $c : \widehat{\mathcal{U}}(n,n) \to \widehat{D}_n$. One can check that c is a monoid map, so we extend it linearly to a $\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge$-algebra map

$$c : \mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge\widehat{\mathcal{U}}(n,n) \to \widehat{D}_n.$$

This algebra map respects the relations of $\widehat{\mathrm{KT}}_{n,\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge}$ (see Remark 2.12 at the the end of Section 2.4), so induces a map

$$c : \widehat{\mathrm{KT}}_{n,\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge} \to \widehat{D}_n.$$

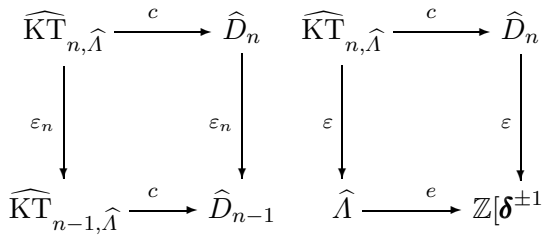
Finally, we have the composition

$$c : \widehat{\mathrm{KT}}_{n,\widehat{\Lambda}} \to \widehat{\mathrm{KT}}_{n,\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge} \to \widehat{D}_n.$$

This map c is given simply by the formula $c(\sum \alpha_i T_i) = \sum e(\alpha_i)c(T_i)$, where $\alpha_i \in \widehat{\Lambda}$, the T_i are affine tangles representing elements of $\widehat{\mathrm{KT}}_{n,\widehat{\Lambda}}$, and e is the homomorphism of $\widehat{\Lambda}$ to $\mathbb{Z}[\boldsymbol{\delta}^{\pm 1}]^\wedge$ determined by $\boldsymbol{\lambda} \mapsto 1$, $\boldsymbol{z} \mapsto 0$, $\boldsymbol{\delta} \mapsto \boldsymbol{\delta}$, and $\boldsymbol{q}_i \mapsto \boldsymbol{q}_i$.

([1]) The color r can be determined combinatorially: traversing the strand from lower numbered vertex to higher numbered vertex, list the over-crossings (+) and under-crossings (−) of the strand with the flagpole. Cancel any two successive +'s or −'s in the list, so the list now consists of alternating +'s and −'s. Then r is $\pm(1/2)$ the length of the list, + if the list begins with a +, and − if the list begins with a −.

PROPOSITION 4.6. *The following diagrams commute*:



Proof. Left to the reader. ∎

The following proposition is adapted from [9] and [10].

PROPOSITION 4.7. *For each colored n-connector d, let T_d be an affine (n,n)-tangle diagram with $c(T_d) = d$. Then the set $\{T_d : d$ a colored n-connector$\}$ is linearly independent over $\widehat{\Lambda}$.*

Proof. Let S be a finite set of colored n-connectors which is closed under reflection $d \mapsto \bar{d}$. Consider the matrices $B_S = (\varepsilon(T_d T_{d'}))_{d,d' \in S}$ and $A_S = (\varepsilon(dd'))_{d,d' \in S}$. One has

$$e(\det(B_S)) = \det(e \circ \varepsilon(T_d T_{d'})) = \det(\varepsilon \circ c(T_d T_{d'})) = \det(\varepsilon(dd')) = \det(A_S).$$

By Lemma 4.5, $\det(A_S) \neq 0$, so $\det(B_S) \neq 0$. Since $\widehat{\Lambda}$ is an integral domain, B_S is invertible over the field of fractions of $\widehat{\Lambda}$. It follows that $\{T_d : d \in S\}$ is linearly independent over $\widehat{\Lambda}$.

Since S is arbitrary, it follows that $\{T_d : d$ a colored n-connector$\}$ is linearly independent over $\widehat{\Lambda}$. ∎

COROLLARY 4.8. *For each ordinary n-connector d, let T_d be an ordinary (n,n)-tangle diagram with $c(T_d) = d$. Then the set $\{T_d : d$ an n-connector$\}$ is linearly independent over Λ.*

Proof. Use the injection $i_n : \mathrm{KT}_{n,\Lambda} \to \widehat{\mathrm{KT}}_{n,\widehat{\Lambda}}$ of Remark 2.11. ∎

5. ISOMORPHISM OF ORDINARY BMW AND KAUFFMAN TANGLE ALGEBRAS

This section is devoted to exhibiting an R-basis of $\mathrm{KT}_{n,R}$ and to proving that $\varphi : W_{n,R} \to \mathrm{KT}_{n,R}$ is an isomorphism. The results and arguments of this section are taken from [10].

All the tangle diagrams in this section will be ordinary (n,n)-tangle diagrams for some n.

5.1. Totally descending tangles and freeness of KT_n

LEMMA 5.1. $\mathrm{KT}_{n,R}$ *is spanned by tangle diagrams without closed strands.*

Proof. The proof is by induction on the number of crossings.

If a tangle diagram T has no crossings, then, by Definition 2.6(3), $T = \delta^k T'$, where k is the number of closed loops of T and T' is the tangle diagram obtained by removing all the closed loops of T.

Let T be a tangle diagram with $l \geq 1$ crossings. Assume that any tangle diagram with fewer than l crossings is in the span of tangle diagrams without closed strands.

By the Kauffman tangle relation, if S is a tangle diagram which differs from T only by reversing one or more crossings, then T and S are congruent modulo the span of tangle diagrams with fewer crossings. Hence T and S are congruent modulo the span of tangle diagrams without closed strands, by the induction hypothesis.

Suppose T has a closed strand s. By changing crossings, one can suppose that s has only over-crossings with other strands of T and that s is unknotted. Then T is ambient isotopic to a tangle diagram in which the closed strand corresponding to s has no crossings with other strands and no self-crossings. If T' is the tangle diagram with s removed, then $T = \delta\lambda^k T'$ for some k, by Definition 2.6(2, 3). ∎

Definition 5.2. An *orientation* of an affine or ordinary (n, n)-tangle diagram is a linear ordering of the strands, a choice of an orientation of each strand, and a choice of an initial point on each closed loop.

Order the boundary points $\{\mathbf{1}, \ldots, \boldsymbol{n}, \overline{\mathbf{1}}, \ldots, \overline{\boldsymbol{n}}\}$ of (n, n)-tangle diagrams by

$$\mathbf{1} < \mathbf{2} < \cdots < \boldsymbol{n} < \overline{\boldsymbol{n}} < \cdots < \overline{\mathbf{2}} < \overline{\mathbf{1}},$$

as in the discussion of colored n-connectors.

Definition 5.3. A *standard orientation* of an ordinary or affine (n, n)-tangle diagram is one in which

(1) each non-closed strand is oriented from its lower numbered endpoint to its higher numbered endpoint,
(2) the non-closed strands are ordered according to the order of their initial endpoints,
(3) the closed loops follow the non-closed strands in the ordering of the strands.

If a tangle diagram has no closed loops, then it has a unique standard orientation.

An orientation determines a way of traversing the tangle diagram; namely, the strands are traversed successively, in the given order and orientation (the closed loops being traversed starting at the assigned initial point).

DEFINITION 5.4. An oriented ordinary (n,n)-tangle diagram is *totally descending* with respect to its orientation if, as the tangle diagram is traversed, each crossing is encountered first as an over-crossing.

PROPOSITION 5.5. $\mathrm{KT}_{n,R}$ *is spanned by ordinary* (n,n)*-tangle diagrams without closed loops that are totally descending with respect to the standard orientation.*

Proof. We already know that $\mathrm{KT}_{n,R}$ is spanned by (n,n)-tangle diagrams without closed loops.

If a tangle diagram has no crossings, it is already totally descending. Let T be a tangle diagram with $l \geq 1$ crossings. Assume that any tangle diagram with fewer than l crossings is in the span of totally descending tangle diagrams. The totally descending tangle diagram S which differs from T only by reversing some number of crossings is congruent to T modulo the span of tangle diagrams with fewer crossings, hence modulo the span of totally descending tangle diagrams. ∎

COROLLARY 5.6. $\mathrm{KT}_{n,R}$ *is spanned by the set of* (n,n)*-tangle diagrams* T *without closed loops that are totally descending with respect to the standard orientation, and such that no strand of* T *has self-crossings.*

Proof. If T is a totally descending (n,n)-tangle diagram, then T is layered; that is, T can be drawn with different strands in different levels above the plane of $R = \mathbb{R} \times I$. The individual strands are unknotted, so they can be changed by level-preserving ambient isotopy to arcs without self-crossings. Thus $T = \lambda^k T'$, where T' is a totally descending tangle whose strands have no self-crossings. ∎

PROPOSITION 5.7. *Suppose* S *and* T *are two* (n,n)*-tangle diagrams without closed loops such that*

(1) S *and* T *have the same connector*,
(2) S *and* T *are both totally descending* (*with respect to the same orientation*),
(3) *the strands of* S *and* T *have no self-crossings.*

Then S *and* T *are regularly isotopic, so they represent the same element of* $\mathrm{KT}_{n,R}$.

Proof. Since S and T have the same connector and are both totally descending with respect to the same ordering of the strands, they can be layered, with the strands connecting corresponding endpoints in the two diagrams lying at the same level above the plane of R. Each strand of S can then be deformed by a level-preserving isotopy to coincide with the corresponding strand of T; this deformation corresponds to regular isotopy of the diagrams. ∎

For each n-connector d, let T_d be an (n,n)-tangle diagram without closed loops such that

(1) $c(T_d) = d$,
(2) T_d is totally descending with respect to the standard orientation,
(3) the strands of T_d have no self-crossings.

By Proposition 5.7, T_d is unique up to regular isotopy. The tangle diagrams T_d (or rather the regular isotopy classes which they represent) can be regarded as elements of $\mathrm{KT}_{n,R}$ for any R. For any R,

$$B_0 = \{T_d : d \text{ is an } n\text{-connector}\}$$

spans $\mathrm{KT}_{n,R}$, by Corollary 5.6. By Corollary 4.8, B_0 is linearly independent in $\mathrm{KT}_{n,\Lambda}$. Thus we have:

THEOREM 5.8. *$\mathrm{KT}_{n,\Lambda}$ is free over Λ with basis B_0.*

COROLLARY 5.9. *For each R, $\mathrm{KT}_{n,R}$ is free over R with basis B_0. Moreover, $\mathrm{KT}_{n,R} \cong \mathrm{KT}_{n,\Lambda} \otimes_\Lambda R$.*

Proof. $\mathrm{KT}_{n,\Lambda} \otimes_\Lambda R$ is free over R with basis $B_0 \otimes 1$. On the other hand, B_0 spans $\mathrm{KT}_{n,R}$ by Corollary 5.6. There is an R-algebra homomorphism from $\mathrm{KT}_{n,R}$ to $\mathrm{KT}_{n,\Lambda} \otimes_\Lambda R$ which sends a tangle T to $T \otimes 1$. Since this map takes a spanning set to a basis, it is an isomorphism. ∎

COROLLARY 5.10. *Let R be a ring with distinguished elements λ, z and δ, and let $S \supseteq R$ be a ring containing R. Then $\mathrm{KT}_{n,R}$ imbeds in $\mathrm{KT}_{n,S}$.*

COROLLARY 5.11. *Let R be a ring with distinguished elements λ, z, and δ, and $S \supseteq R$ a ring with additional elements $q_1, q_2, \ldots$. The R-algebra homomorphism $i_n : \mathrm{KT}_{n,R} \to \widehat{\mathrm{KT}}_{n,S}$ of Remark 2.11 is injective.*

Proof. This follows from the previous corollary and point (1) of Remark 2.11. ∎

5.2. Positive permutation braids

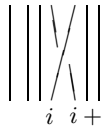
DEFINITION 5.12. An *n-braid diagram* is an (n,n)-tangle diagram all of whose strands are monotone. That is, each strand decreases monotonically from a top vertex to a bottom vertex.

DEFINITION 5.13. The (geometric) *braid group* B_n is the group of n-braid diagrams modulo ambient isotopy.

The group B_n has the well known presentation (due to Artin) with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

(1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$,
(2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$.

The generator σ_i is the n-braid diagram with a single positive crossing between the ith and $(i+1)$st strand,

$\sigma_i =$ 

Since the generators g_i of the ordinary BMW algebra satisfy the braid relations, $\psi : \sigma_i \mapsto g_i$ determines a group homomorphism from B_n into the group of invertible elements of W_n. Denote by perm the homomorphism from B_n to the symmetric group $\mathfrak{S}_n$: $\mathrm{perm}(\beta)(i) = j$ if the braid diagram β connects the top vertex $\boldsymbol{i}$ with the bottom vertex $\overline{\boldsymbol{j}}$. In particular $\mathrm{perm}(\sigma_i)$ is the adjacent transposition $s_i = (i, i+1)$. Note that $\mathrm{perm} = c \circ \varphi \circ \psi$, where we identify permutations with their diagrams in the Brauer algebra,

$$\mathrm{perm} : \sigma_i \mapsto g_i \mapsto G_i \mapsto c(G_i) = s_i.$$

PROPOSITION 5.14. *The following are equivalent for an element β of the braid group*:

(1) *Two strands of β cross at most once, and all crossings are positive (that is, β is in the monoid generated by the σ_i).*
(2) *β is the product of r generators σ_i, where r is the length of* $\mathrm{perm}(\beta)$.

Moreover, if β satisfies these conditions, and $s_{i_r} \cdots s_{i_1}$ is any reduced expression for $\mathrm{perm}(\beta)$, *then $\beta = \sigma_{i_r} \cdots \sigma_{i_1}$.*

Proof. Suppose that $\beta = \sigma_{i_r} \cdots \sigma_{i_1}$, but that two strands of β cross twice. Then $\mathrm{perm}(\beta) = s_{i_r} \cdots s_{i_1}$ has a subword $s_a \pi_0 s_b$ with the property that $\pi_0(b) = a$ and $\pi_0(b+1) = a+1$. But then $\pi_0 s_b = s_a \pi_0$, so $s_a \pi_0 s_b = \pi_0$, and the length of $\mathrm{perm}(\beta)$ is less than r. This proves (2)⇒(1).

Now suppose that β satisfies (1). Let $\mathrm{perm}(\beta) = s_{i_r} \cdots s_{i_1}$ be a reduced expression for $\mathrm{perm}(\beta)$ and set $\beta' = \sigma_{i_r} \cdots \sigma_{i_1}$. Then β and β' are two braid diagrams both satisfying condition (1) with $\mathrm{perm}(\beta) = \mathrm{perm}(\beta')$. But a braid diagram satisfying condition (1) is totally descending (with respect to the orientation in which the strands are oriented from top to bottom and ordered in the reversed order of their top vertices). Therefore, by Proposition 5.7, such a braid diagram β is determined up to ambient isotopy by its connector, that is, by $\mathrm{perm}(\beta)$. ∎

DEFINITION 5.15. A braid diagram satisfying the conditions of the previous proposition is called a *positive permutation braid*.

For each $\pi \in \mathfrak{S}_n$, there is a unique positive permutation braid $\beta_\pi \in B_n$ with $\mathrm{perm}(\beta_\pi) = \pi$. We write g_π for the image of β_π in W_n and G_π for the image of β_π in KT_n. We also call these elements (which are determined by π) positive permutation braids. Note that $g_{\pi^{-1}} = \alpha(g_\pi)$.

For $\pi \in \mathfrak{S}_n$ and $1 \le i < n$, we have $\ell(\pi s_i) = \ell(\pi) + 1 \Leftrightarrow \pi(i) < \pi(i+1)$. In this case, $g_{\pi s_i} = g_\pi g_i$. Otherwise, $\ell(\pi s_i) = \ell(\pi) - 1$, $\pi(i) > \pi(i+1)$, and $g_{\pi s_i} = g_\pi g_i^{-1}$. Likewise, $\ell(s_i \pi) = \ell(\pi) + 1 \Leftrightarrow \pi^{-1}(i) < \pi^{-1}(i+1)$. In this case, $g_{s_i \pi} = g_i g_\pi$. Otherwise, $\ell(\pi s_i) = \ell(\pi) - 1$, $\pi^{-1}(i) > \pi^{-1}(i+1)$, and $g_{s_i \pi} = g_i^{-1} g_\pi$.

DEFINITION 5.16. An (a,b)-*shuffle* is an element $\pi \in \mathfrak{S}_{a+b}$ such that $\pi(i) < \pi(j)$ if $1 \le i < j \le a$ or $a+1 \le i < j \le a+b$.

LEMMA 5.17. *If $\pi \in \mathfrak{S}_{a+b}$ then $\pi = \pi_1 \pi_2$ where π_1 is an (a,b)-shuffle, $\pi_2 \in \mathfrak{S}_a \times \mathfrak{S}_b \subseteq \mathfrak{S}_{a+b}$, and $\ell(\pi) = \ell(\pi_1) + \ell(\pi_2)$. It follows that $g_\pi = g_{\pi_1} g_{\pi_2}$.*

Proof. The proof is by induction on the length of π. The result is evident if π is the identity permutation. If π is not already an (a,b)-shuffle, then there exists an i with $i \ne a$ such that $\pi(i) > \pi(i+1)$. Consequently, $\pi = \pi' s_i$ with $\ell(\pi) = \ell(\pi') + 1$. The result follows by applying the induction hypothesis to π'. ∎

LEMMA 5.18. *Let $1 \le i \le n-1$ and suppose $\pi \in \mathfrak{S}_n$ satisfies $\pi^{-1}(i+1) = \pi^{-1}(i) + 1$. Then $g_i g_\pi = g_\pi g_{\pi^{-1}(i)}$ and $e_i g_\pi = g_\pi e_{\pi^{-1}(i)}$.*

Proof. Put $j = \pi^{-1}(i)$. We have $s_i \pi = \pi s_j$, and $\ell(s_i \pi) = \ell(\pi) + 1$. Hence

$$g_\pi g_j = g_{\pi s_j} = g_{s_i \pi} = g_i g_\pi.$$

The second equality is proved by induction on the length of π. If $\ell(\pi) = 0$, the assertion is trivial. Suppose that $\ell(\pi) > 1$. Choose k such that $\pi = s_k \pi_1$, and $\ell(\pi) = \ell(\pi_1) + 1$. To prove the induction step, we consider three cases:

(1) $k = i+1$. In this case $\pi^{-1}(i+2) < \pi^{-1}(i+1) = \pi^{-1}(i) + 1$. It follows that $\pi^{-1}(i+2) < \pi^{-1}(i)$, and $\pi = s_{i+1} s_i \pi'$, with $\ell(\pi) = \ell(\pi') + 2$. Therefore $g_\pi = g_{i+1} g_i g_{\pi'}$ and $e_i g_\pi = e_i g_{i+1} g_i g'_\pi = g_{i+1} g_i e_{i+1} g_{\pi'}$. Since ${\pi'}^{-1}(i+2) = j+1 = {\pi'}^{-1}(i+1) + 1$, we have $e_{i+1} g_{\pi'} = g_{\pi'} e_j$, by the induction hypothesis.

(2) $k = i-1$. This case is similar.

(3) $|k-i| \ge 2$. Then $e_i g_\pi = e_i g_k g_{\pi_1} = g_k e_i g_{\pi_1}$. Now ${\pi_1}^{-1}(i+1) = j+1 = {\pi_1}^{-1}(i) + 1$, so by the induction hypothesis, $e_i g_{\pi_1} = g_{\pi_1} e_j$. ∎

5.3. Surjectivity of $\varphi : W_n \to \mathrm{KT}_n$. We will prove that $\mathrm{KT}_{n,R}$ is generated as a unital algebra by $\{E_i, G_i^{\pm 1} : 1 \le i \le n-1\}$, which is equivalent to the surjectivity of $\varphi : W_{n,R} \to \mathrm{KT}_{n,R}$.

The tensor product $T_1 \otimes T_2$ of a (k,k)-tangle diagram and an (l,l)-tangle diagram is the $(k+l, k+l)$-tangle diagram obtained by placing T_1 and T_2 side by side.

The tensor product of tangle diagrams clearly extends to a bilinear product $\mathrm{KT}_{k,R} \times \mathrm{KT}_{l,R} \to \mathrm{KT}_{k+l,R}$. If T_1 and T_2 are both in the unital subalgebra generated by the E_i's and G_i's, then so is $T_1 \otimes T_2$.

LEMMA 5.19. *Any element of* $\mathrm{KT}_{n,R}$ *which is represented by an* n*-braid diagram is in the group generated by* $\{G_i^{\pm 1} : 1 \le i \le n-1\}$.

Proof. Induction on the number of crossings. ∎

THEOREM 5.20. $\varphi : W_{n,R} \to \mathrm{KT}_{n,R}$ *is surjective.*

Proof. For $n = 0$ and $n = 1$, $\mathrm{KT}_{n,R} \cong R$ by Corollary 5.9, and the statement is trivially valid.

Fix $n \ge 2$. We have to show that $\mathrm{KT}_{n,R}$ is generated as a unital algebra by $\{E_i, G_i^{\pm 1} : 1 \le i \le n-1\}$. By Theorem 5.8, it suffices to show that each totally descending (n,n)-tangle diagram T whose strands have no self-crossings is in the unital subalgebra generated by the E_i's and G_i's.

If the connector $c(T)$ is a permutation diagram (i.e. top vertices are connected only to bottom vertices), then T is regularly isotopic to an n-braid diagram, and thus T is in the monoid generated by $\{G_i^{\pm 1} : 1 \le i \le n-1\}$.

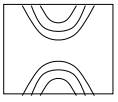
Otherwise, for some $k \ge 1$, the connector of T has k horizontal strands connecting pairs of vertices at the top and k horizontal strands connecting vertices at the bottom. In this case there exist n-braid diagrams B_1 and B_2 and an $(n-2k)$-braid diagram B such that

$$T = B_1[(E_1 E_3 \cdots E_{2k-1}) \otimes B]B_2.$$

Since each of B_1, B_2, and B are in the group generated by the $\{G_i^{\pm 1}\}$, it follows that T is in the monoid generated by $\{E_i, G_i^{\pm 1}\}$. ∎

5.4. The elements f_k and a filtration of W_n. Fix a ring R and write W_n for $W_{n,R}$ and KT_n for $\mathrm{KT}_{n,R}$.

Consider the element $F_k \in \mathrm{KT}_{2k}$ represented by the tangle diagram with no crossings, in which the points $\boldsymbol{i}$ and $\boldsymbol{2k+1-i}$ at the top of the diagram are connected, and likewise the points $\overline{\boldsymbol{i}}$ and $\overline{\boldsymbol{2k+1-i}}$ at the bottom of the diagram are connected. For example

$F_3 =$ 

The element F_k is evidently fixed by the symmetries α, β, and ϱ_{2k} of KT_{2k}.

The following proposition is from [10] and is easily verified by picture proofs.

PROPOSITION 5.21.

(1) *For all* $i < k$, $G_i^{\pm 1} F_k = G_{2k-i}^{\pm 1} F_k$ *and* $F_k G_i^{\pm 1} = F_k G_{2k-i}^{\pm 1}$.
(2) *For all* $i < k$, $E_i F_k = E_{2k-i} F_k$ *and* $F_k E_i = F_k E_{2k-i}$.
(3) $F_k = (G_1 G_2 \cdots G_{2k-1}) F_{k-1} E_{2k-1} (G_{2k-1} \cdots G_2 G_1)$.

Following [10], we recursively define elements f_k of W_{2k} such that $\varphi(f_k) = F_k$, as follows:

DEFINITION 5.22. Define $f_1 = e_1$ and

$$f_k = (g_1 g_2 \cdots g_{2k-2}) f_{k-1} e_{2k-1} (g_{2k-2} \cdots g_2 g_1)$$

for $k \geq 2$.

We want to find an expression for f_k that does not involve the g_i's (since the corresponding tangle diagrams have no crossings), and that makes manifest the symmetries $\alpha(f_k) = \beta(f_k) = \varrho_{2k}(f_k) = f_k$ (which the definition does not).

LEMMA 5.23.

$$(g_1 g_2 \cdots g_{2k})(e_1 e_3 \cdots e_{2k-1}) = (e_2 e_4 \cdots e_{2k})(e_1 e_3 \cdots e_{2k-1}),$$
$$(e_1 e_3 \cdots e_{2k-1})(g_{2k} \cdots g_2 g_1) = (e_1 e_3 \cdots e_{2k-1})(e_2 e_4 \cdots e_{2k}).$$

Proof. The second equation follows from the first by applying the anti-automorphism α. To prove the first equation, rewrite the left hand side as

$$(g_1 g_2 e_1)(g_3 g_4 e_3) \cdots (g_{2k-1} g_{2k} e_{2k-1}),$$

which equals

$$(e_2 e_1)(e_4 e_3) \cdots (e_{2k} e_{2k-1}),$$

by use of (5b) of Definition 3.6. Finally, the even terms can be shuffled to the left. ∎

PROPOSITION 5.24. *For $k \geq 1$,*

$$f_k = (e_k)(e_{k-1} e_{k+1}) \cdots (e_1 e_3 \cdots e_{2k-1}) \cdots (e_{k-1} e_{k+1})(e_k).$$

For example,

$$f_4 = e_4 (e_3 e_5)(e_2 e_4 e_6)(e_1 e_3 e_5 e_7)(e_2 e_4 e_6)(e_3 e_5) e_4.$$

Proof. The proof goes by induction on k, the assertion being evident for $k = 1$. Consider $k > 2$. We have

$$\begin{aligned} f_k &= (g_1 g_2 \cdots g_{2k-2}) f_{k-1} e_{2k-1} (g_{2k-2} \cdots g_2 g_1) \\ &= (g_1 g_2 \cdots g_{2k-2})[e_{k-1}(e_{k-2} e_k) \cdots (e_1 e_3 \cdots e_{2k-3}) \cdots (e_{k-2} e_k) e_{k-1})] \\ &\qquad \cdot e_{2k-1} (g_{2k-2} \cdots g_2 g_1) \end{aligned}$$

by the induction hypothesis. Move e_{2k-1} to the left to get

$$\begin{aligned} f_k = (g_1 g_2 \cdots g_{2k-2})[e_{k-1}(e_{k-2} e_k) \cdots (e_1 e_3 \cdots e_{2k-3} e_{2k-1}) \cdots (e_{k-2} e_k) e_{k-1})] \\ \cdot (g_{2k-2} \cdots g_2 g_1). \end{aligned}$$

Now move all of the e_i's, except those in the product $(e_1e_3\cdots e_{2k-1})$, to the left or right past a string of g_i's, making use of Lemma 3.9; this yields

$$f_k = e_k(e_{k-1}e_{k+1})\cdots(e_3e_5\cdots e_{2k-3})(g_1g_2\cdots g_{2k-2})(e_1e_3\cdots e_{2k-3}e_{2k-1}) \cdot(g_{2k-2}\cdots g_2g_1)(e_3e_5\cdots e_{2k-3})\cdots(e_{k-1}e_{k+1})e_k.$$

Both strings of g_i's in the middle of the expression can be replaced by $e_2e_4\cdots e_{2k-2}$, by the previous lemma, and this gives the desired expression. ∎

Remark 5.25. The elements in each group (indicated by parentheses) commute. It can be helpful to view the entire expression as a diamond-shaped grid, for example

$$f_3 = \begin{array}{ccccc} & & e_1 & & \\ & e_2 & & e_2 & \\ e_3 & & e_3 & & e_3 \\ & e_4 & & e_4 & \\ & & e_5 & & \end{array}$$

This is read from left to right by columns, with the elements in each column commuting. This expression for f_k makes evident the invariance of f_k under the maps ϱ_{2k}, α, and β of W_{2k}.

Analogous elements in the Temperley–Lieb algebras were introduced in subfactor theory by M. Pimsner and S. Popa [16] to study iterations of the Jones basic construction; see also [7]. The diamond grid representation of these elements is due to A. Ocneanu (personal communication).

Note that one can also read the diamond grid by diagonals, so we have

$$f_k = (e_ke_{k+1}\cdots e_{2k-1})(e_{k-1}e_k\cdots e_{2k-2})\cdots(e_1e_2\cdots e_k).$$

Moreover, f_k satisfies

$$f_k = (e_ke_{k+1}\cdots e_{2k-2})f_{k-1}e_{2k-1}(e_{2k-2}\cdots e_{k+1}e_k).$$

The following proposition from [10] establishes the analogue of Proposition 5.21 for the elements f_k. In points (3) and (4), S denotes the shift homomorphism (see Remark 3.3).

Proposition 5.26.

(1) *For $i<k$, $g_i^{\pm1}f_k = g_{2k-i}^{\pm1}f_k$ and $f_kg_i^{\pm1} = f_kg_{2k-i}^{\pm1}$.*
(2) *For $i<k$, $e_if_k = e_{2k-i}f_k$ and $f_ke_i = f_ke_{2k-i}$.*
(3) *For $2\le i<k+1$, $g_i^{\pm1}S(f_k) = g_{2k+2-i}^{\pm1}S(f_k)$ and $S(f_k)g_i^{\pm1} = S(f_k)g_{2k+2-i}^{\pm1}$.*
(4) *For $2\le i<k+1$, $e_iS(f_k)=e_{2k+2-i}S(f_k)$ and $S(f_k)e_i=S(f_k)e_{2k+2-i}$.*

Proof. The second part of each assertion follows from the first part by applying the anti-automorphism α.

Note that i and $2k-i$ have the same parity, so g_i and g_{2k-i} commute. Therefore, multiplying the equality

$$g_i f_k = g_{2k-i} f_k \tag{5.1}$$

by $g_i^{-1} g_{2k-i}^{-1}$ gives

$$g_i^{-1} f_k = g_{2k-i}^{-1} f_k. \tag{5.2}$$

Thus, for statement (1), it suffices to prove equation (5.1).

We prove statement (1) by induction on k (following the proof in [10]). For $k = 2$, the only instance to check is $i = 1$. Using Lemma 3.9(3), we get

$$g_1 f_2 = g_1(g_1 g_2 e_1 e_3) g_2 g_1 = g_1(g_3 g_2 e_3 e_1) g_2 g_1 = g_3(g_1 g_2 e_1 e_3 g_2 g_1) = g_3 f_k.$$

Consider $k > 2$. If $2 \le i < k$, then, using Lemma 3.9, we have

$$\begin{aligned} g_i f_k &= g_i(g_1 g_2 \cdots g_{2k-2}) f_{k-1} e_{2k-1}(g_{2k-2} \cdots g_2 g_1) \\ &= (g_1 g_2 \cdots g_{2k-2}) g_{i-1} f_{k-1} e_{2k-1}(g_{2k-2} \cdots g_2 g_1) \\ &= (g_1 g_2 \cdots g_{2k-2}) g_{2k-i-1} f_{k-1} e_{2k-1}(g_{2k-2} \cdots g_2 g_1), \end{aligned}$$

which by the induction hypothesis equals

$$g_{2k-i}(g_1 g_2 \cdots g_{2k-2}) f_{k-1} e_{2k-1}(g_{2k-2} \cdots g_2 g_1) = g_{2k-i} f_k.$$

The last case to check is $k > 2$ and $i = 1$. We have

$$\begin{aligned} g_1 f_k &= g_1(g_1 g_2 \cdots g_{2k-2}) f_{k-1} e_{2k-1}(g_{2k-2} \cdots g_2 g_1) \\ &= g_1(g_1 g_2 \cdots g_{2k-2})((g_1 g_2 \cdots g_{2k-4}) f_{k-2} e_{2k-3}(g_{2k-4} \cdots g_2 g_1)) \\ &\qquad \cdot e_{2k-1}(g_{2k-2} \cdots g_2 g_1) \\ &= g_1(g_2 g_3 \cdots g_{2k-3})(g_1 g_2 \cdots g_{2k-2}) f_{k-2} e_{2k-3}(g_{2k-4} \cdots g_2 g_1) \\ &\qquad \cdot e_{2k-1}(g_{2k-2} \cdots g_2 g_1), \end{aligned}$$

by repeated use of Lemma 3.9(4). Moving e_{2k-3} and e_{2k-1} to the left and applying Lemma 3.9(3), we get

$$\begin{aligned} &g_1(g_2 g_3 \cdots g_{2k-3})(g_1 g_2 \cdots g_{2k-4}(g_{2k-3} g_{2k-2} e_{2k-3} e_{2k-1})) \\ &\qquad \cdot f_{k-2}(g_{2k-4} \cdots g_2 g_1)(g_{2k-2} \cdots g_2 g_1) \\ &= g_1(g_2 g_3 \cdots g_{2k-3})(g_1 g_2 \cdots g_{2k-4}(g_{2k-1} g_{2k-2} e_{2k-3} e_{2k-1})) \\ &\qquad \cdot f_{k-2}(g_{2k-4} \cdots g_2 g_1)(g_{2k-2} \cdots g_2 g_1). \end{aligned}$$

Now g_{2k-1} and g_{2k-2} can be moved to the left and e_{2k-1} and e_{2k-3} to the right to yield

$$\begin{aligned} &g_{2k-1} g_1(g_2 g_3 \cdots g_{2k-3} g_{2k-2})(g_1 g_2 \cdots g_{2k-4}) f_{k-2}(e_{2k-3} e_{2k-1}) \\ &\qquad \cdot (g_{2k-4} \cdots g_2 g_1)(g_{2k-2} \cdots g_2 g_1) \\ &\qquad = g_{2k-1} f_k. \end{aligned}$$

This completes the proof of (1).

Statement (2) is also proved by induction on k. For $k = 2$ and $i = 1$, we have

$$e_1 f_2 = e_1 e_2 e_1 e_3 e_2 = e_1 e_3 e_2,$$

and similarly $e_3 f_2 = e_1 e_3 e_2$.

For $2 \le i < k$, exactly the same induction step can be used as in the proof of statement (1). The only remaining case to check is $k > 2$ and $i = 1$.

We claim that $e_1 f_k$ is the word w_k in the e_i's obtained by deleting the two columns to the left of the middle column in the diamond expression for f_k. For example,

$$e_1 f_4 = e_1 \begin{bmatrix} & & e_1 & & \\ & e_2 & & e_2 & \\ e_3 & & e_3 & & e_3 \\ & e_4 & & e_4 & \\ e_5 & & e_5 & & e_5 \\ & e_6 & & e_6 & \\ & & e_7 & & \end{bmatrix} = e_4 \begin{bmatrix} e_1 & & \\ & e_2 & \\ e_3 & & e_3 \\ & e_4 & \\ e_5 & & e_5 \\ & e_6 & \\ e_7 & & \end{bmatrix}.$$

This claim can be proved by induction on k. Moreover, both f_k and w_k are invariant under the automorphism ϱ_{2k} of W_{2k}. Applying this automorphism to the equation $e_1 f_k = w_k$ gives

$$e_{2k-1} f_k = w_k = e_1 f_k.$$

This completes the proof of (2).

Statements (3) and (4) follow from (1) and (2) by applying the shift homomorphism S, for example

$$e_i S(f_k) = S(e_{i-1} f_k) = S(e_{2k+1-i} f_k) = e_{2k+2-i} S(f_k). \blacksquare$$

REMARK 5.27. One can give a similar explicit expression for $e_i f_k$ for all $i < k$. One obtains $e_{k-1} f_k$ by deleting the leftmost e_k from the diamond expression for f_k. Moreover, if $i < k - 1$, then $e_i f_k$ is the word in the e_j's obtained by deleting the two columns just to the left of the first column beginning with e_i in the diamond expression for f_k. For example,

$$e_2 f_4 = \begin{bmatrix} & e_1 & & \\ e_2 & & e_2 & \\ & e_3 & & e_3 \\ e_4 & & e_4 & & e_4 \\ & e_5 & & e_5 & \\ e_6 & & e_6 & & \\ & e_7 & & & \end{bmatrix}.$$

The proof of statement (2) in the previous proposition can be based entirely on this observation.

PROPOSITION 5.28.

(1) *For all* k, $W_{2k}f_k = W_k f_k$ *and* $f_k W_{2k} = f_k W_k$.
(2) *For all* k, $W_{2k+1}S(f_k) = W_{k+1}S(f_k)$ *and* $S(f_k)W_{2k+1} = S(f_k)W_{k+1}$.

Proof. The second part of each statement follows from the first part by applying the symmetry α.

To prove (1), it suffices to show that $W_k f_k$ is a left ideal in W_{2k}. For this, it is enough to show that $e_i W_k f_k \subseteq W_k f_k$ and $g_i^{\pm 1} W_k f_k \subseteq W_k f_k$ for $k \le i \le 2k-1$. For $k < i < 2k-1$, this follows from Proposition 5.26, since e_i and $g_i^{\pm 1}$ commute with W_k. Finally, by Proposition 5.24, we can write $f_k = e_k r_k$ for some $r_k \in W_{2k}$. Hence for $\chi \in \{e_k, g_k^{\pm 1}\}$,

$$\chi W_k f_k = \chi W_k e_k r_k \subseteq W_{k+1} e_k r_k = W_k e_k r_k = W_k f_k,$$

using Corollary 3.21. The proof of part (2) is similar. ∎

DEFINITION 5.29. Let $n \ge 2k$ and put $r = n - 2k$. Define $W_n^{(r)}$ to be the ideal in W_n generated by $e_1 e_3 \cdots e_{2k-1}$.

Evidently, one has

$$W_n^{(0)} \subseteq W_n^{(2)} \subseteq \cdots \subseteq W_n^{(n)} = W_n \quad \text{if } n \text{ is even},$$
$$W_n^{(1)} \subseteq W_n^{(3)} \subseteq \cdots \subseteq W_n^{(n)} = W_n \quad \text{if } n \text{ is odd}.$$

Morton and Wassermann show that $\varphi(W_n^{(r)})$ is the ideal $KT_n^{(r)}$ in KT_n spanned by tangle diagrams of rank no more than r, that is, tangle diagrams of the form ST, where T is an (n,r)-tangle diagram and S is an (r,n)-tangle diagram. We will not need this observation, but it provides the motivation for the definition of $W_n^{(r)}$.

LEMMA 5.30. *Let* $n \ge 2k$ *and put* $r = n - 2k$.

(1) $W_n^{(r)}$ *is the ideal in* W_n *generated by* f_k.
(2) *If* $n > 2k$ *then* $W_n^{(r)}$ *is the ideal in* W_n *generated by* $S(f_k)$.

Proof. It follows from the definition of f_k, and induction on k, that $f_k = a(e_1 e_3 \cdots e_{2k-1})b$ where a and b are in the monoid generated by $\{g_j\}$, and in particular are invertible. Likewise, if $n > 2k$, then $S(f_k) = \mathrm{Ad}(a)(f_k)$ where a is invertible. ∎

COROLLARY 5.31.

(1) $W_{2k}^{(0)} = W_k f_k W_k$.
(2) $W_{2k+1}^{(1)} = W_{k+1} S(f_k) W_{k+1}$.

Proof. Immediate from Proposition 5.28 and Lemma 5.30. ∎

5.5. Injectivity of $\varphi : W_n \to \mathrm{KT}_n$. Fix a ring R and write W_n for $W_{n,R}$ and KT_n for $\mathrm{KT}_{n,R}$. This section contains Morton and Wassermann's proof of the injectivity of $\varphi : W_n \to \mathrm{KT}_n$. The strategy is to show that φ is injective on $W_n^{(r)}$ for all n and r, by double induction.

PROPOSITION 5.32. *Suppose $\varphi|_{W_{n-1}}$ is injective for some n. Then $\varphi|_{W_n^{(0)}}$ is injective if n is even, and $\varphi|_{W_n^{(1)}}$ is injective if n is odd.*

Proof. Consider the case that $n = 2k$. By Corollary 5.31, $W_n^{(0)} = W_k f_k W_k$. By Theorem 5.8, KT_k has a basis $\{T_c : c \text{ a } k\text{-connector}\}$ consisting of totally descending tangle diagrams with distinct connectors. Since by hypothesis, $\varphi : W_k \to \mathrm{KT}_k$ is an isomorphism, it follows that $\{\varphi^{-1}(T_c) : c \text{ a } k\text{-connector}\}$ is a basis of W_k. Therefore $W_n^{(0)}$ is spanned by the set

$$\{\varphi^{-1}(T_c) f_k \varphi^{-1}(T_d) : c, d \ k\text{-connectors}\}.$$

The image of this spanning set under φ is $\{T_c F_k T_d : c, d \ k\text{-connectors}\}$. This set of (n,n)-tangle diagrams has distinct connectors, and therefore is linearly independent by Corollary 4.8. It follows that φ is injective on $W_n^{(0)}$. The proof for the case $n = 2k+1$ is similar. ∎

Next we introduce a linear complement of $W_n^{(r-2)}$ in $W_n^{(r)}$.

DEFINITION 5.33. Let $n > 2k$ and put $r = n - 2k$. Then $V_n^{(r)}$ is the span of $\{g_\pi[w \otimes g_\tau]\alpha(g_\sigma)\}$, where $w \in W_{2k}^{(0)}$, π and σ are $(2k,r)$-shuffles, τ is an r-permutation, and g_π, g_σ, and g_τ are the corresponding positive permutation braids.

LEMMA 5.34. *Assume that $\varphi|_{W_{n-1}}$ is injective for a particular n.*

(1) *$\varphi|_{V_n^{(r)}}$ is injective for $r > 0$.*

(2) *$V_n^{(r)} \cap W_n^{(r-2)} = (0)$.*

Proof. By assumption, $\varphi|_{W_{2k}^{(0)}}$ is injective, and moreover, $W_{2k}^{(0)}$ has a basis consisting of elements $\varphi^{-1}(T_c) f_k \varphi^{-1}(T_d)$, where T_c and T_d vary independently over a set of totally descending (k,k)-tangle diagrams with distinct connectors. Thus $V_n^{(r)}$ has a spanning set

$$\{g_\pi[\varphi^{-1}(T_c) f_k \varphi^{-1}(T_d) \otimes g_\tau]\alpha(g_\sigma)\}.$$

The image of this set,

$$\{G_\pi[T_c F_k T_d \otimes G_\tau]\alpha(G_\sigma)\},$$

is a family of tangle diagrams with distinct connectors $\pi[cF_k d \otimes \tau]\sigma^{-1}$, and is therefore linearly independent by Corollary 4.8. This proves statement (1).

For (2), $\varphi(W_n^{(r-2)})$ is spanned by totally descending tangle diagrams with no more than $r-2$ through strands (i.e. strands connecting top to

bottom), while $\varphi(V_n^{(r)})$ is spanned by totally descending tangle diagrams with exactly r through strands. By Corollary 4.8, $\varphi(V_n^{(r)}) \cap \varphi(W_n^{(r-2)}) = (0)$, so also $V_n^{(r)} \cap W_n^{(r-2)} = (0)$. ∎

COROLLARY 5.35. *If φ is injective on W_{n-1} and also on $W_n^{(r-2)}$, then φ is injective on $W_n^{(r-2)} \oplus V_n^{(r)}$.*

The remainder of the proof consists of showing that $W_n^{(r-2)} \oplus V_n^{(r)} = W_n^{(r)}$. Since $e_1 e_3 \cdots e_{2k-1} \in V_n^{(r)}$, it suffices to show that $W_n^{(r-2)} \oplus V_n^{(r)}$ is an ideal in W_n. Because of the invariance of $W_n^{(r-2)} \oplus V_n^{(r)}$ under the anti-automorphism α, it is enough to show that $W_n^{(r-2)} \oplus V_n^{(r)}$ is a left ideal. It suffices to show that $\chi V_n^{(r)} \subseteq W_n^{(r-2)} \oplus V_n^{(r)}$ for χ a generator of W_n, because $W_n^{(r-2)}$ is already an ideal. Moreover, since $g_i^{-1} = g_i - z e_i + z$, it suffices to show this for $\chi \in \{e_i, g_i : 1 \le i \le n-1\}$.

Note that $V_n^{(n)}$ is the span of $\{g_\pi : \pi \in \mathfrak{S}_n\}$ and $W_n^{(n-2)} = W_n e_1 W_n$, so a particular case of our assertion is that $V_n^{(n)} \oplus W_n e_1 W_n = W_n$.

LEMMA 5.36. *Let $n > 2k$ and $r = n - 2k$.*

(1) $V_n^{(n)} \oplus W_n e_1 W_n = W_n$.
(2) $W_{2k}^{(0)} \otimes W_r \subseteq W_n^{(r-2)} + W_{2k}^{(0)} \otimes V_r^{(r)} \subseteq W_n^{(r-2)} + V_n^{(r)}$.
(3) $g_\pi (W_{2k}^{(0)} \otimes W_r) g_\sigma \subseteq W_n^{(r-2)} + V_n^{(r)}$ *for any positive permutation braids* g_π, g_σ.

Proof. For (1), we have to show that $\chi g_\pi \in V_n^{(n)} + W_n e_1 W_n$ for $\chi \in \{e_i, g_i : 1 \le i \le n-1\}$. We have $e_i g_\pi \in W_n e_i W_n = W_n e_1 W_n$ for all i. If $\ell(s_i \pi) = \ell(\pi) + 1$, then $g_i g_\pi = g_{s_i \pi} \in V_n^{(n)}$; on the other hand, if $\ell(s_i \pi_i) = \ell(\pi) - 1$, then $g_\pi = g_i g_{s_i \pi}$, and

$$g_i g_\pi = (g_i^2) g_{s_i \pi} = g_{s_i \pi} - z g_\pi + z \lambda^{-1} e_i g_{s_i \pi} \in V_n^{(n)} + W_n e_1 W_n.$$

Applying (1) to W_r, we get $W_r = W_r e_1 W_r + V_r^{(r)}$, so

$$W_{2k}^{(0)} \otimes W_r = W_{2k}^{(0)} \otimes W_r e_1 W_r + W_{2k}^{(0)} \otimes V_r^{(r)} \subseteq W_n^{(r-2)} + V_n^{(r)}.$$

This shows (2).

Using Lemma 5.17, the positive permutation braid g_π can be written as $g_{\pi_1} w_1$, where π_1 is a $(2k, r)$-shuffle and $w_1 \in W_{2k} \otimes W_r$. Applying the same result to $\alpha(g_\sigma)$, we get $g_\sigma = w_2 \alpha(g_{\sigma_1})$, where σ_1 is a $(2k, r)$-shuffle and $w_2 \in W_{2k} \otimes W_r$. Thus

$$g_\pi (W_{2k}^{(0)} \otimes W_r) g_\sigma \subseteq g_{\pi_1} (W_{2k}^{(0)} \otimes W_r) \alpha(g_{\sigma_1}).$$

By statement (2), this lies in

$$W_n^{(r-2)} + g_{\pi_1} (W_{2k}^{(0)} \otimes V_r^{(r)}) \alpha(g_{\sigma_1}) \subseteq W_n^{(r-2)} + V_n^{(r)}. \ ∎$$

LEMMA 5.37. *$e_i V_n^{(r)} \subseteq W_n^{(r-2)} + V_n^{(r)}$ and $g_i V_n^{(r)} \subseteq W_n^{(r-2)} + V_n^{(r)}$ for $1 \le i \le n-1$.*

Proof. Consider $x = g_\pi[w \otimes g_\tau]\alpha(g_\sigma)$, where π and σ are $(2k, r)$-shuffles, $w \in W_{2k}^{(0)}$, and $\tau \in \mathfrak{S}_r$. We have to show that $e_i x$ and $g_i x$ lie in $W_n^{(r-2)} + V_n^{(r)}$ for all $1 \le i \le n-1$.

Suppose that $\pi^{-1}(i) > \pi^{-1}(i+1)$. Then $g_\pi = g_i g_{\pi_1}$, where ${\pi_1}^{-1}(i) < \pi_1^{-1}(i+1)$, and π_1 is also a $(2k, r)$-shuffle. Thus $e_i g_\pi = e_i g_i g_{\pi_1} = \lambda^{-1} e_i g_{\pi_1}$. Likewise, $g_i g_\pi = (g_i)^2 g_{\pi_1} = g_{\pi_1} - z g_\pi + z\lambda^{-1} e_i g_{\pi_1}$. We are therefore reduced to considering the case that $\pi^{-1}(i) < \pi^{-1}(i+1)$.

If $\pi^{-1}(i) < \pi^{-1}(i+1)$, then

$$g_i x = g_{s_i \pi}[w \otimes g_\tau]\alpha(g_\sigma) \subseteq W_n^{(r-2)} + V_n^{(r)},$$

by Lemma 5.36.

It remains to consider $e_i x$ when $\pi^{-1}(i) < \pi^{-1}(i+1)$.

If $\pi^{-1}(i+1) \le 2k$ or $\pi^{-1}(i) \ge 2k+1$, then $\pi^{-1}(i+1) = \pi^{-1}(i) + 1$. By Lemmas 5.18 and 5.36,

$$e_i x = g_\pi e_{\pi^{-1}(i)}[w \otimes g_\tau]\alpha(g_\sigma) \in g_\pi[W_{2k}^{(0)} \otimes W_r]\alpha(g_\sigma) \subseteq W_n^{(r-2)} + V_n^{(r)}.$$

The remaining case to consider is $\pi^{-1}(i) \le 2k$ and $\pi^{-1}(i+1) \ge 2k+1$. Define a permutation ϱ by

$$\varrho(j) = \begin{cases} j & \text{if } j < \pi^{-1}(i), \\ j+1 & \text{if } \pi^{-1}(i) \le j < 2k, \\ \pi^{-1}(i) & \text{if } j = 2k, \\ \pi^{-1}(i+1) & \text{if } j = 2k+1, \\ j-1 & \text{if } 2k+1 < j \le \pi^{-1}(i+1), \\ j & \text{if } j > \pi^{-1}(i+1). \end{cases}$$

Since $\varrho \in \mathfrak{S}_{2k} \times \mathfrak{S}_r$, $\ell(\pi\varrho) = \ell(\pi) + \ell(\varrho)$ and $g_{\pi\varrho} = g_\pi g_\varrho$. The permutation $\pi\varrho$ has the following properties: $\pi\varrho(2k) = i$; $\pi\varrho(2k+1) = i+1$; if $1 \le a < b \le 2k-1$ or $2k+2 \le a < b \le n$, then $\pi\varrho(a) < \pi\varrho(b)$. We have

$$e_i g_\pi = e_i g_\pi g_\varrho g_\varrho^{-1} = e_i g_{\pi\varrho} g_\varrho^{-1} = g_{\pi\varrho} e_{2k} g_\varrho^{-1}.$$

Therefore

$$e_i x \in g_{\pi\varrho} e_{2k}(W_{2k}^{(0)} \otimes W_r)\alpha(g_\sigma).$$

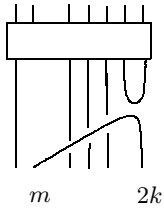
We concentrate on $e_{2k}(W_{2k}^{(0)} \otimes W_r)$.

LEMMA 5.38. *Suppose φ is injective on $W_{2k}^{(0)}$. Then $W_{2k}^{(0)}$ is spanned by elements of the form*

$$g_m g_{m+1} \cdots g_{2k-2} e_{2k-1} w,$$

where $1 \le m \le 2k-2$ and $w \in W_{2k}^{(0)}$.

Proof. By assumption, φ is an isomorphism from $W_{2k}^{(0)}$ to the ideal $KT_{2k}^{(0)}$ of KT_{2k} spanned by tangle diagrams of rank 0. Now, $KT_{2k}^{(0)}$ is spanned by totally descending tangle diagrams, and we can choose the order of the strands so that the strand incident with the vertex $\overline{\mathbf{2k}}$ lies above all other strands. Such a totally descending tangle diagram is isotopic to a tangle diagram of the following form:

$T =$ 

Such a tangle diagram has the factorization $T = G_m \cdots G_{2k-2} E_{2k-1} T'$. ∎

LEMMA 5.39. *$g_{\pi\varrho} e_{2k} g_m g_{m+1} \cdots g_{2k-2} e_{2k-1} = g_{\pi'} e_{2k-1}$ for some positive permutation braid $g_{\pi'}$.*

Proof. First, e_{2k} can be moved to the right of $g_m g_{m+1} \cdots g_{2k-2}$ and $e_{2k} e_{2k-1}$ can be written as $g_{2k-1} g_{2k} e_{2k-1}$ or as $g_{2k-1}^{-1} g_{2k}^{-1} e_{2k-1}$. Therefore

$$\begin{aligned} g_{\pi\varrho} e_{2k} g_m g_{m+1} \cdots g_{2k-2} e_{2k-1} &= g_{\pi\varrho}(g_m g_{m+1} \cdots g_{2k-2})(g_{2k-1} g_{2k}) e_{2k-1} \\ &= g_{\pi\varrho}(g_m g_{m+1} \cdots g_{2k-2})(g_{2k-1}^{-1} g_{2k}^{-1}) e_{2k-1}. \end{aligned}$$

Since $\pi\varrho(a) < \pi\varrho(b)$ if $1 \le a < b \le 2k-1$, $g_{\pi\varrho} g_m g_{m+1} \cdots g_{2k-2}$ is the positive permutation braid corresponding to the permutation $\pi\varrho s_m \cdots s_{2k-2}$. Moreover,

$$\begin{aligned} \pi\varrho s_m \ldots s_{2k-2}(2k-1) &= \pi\varrho(m), \\ \pi\varrho s_m \ldots s_{2k-2}(2k) &= \pi\varrho(2k) = i, \\ \pi\varrho s_m \ldots s_{2k-2}(2k+1) &= \pi\varrho(2k+1) = i+1. \end{aligned}$$

It follows that if $\pi\varrho(m) < i$, then $g_{\pi\varrho}(g_m g_{m+1} \cdots g_{2k-2})(g_{2k-1} g_{2k})$ is a positive permutation braid, while if $\pi\varrho(m) > i+1$, then $g_{\pi\varrho}(g_m g_{m+1} \cdots g_{2k-2}) \cdot (g_{2k-1}^{-1} g_{2k}^{-1})$ is a positive permutation braid. In either case, the desired conclusion follows. ∎

End of the proof of Lemma 5.37. We have to show that

$$g_{\pi\varrho} e_{2k} (W_{2k}^{(0)} \otimes W_r) \alpha(g_\sigma) \subseteq W_n^{(r-2)} + V_n^{(r)}.$$

By Lemma 5.38, it is enough to show

$$g_{\pi\varrho} e_{2k} (g_m \cdots g_{2k-2}) e_{2k-1} (W_{2k}^{(0)} \otimes W_r) \alpha(g_\sigma) \subseteq W_n^{(r-2)} + V_n^{(r)}.$$

But by Lemmas 5.39 and 5.36,

$$\begin{aligned} g_{\pi\varrho} e_{2k} (g_m \cdots g_{2k-2}) e_{2k-1} (W_{2k}^{(0)} \otimes W_r) \alpha(g_\sigma) \\ = g_{\pi'} e_{2k-1} (W_{2k}^{(0)} \otimes W_r) \alpha(g_\sigma) \subseteq W_n^{(r-2)} + V_n^{(r)}. \end{aligned}$$ ∎

PROPOSITION 5.40. *If φ is injective on W_{n-1} then*

$$V_n^{(r)} \oplus W_n^{(r-2)} = W_n^{(r)}.$$

Proof. We have shown that $V_n^{(r)} \oplus W_n^{(r-2)}$ is an ideal containing $e_1 \cdots e_{2k-1}$. Therefore $V_n^{(r)} \oplus W_n^{(r-2)} = W_n^{(r)}$. ∎

THEOREM 5.41. *For all n, $\varphi : W_n \to \mathrm{KT}_n$ is an isomorphism.*

Proof. Surjectivity was shown in Theorem 5.20. Injectivity is evident for $n = 0, 1$. We show φ is injective on all ideals $W_n^{(r)}$ by double induction on n and r. By Proposition 5.32, if φ is injective on W_{n-1}, then φ is injective on $W_n^{(0)}$ (if n is even) or on $W_n^{(1)}$ (if n is odd). By Corollary 5.35 and Proposition 5.40, if φ is injective on W_{n-1} and on $W_n^{(r-2)}$, then φ is injective on $W_n^{(r)}$. ∎

COROLLARY 5.42. *For all R and n, $W_{n,R}$ is a free R-module with basis $\{\varphi^{-1}(T_c) : c$ is an n-connector$\}$.*

COROLLARY 5.43. *For all R and n, $W_{n,R} \cong W_{n,\Lambda} \otimes_\Lambda R$.*

Proof. $W_{n,R} \cong \mathrm{KT}_{n,R}$ and $W_{n,\Lambda} \cong \mathrm{KT}_{n,\Lambda}$. But $\mathrm{KT}_{n,R} \cong \mathrm{KT}_{n,\Lambda} \otimes_\Lambda R$, by Corollary 5.9. ∎

COROLLARY 5.44. *If R is a ring with distinguished elements λ, z and δ, and $S \supseteq R$ is a ring containing R, then $W_{n,R}$ imbeds in $W_{n,S}$.*

COROLLARY 5.45. *If S is a ring with distinguished elements λ, z, δ, and q_r $(r \geq 1)$, then $i_n : W_{n,S} \to \widehat{W}_{n,S}$ is injective.*

Proof. Consider the commuting diagram

$$\begin{array}{ccc} \widehat{W}_{n,S} & \xrightarrow{\varphi} & \widehat{\mathrm{KT}}_{n,S} \\ \uparrow{\scriptstyle i_n} & & \uparrow{\scriptstyle i'_n} \\ W_{n,S} & \xrightarrow{\varphi} & \mathrm{KT}_{n,S} \end{array}$$

Since $\varphi : W_{n,S} \to \mathrm{KT}_{n,S}$ is an isomorphism, $i'_n = \varphi \circ i_n \circ \varphi^{-1}$. But i'_n is injective by Remark 2.11, and therefore i_n is injective as well. ∎

COROLLARY 5.46. *For all R and n,*

(1) *There is a conditional expectation $\varepsilon_n : W_{n,R} \to W_{n-1,R}$ such that $e_n x e_n = \delta \varepsilon_n(x)$ for $x \in W_{n,R}$.*
(2) *The map $\iota : W_{n-1,R} \to W_{n,R}$ is injective (and $\varepsilon_n \circ \iota(x) = x$ for $x \in W_{n-1,R}$).*
(3) *There is a trace $\varepsilon : W_{n,R} \to R$ such that $\varepsilon \circ \varepsilon_{n+1}(x) = \varepsilon(x)$ for $x \in W_{n+1,R}$ and $\varepsilon \circ \iota(x) = \varepsilon(x)$ for $x \in W_{n-1,R}$.*

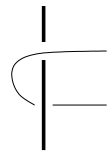
Proof. This follows from the isomorphism $W_{n,R} \cong \mathrm{KT}_{n,R}$ and the corresponding properties of $\mathrm{KT}_{n,R}$. ∎

6. ISOMORPHISM OF AFFINE BMW AND KAUFFMAN TANGLE ALGEBRAS

This section contains our main result, the isomorphism of $\widehat{W}_{n,S}$ and $\widehat{\mathrm{KT}}_{n,S}$ for any ring S with distinguished elements λ, z, δ, and q_r $(r \geq 1)$. We also give a basis of $\widehat{\mathrm{KT}}_{n,S}$ over S.

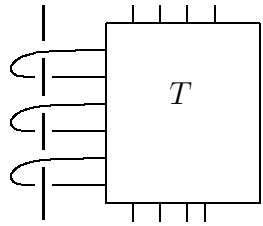
6.1. Surjectivity of $\varphi : \widehat{W}_n \to \widehat{\mathrm{KT}}_n$. Fix a ring S with distinguished elements λ, z, δ, and q_r $(r \geq 1)$.

Definition 6.1. A *simple winding* is a piece of an affine tangle diagram with one ordinary strand, without self-crossings, regularly isotopic to the intersection of one of the affine tangle diagrams X_1 or X_1^{-1} with a small neighborhood of the flagpole, as in the following figure:



Definition 6.2. An affine tangle diagram is in *standard position* if:

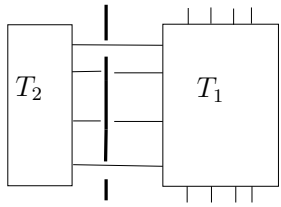
(1) It has no crossings to the left of the flagpole.
(2) There is a neighborhood of the flagpole whose intersection with the tangle diagram is a union of simple windings.
(3) The simple windings have no crossings and are not nested. That is, between the two crossings of a simple winding with the flagpole, there is no other crossing of a strand with the flagpole.



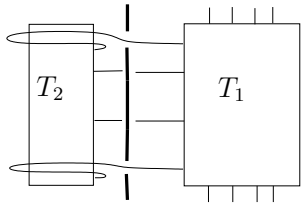
Lemma 6.3. *Any affine tangle diagram is regularly isotopic to an affine tangle diagram in standard position.*

Proof. An affine (n, n)-tangle can be viewed as the union of

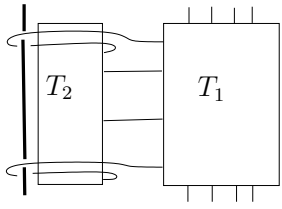
(1) an ordinary tangle diagram $T_1$with $2n + 2l$ boundary points to the right of the flagpole,
(2) an ordinary tangle diagram T_2 with $2l$ boundary points to the left of the flagpole, and
(3) $2l$ horizontal strands connecting T_1 and T_2 and crossing above or below the flagpole.



From each horizontal strand crossing over the flagpole, pull a finger over the tangle diagram T_2.



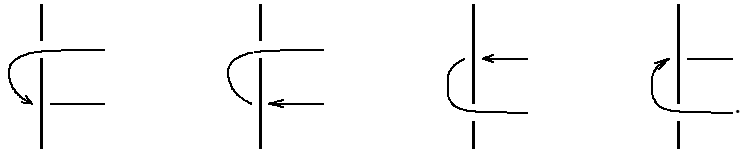
Now slide T_1 and T_2 to the right, with the diagram T_2 sliding under the flagpole. The result is a diagram in standard position.



■

We give each affine tangle diagram in standard position a standard orientation (see Section 5.1).

In the oriented diagram, the simple windings are of the four possible types pictured below:

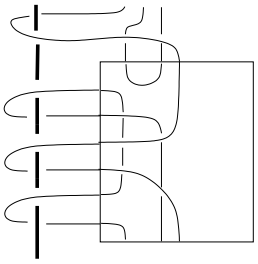


(a) (b) (c) (d)

LEMMA 6.4. *Let T be an affine tangle diagram in standard position and with a standard orientation. Suppose T has l simple windings and K crossings of ordinary strands. If T has a simple winding on a non-closed strand, then T can be written as $T = AT_0 + B$ or $T = T_0A + B$, where A is in the monoid generated by $\{X_1^{\pm 1}, G_i^{\pm 1}\}$, T_0 is an affine tangle diagram in standard position with $l-1$ simple windings, and B is in the span of affine tangle diagrams in standard position with fewer than K crossings of ordinary strands and at most l simple windings.*

Proof. Let s be a non-closed strand of T with a simple winding. Suppose that the initial endpoint of s is on the top boundary of R, say the jth vertex on the top boundary. (Otherwise, the final endpoint of s is on the bottom boundary of R, and this case can be handled similarly.) Consider the first simple winding on s.

If the simple winding is of type (a) or (d), then consider the affine tangle diagram T' obtained from T by reversing crossings of s as necessary so that all crossings of s with other ordinary strands are over-crossings, and all self-crossings of s are encountered first as over-crossings. By the Kauffman skein relation, T and T' are congruent modulo the span of affine tangle diagrams in standard position with fewer than K crossings (and at most l simple windings). Consider the diagram $T_0 = T'G_j^{-1} \cdots G_1^{-1}X_1^{-1}$. It has two simple windings, one of type (c) and the other of type (a) or (d), connected by an unknotted arc which has only over-crossings with other arcs; thus T_0 is regularly isotopic to an affine tangle diagram in standard position with $l-1$ simple windings:



We have $T = T' + B = T_0X_1G_1 \cdots G_j + B$, where B is in the span of affine tangle diagrams in standard position with fewer than K crossings and at most ℓ simple windings.

If the first simple winding on s is of type (b) or (c), then take T' to be the diagram obtained from T by changing crossings of s so that all crossings of s with other ordinary strands are under-crossings, and all self-crossings of s are encountered first as under-crossings, and take $T_0 = T'G_j \cdots G_1X_1$. Now T_0 has two simple windings, one of type (a) and the other of type (b) or (c), connected by an unknotted arc which has only under-crossings with other

arcs; again, T_0 is regularly isotopic to an affine tangle diagram in standard position with $l-1$ simple windings. ∎

Say that the oriented affine tangle diagram T is *totally descending* if, as T is traversed in accordance with the orientation, each crossing of ordinary strands is encountered the first time as an over-crossing. By the inductive argument of Proposition 5.5, $\widehat{\mathrm{KT}}_n$ is spanned by totally descending affine tangle diagrams in standard position.

THEOREM 6.5. *$\varphi : \widehat{W}_{n,S} \to \widehat{\mathrm{KT}}_{n,S}$ is surjective.*

Proof. We have to show that each totally descending oriented affine tangle diagram T in standard position is in the subalgebra of $\widehat{\mathrm{KT}}_n$ generated by $X_1^{\pm 1}$ and $\{E_i, G_i^{\pm 1}\}$.

The proof is by induction first on the number l of simple windings of T on non-closed strands, and then on the number K of crossings of ordinary strands of T.

Suppose that T has no simple windings on non-closed strands. Because T is totally descending and all simple windings are located on closed loops of T, T is regularly isotopic to a diagram in which the closed loops are confined to a neighborhood of the flagpole and the non-closed strands do not enter this neighborhood. It follows that $T \in \widehat{\mathrm{KT}}_{0,S} \otimes KT_{n,S}$, which is isomorphic to $KT_{n,S}$ by Turaev's theorem [19]. By Theorem 5.20, T is in the subalgebra generated by $\{E_i, G_i^{\pm 1}\}$.

Suppose T has $l \geq 1$ simple windings on non-closed strands and that any totally descending affine tangle diagram in standard position with fewer than l simple windings on non-closed strands is in the subalgebra of $\widehat{\mathrm{KT}}_{n,S}$ generated by $X_1^{\pm 1}$ and $\{E_i, G_i^{\pm 1}\}$. If the number K of crossings of ordinary strands is zero, then by Lemma 6.4, $T = AT_0$ or $T = T_0A$, where A is in the monoid generated by $\{X_1^{\pm 1}, G_i^{\pm 1}\}$, and T_0 is an affine tangle diagram in standard position with $l-1$ simple windings. By the induction assumption on l, the affine tangle T_0 and hence T is in the subalgebra of $\widehat{\mathrm{KT}}_{n,S}$ generated by $X_1^{\pm 1}$ and $\{E_i, G_i^{\pm 1}\}$.

Now suppose T has $l \geq 1$ simple windings on non-closed strands and $K \geq 1$ crossings of ordinary strands. Then, by Lemma 6.4, T can be written as $T = AT_0 + B$ or $T = T_0A + B$ where B is has fewer than K crossings, T_0 has fewer than l simple windings, and A is in the monoid generated by $\{X_1^{\pm 1}, G_i^{\pm 1}\}$. It follows from the induction assumptions on l and K that T is in the subalgebra of $\widehat{\mathrm{KT}}_n$ generated by $X_1^{\pm 1}$ and $\{E_i, G_i^{\pm 1}\}$. ∎

6.2. Freeness of $\widehat{W}_n$ and injectivity of $\varphi : \widehat{W}_n \to \widehat{\mathrm{KT}}_n$. Fix a ring S with distinguished elements λ, z, δ, and q_r ($r \geq 1$). Write W_n for $W_{n,S}$, KT_n for $\mathrm{KT}_{n,S}$, $\widehat{W}_n$ for $\widehat{W}_{n,S}$, and $\widehat{\mathrm{KT}}_n$ for $\widehat{\mathrm{KT}}_{n,S}$.

We can identify W_n with KT_n via φ, and we can regard W_n as imbedded in $\widehat{W}_n$ by Corollary 5.45 and KT_n imbedded in $\widehat{\mathrm{KT}}_n$ by Remark 2.11.

LEMMA 6.6. *Let $T' \in \mathrm{KT}_n$ be a totally descending (n,n)-tangle diagram whose strands have no self-crossings. Suppose that T' has a strand connecting a top vertex $\boldsymbol{a}$ with a bottom vertex $\overline{\boldsymbol{b}}$, and that this strand lies above all other strands (i.e. has only over-crossings with other strands). Let T'' be the otherwise identical tangle diagram in which the strand connecting $\boldsymbol{a}$ and $\overline{\boldsymbol{b}}$ lies under all other strands. Then*

$$x_b \varphi^{-1}(T') = \varphi^{-1}(T'') x_a.$$

Proof. If T' is a braid diagram, then the assertion follows from Propositions 3.10 and 3.11 and induction on the number of crossings.

Otherwise, T' is regularly isotopic to a tangle diagram of the form

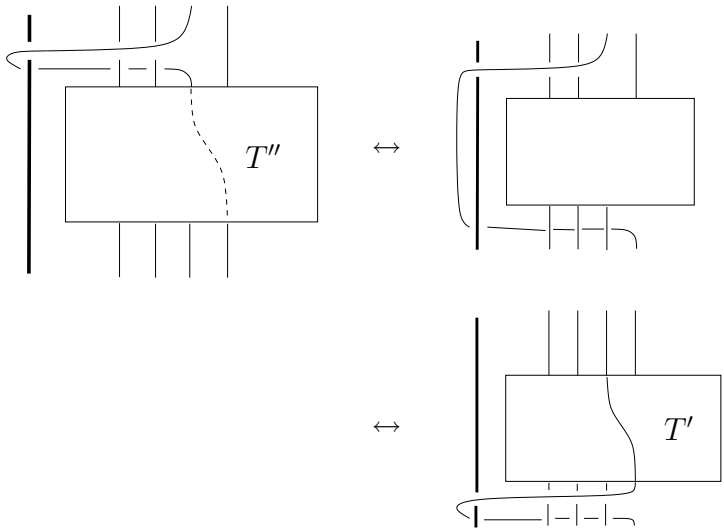
$$T' = B_1[(E_1 E_3 \cdots E_{2k-1}) \otimes B] B_2,$$

where B_1, B_2, and B are totally descending braid diagrams, as described in the proof of Theorem 5.20. The x_* can be successively passed through $\varphi^{-1}(B_1)$, $\varphi^{-1}(B)$, and $\varphi^{-1}(B_2)$, by the result for braid diagrams. ∎

REMARK 6.7. Applying φ, we get

$$X_b T' = T'' X_a.$$

This statement can also be verified by the following picture proof (which provides the motivation for Lemma 6.6):



LEMMA 6.8. *Let $T' \in \mathrm{KT}_n$ be a totally descending (n,n)-tangle diagram whose strands have no self-crossings. Suppose that T' has a strand connecting a bottom vertex $\overline{\boldsymbol{a}}$ with a bottom vertex $\overline{\boldsymbol{b}}$, and that this strand lies above all*

other strands (*i.e. has only over-crossings with other strands*). *Let* T'' *be the otherwise identical tangle diagram in which the strand connecting* $\overline{\boldsymbol{a}}$ *and* $\overline{\boldsymbol{b}}$ *lies under all other strands. Then*

$$x_a\varphi^{-1}(T') = \lambda^{-2}x_b^{-1}\varphi^{-1}(T'').$$

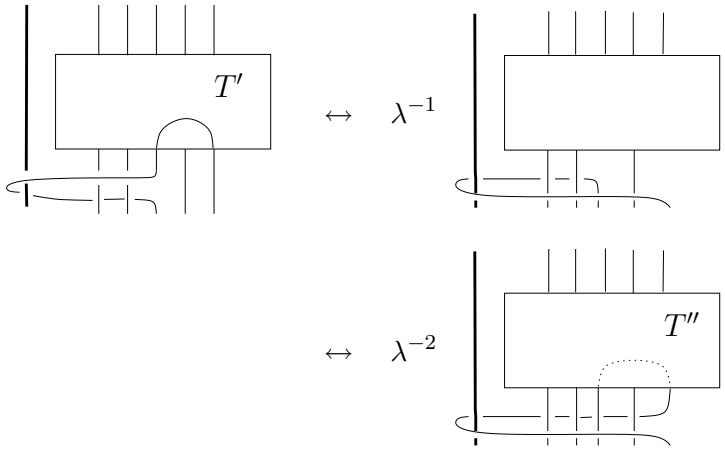
Proof. Assume without loss of generality that $a < b$. As in the proof of Lemma 5.38, T' has a factorization $T' = G_aG_{a+1}\cdots G_{b-2}E_{b-1}T_0$. Thus

$$\begin{aligned} x_a\varphi^{-1}(T') &= x_ag_a\cdots g_{b-2}e_{b-1}\varphi^{-1}(T_0)\\ &= g_a^{-1}\cdots g_{b-2}^{-1}x_{b-1}e_{b-1}\varphi^{-1}(T_0)\\ &= \lambda^{-2}g_a^{-1}\cdots g_{b-2}^{-1}x_b^{-1}e_{b-1}\varphi^{-1}(T_0)\\ &= \lambda^{-2}x_b^{-1}g_a^{-1}\cdots g_{b-2}^{-1}e_{b-1}\varphi^{-1}(T_0)\\ &= \lambda^{-2}x_b^{-1}\varphi^{-1}(T''). \ \blacksquare \end{aligned}$$

Remark 6.9. Applying φ, we get

$$X_aT' = \lambda^{-2}X_b^{-1}T''.$$

This statement can also be verified by the following picture proof (which provides the motivation for Lemma 6.8):



In the following we write $x^\mu = x_1^{\mu_1}\cdots x_n^{\mu_n}$ for a Laurent monomial in the commuting elements x_i.

Lemma 6.10 (Transport Lemma). *Let* $T \in \mathrm{KT}_n$ *be an ordinary* (n,n)*-tangle diagram.*

(1) *Suppose* T *has a strand connecting a top vertex* $\boldsymbol{a}$ *with a bottom vertex* $\overline{\boldsymbol{b}}$. *Then*

$$x_b\varphi^{-1}(T) \equiv \varphi^{-1}(T)x_a, \qquad x_b^{-1}\varphi^{-1}(T) \equiv \varphi^{-1}(T)x_a^{-1}$$

modulo the span of elements of the form $x^\mu\varphi^{-1}(T_0)x^\nu$, *where* T_0 *is an ordinary* (n,n)*-tangle diagram with fewer crossings than* T.

(2) *Suppose* T *has a strand connecting two bottom vertices* $\overline{\boldsymbol{a}}$ *and* $\overline{\boldsymbol{b}}$. *Then*

$$x_a\varphi^{-1}(T) \equiv \lambda^{-2}x_b^{-1}\varphi^{-1}(T)$$

modulo the span of elements of the form $x^\mu\varphi^{-1}(T_0)x^\nu$, *where* T_0 *is an ordinary* (n,n)*-tangle diagram with fewer crossings than* T.

(3) *Suppose* T *has a strand connecting two top vertices* $\boldsymbol{a}$ *and* $\boldsymbol{b}$. *Then*

$$\varphi^{-1}(T)x_a \equiv \lambda^{-2}\varphi^{-1}(T)x_b^{-1}$$

modulo the span of elements of the form $x^\mu\varphi^{-1}(T_0)x^\nu$, *where* T_0 *is an ordinary* (n,n)*-tangle diagram with fewer crossings than* T.

Proof. By the Kauffman skein relation, if T' is an ordinary (n,n)-tangle diagram which differs from T by changing one or more crossings, then T and T' are congruent modulo the span of (n,n)-tangle diagrams with fewer crossings.

Let T' be a diagram obtained from T by changing crossings, in which the strand connecting $\boldsymbol{a}$ and $\overline{\boldsymbol{b}}$ has only over-crossings with other strands, and let T'' be the otherwise identical diagram in which the strand connecting $\boldsymbol{a}$ and $\overline{\boldsymbol{b}}$ has only under-crossings with other strands. Then $T \equiv T' \equiv T''$ modulo the span of tangle diagrams with fewer crossings. By Lemma 6.6, $x_b\varphi^{-1}(T') = \varphi^{-1}(T'')x_a$, and the first assertion in statement (1) follows. The second assertion results from multiplying the congruence on the left by x_b^{-1} and on the right by x_a^{-1}.

Statement (2) follows similarly using Lemma 6.6, and statement (3) is obtained from (2) by applying the anti-automorphism α. ∎

PROPOSITION 6.11. $\widehat{W}_n$ *is spanned by elements of the form* $x^\mu w x^\nu$, *where* x^μ, x^ν *are Laurent monomials in the elements* x_i *and* $w \in W_n$.

Proof. Let M denote the span of elements of the form $x^\mu w x^\nu$. Because M contains the identity element, it suffices to show that M is a left ideal in $\widehat{W}_n$.

Since $\widehat{W}_n$ is generated as a unital algebra by $x_1^{\pm 1}$ and $\{g_i^{\pm 1}, e_i\}$, it suffices to show that M is invariant under left multiplication by these generators. Invariance under left multiplication by $x_1^{\pm 1}$ is obvious (since the x_i's commute). Because of Propositions 3.10 and 3.11, invariance under left multiplication by $g_i^{\pm 1}$ will follow from invariance under left multiplication by e_i.

For $k \geq 0$, let M_k be the span of elements of the form $x^\mu\varphi^{-1}(T)x^\nu$, where x^μ, x^ν are monomials in the elements x_i and T is an ordinary (n,n)-tangle diagram with no more than k crossings. Let $M_{-1} = (0)$. We show by induction first on i and then on k that $e_iM_k \subseteq M$.

Suppose T is an ordinary (n,n)-tangle diagram with no more than k crossings and $\overline{\boldsymbol{b}}$ and $\boldsymbol{a}$ are connected by a strand of T. Then, by Lemma 6.10, $x_b\varphi^{-1}(T) \equiv \varphi^{-1}(T)x_a$ and $x_b^{-1}\varphi^{-1}(T) \equiv \varphi^{-1}(T)x_a^{-1}$ modulo M_{k-1}. In particular, if $k = 0$, then $x_b\varphi^{-1}(T) = \varphi^{-1}(T)x_a$ and $x_b^{-1}\varphi^{-1}(T) = \varphi^{-1}(T)x_a^{-1}$. Likewise, if $\overline{\boldsymbol{b}}$ and $\overline{\boldsymbol{a}}$ are connected by a strand of T, then $x_b\varphi^{-1}(T) \equiv \lambda^{-2}x_a^{-1}\varphi^{-1}(T)$ modulo M_{k-1}, and in particular, if $k = 0$, then $x_b\varphi^{-1}(T) = \lambda^{-2}x_a^{-1}\varphi^{-1}(T)$.

It follows from these observations that, for any i,

$$x^\mu\varphi^{-1}(T) \equiv \varrho x^{\mu'}\varphi^{-1}(T)x^{\nu'} \bmod M_{k-1},$$

where $\varrho \in S$ and

(1) if $\overline{\boldsymbol{i}}$ and $\overline{\boldsymbol{i+1}}$ are not connected by a strand of T, then $\mu'_i = \mu'_{i+1} = 0$,
(2) if $\overline{\boldsymbol{i}}$ and $\overline{\boldsymbol{i+1}}$ are connected by a strand of T, then $\mu'_{i+1} = 0$.

For all i, if $\overline{\boldsymbol{i}}$ and $\overline{\boldsymbol{i+1}}$ are not connected by a strand of T, then

$$e_ix^\mu\varphi^{-1}(T) \equiv \varrho e_ix^{\mu'}\varphi^{-1}(T)x^{\nu'} = \varrho x^{\mu'}e_i\varphi^{-1}(T)x^{\nu'} \bmod e_iM_{k-1}. \tag{6.1}$$

Suppose $\overline{\boldsymbol{i}}$ and $\overline{\boldsymbol{i+1}}$ are connected by a strand s of T. We have

$$e_ix^\mu\varphi^{-1}(T) \equiv \varrho e_ix^{\mu'}x_i^r\varphi^{-1}(T) \bmod e_iM_{k-1} = \varrho x^{\mu'}e_ix_i^r\varphi^{-1}(T), \tag{6.2}$$

where $\mu'_i = \mu'_{i+1} = 0$. Modulo the span of diagrams with fewer crossings, T is congruent to a totally descending tangle diagram T'. Moreover, $T' = E_iT''$ (where T'' may have more than k crossings). In fact, T' must have a strand s' connecting two points on the top boundary of $\mathbb{R} \times I$. By regular isotopy, s can be contracted and s' pulled down so that the minimum on s' lies just above the maximum on s and both lie below any other crossing or local extremum of the tangle diagram. Thus

$$\begin{aligned} e_ix^\mu\varphi^{-1}(T) &\equiv \varrho x^{\mu'}e_ix_i^re_i\varphi^{-1}(T'') \bmod e_iM_{k-1} \\ &= \varrho x^{\mu'}be_i\varphi^{-1}(T''), \end{aligned} \tag{6.3}$$

where $b \in \widehat{W}_{n-1}$, by Proposition 3.17. It follows from equations (6.1) and (6.3) that for all i and k,

$$e_iM_k \subseteq M_k + e_iM_{k-1} + A\widehat{W}_{i-1}M, \tag{6.4}$$

where A is the algebra of Laurent polynomials in $\{x_j : 1 \le j \le n\}$. For $i = 1$, we have

$$e_1M_k \subseteq M_k + e_1M_{k-1}.$$

It follows by induction on k that $e_1M_k \subseteq M_k$ for all k; that is, $e_1M \subseteq M$. Now fix $i > 1$ and assume that $e_jM \subseteq M$ for all $j < i$; it follows that $\widehat{W}_{i-1}M \subseteq M$, and thus from (6.4), for all k,

$$e_iM_k \subseteq M + e_iM_{k-1}.$$

By induction on k, $e_iM_k \subseteq M$ for all k. ∎

Let $B_0 = \{T_d : d \text{ is an } n\text{-connector}\}$ be the common basis of $\mathrm{KT}_{n,R}$ for all R, as described at the end of Section 5.1. Let B be the set of elements $X^\mu T X^\nu$, where $T \in B_0$ and $\nu_i = 0$ unless T has a strand with initial point $\boldsymbol{i}$, and $\mu_j = 0$ unless T has a strand with initial point $\overline{\boldsymbol{j}}$. The set B may be regarded as a subset of $\widehat{\mathrm{KT}}_{n,S}$ for any S.

For any S, we lift B to a subset of $\widehat{W}_{n,S}$ as follows: $\varphi : W_{n,S} \to \mathrm{KT}_{n,S}$ is an isomorphism. We take $A = A(n,S)$ to be the set of elements of $\widehat{W}_{n,S}$ of the form $x^\mu \varphi^{-1}(T) x^\nu$, where $T \in B_0$ and $\nu_i = 0$ unless T has a strand with initial point $\boldsymbol{i}$, and $\mu_j = 0$ unless T has a strand with initial point $\overline{\boldsymbol{j}}$.

Proposition 6.12. *For every S and n,*

(1) *$A(n,S)$ spans $\widehat{W}_{n,S}$.*
(2) *B spans $\widehat{\mathrm{KT}}_{n,S}$.*

Proof. Because of Proposition 6.11, to prove statement (1) it suffices to show that every element of the form $x^\mu \varphi^{-1}(T) x^\nu$, where T is an ordinary (n,n)-tangle diagram, is in the span of $A = A(n,S)$. The basis B_0 has the following triangular property: any ordinary (n,n)-tangle diagram is congruent to a multiple of an element of B_0 modulo the span of diagrams with fewer crossings. Combining this with the Transport Lemma 6.10, we can prove that $x^\mu \varphi^{-1}(T) x^\nu$ is in the span of A by induction on the number of crossings of T.

Statement (2) follows from statement (1) by applying the surjective map $\varphi : \widehat{W}_{n,S} \to \widehat{\mathrm{KT}}_{n,S}$. ∎

Theorem 6.13. *For all n,*

(1) *$\widehat{W}_{n,\widehat{\Lambda}}$ is a free $\widehat{\Lambda}$-module with basis A.*
(2) *$\widehat{\mathrm{KT}}_{n,\widehat{\Lambda}}$ is a free $\widehat{\Lambda}$-module with basis B.*
(3) *$\varphi : \widehat{W}_{n,\widehat{\Lambda}} \to \widehat{\mathrm{KT}}_{n,\widehat{\Lambda}}$ is an isomorphism.*

Proof. $A = A(n,\widehat{\Lambda})$ spans $\widehat{W}_{n,\widehat{\Lambda}}$, and $B = \varphi(A)$ spans $\widehat{\mathrm{KT}}_{n,\widehat{\Lambda}}$, by Proposition 6.12. Since c is injective on B, Proposition 4.7 implies that B is linearly independent over $\widehat{\Lambda}$. All the conclusions follow. ∎

Corollary 6.14.

(1) *$\widehat{\mathrm{KT}}_{n,S}$ is a free S module with basis B and $\widehat{\mathrm{KT}}_{n,S} \cong \widehat{\mathrm{KT}}_{n,\widehat{\Lambda}} \otimes_{\widehat{\Lambda}} S$.*
(2) *$\varphi : \widehat{W}_{n,S} \to \widehat{\mathrm{KT}}_{n,S}$ is an isomorphism.*
(3) *$\widehat{W}_{n,S} \cong \widehat{W}_{n,\widehat{\Lambda}} \otimes_{\widehat{\Lambda}} S$.*

Proof. By Proposition 6.12, B spans $\widehat{\mathrm{KT}}_{n,S}$. On the other hand, $B \otimes 1$ is a basis of $\widehat{\mathrm{KT}}_{n,\widehat{\Lambda}} \otimes_{\widehat{\Lambda}} S$. Moreover, we have an S-algebra homomorphism

from $\widehat{\mathrm{KT}}_{n,S}$ to $\widehat{\mathrm{KT}}_{n,\widehat{\Lambda}}\otimes_{\widehat{\Lambda}}S$ which takes a tangle T to $T\otimes 1$. Since this map takes a spanning set to a basis, it is an isomorphism. This proves (1).

Once we know that B is a basis of $\widehat{\mathrm{KT}}_{n,S}$, it follows at once that $\varphi : \widehat{W}_{n,S} \to \widehat{\mathrm{KT}}_{n,S}$ is an isomorphism. Namely, φ carries the spanning set $A(n,S)$ to the basis B, and so is an isomorphism.

Finally, (3) follows by combining the isomorphisms

$$\widehat{W}_{n,S}\cong\widehat{\mathrm{KT}}_{n,S},\qquad \widehat{\mathrm{KT}}_{n,S}\cong\widehat{\mathrm{KT}}_{n,\widehat{\Lambda}}\otimes_{\widehat{\Lambda}}S,\qquad \widehat{W}_{n,\widehat{\Lambda}}\otimes_{\widehat{\Lambda}}S\cong\widehat{\mathrm{KT}}_{n,\widehat{\Lambda}}\otimes_{\widehat{\Lambda}}S.\ \blacksquare$$

COROLLARY 6.15. *The map $\iota : \widehat{W}_{n-1,S}\to\widehat{W}_{n,S}$ is injective.*

COROLLARY 6.16.

(1) *For each $n\ge 1$, there is a conditional expectation $\varepsilon_n : \widehat{W}_{n,S}\to\widehat{W}_{n-1,S}$ satisfying*
$$e_nxe_n=\delta\varepsilon_n(x)e_n$$
for $x\in\widehat{W}_{n,S}$.

(2) *There is a trace $\varepsilon:\widehat{W}_{n,S}\to S$ defined by*
$$\varepsilon=\varepsilon_1\circ\cdots\circ\varepsilon_n.$$

(3) *The trace ε has the Markov property: for $b\in\widehat{W}_{n-1,S}$,*
   (a) $\varepsilon(bg_{n-1}^{\pm1})=(\lambda^{\pm1}/\delta)\varepsilon(x)$,
   (b) $\varepsilon(be_{n-1})=(1/\delta)\varepsilon(x)$,
   (c) $\varepsilon(b(x_n')^r)=q_r\varepsilon(b)$ *and* $\varepsilon(b(x_n')^{-r})=f_r(q_1,\dots,q_r)\varepsilon(b)$ *for $r\ge1$, where* $x_n'=(g_{n-1}\cdots g_1)x_1(g_1^{-1}\cdots g_{n-1}^{-1})$.

Proof. All statements follow from the isomorphism $\widehat{W}_{n,S}\cong\widehat{\mathrm{KT}}_n$ and the properties of $\widehat{\mathrm{KT}}_{n,S}$ discussed in Section 2.7. ∎

REMARK 6.17. The trace ε is the unique trace on $\widehat{W}_{n,S}$ satisfying $\varepsilon(1)=1$ and the Markov properties enumerated in statement (3) of the previous corollary. To prove this uniqueness statement, one needs an analogue of Proposition 3.16 with x_n replaced by x_n'.

COROLLARY 6.18. *For $x\in\widehat{W}_{n+1}$,*
$$xe_n=\delta\varepsilon_{n+1}(xe_n)e_n.$$

Proof. By Proposition 3.20, there exists a $y\in\widehat{W}_n$ such that $xe_n=ye_n$. Applying the conditional expectation ε_{n+1} to both sides gives $\varepsilon_{n+1}(xe_n)=y\varepsilon_{n+1}(e_n)=\delta^{-1}y$. ∎

COROLLARY 6.19. *Every element of $\widehat{\mathrm{KT}}_{n,S}$ is a linear combination of elements of the form $a\chi b$, where $a,b\in\widehat{\mathrm{KT}}_{n-1,S}$ and*
$$\chi\in\{E_{n-1},G_{n-1}^{\pm1}\}\cup\{X_n^r : r\in\mathbb{Z}\}.$$

Proof. This follows from the isomorphism $\widehat{W}_{n,S} \cong \widehat{\mathrm{KT}}_{n,S}$ and Proposition 3.16. ∎

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