FUNDAMENTA
MATHEMATICAE
192 (2006)

Generalized E-algebras via λ-calculus I

by

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Abstract. An R-algebra A is called an $E(R)$-algebra if the canonical homomorphism from A to the endomorphism algebra $\mathrm{End}_R A$ of the R-module ${}_RA$, taking any $a \in A$ to the right multiplication $a_r \in \mathrm{End}_R A$ by a, is an isomorphism of algebras. In this case ${}_RA$ is called an $E(R)$-module. There is a proper class of examples constructed in [4]. $E(R)$-algebras arise naturally in various topics of algebra. So it is not surprising that they were investigated thoroughly in the last decades; see [3, 5, 7, 8, 10, 13, 14, 15, 18, 19]. Despite some efforts ([14, 5]) it remained an open question whether proper generalized $E(R)$-algebras exist. These are R-algebras A isomorphic to $\mathrm{End}_R A$ but not under the above canonical isomorphism, so not $E(R)$-algebras. This question was raised about 30 years ago (for $R = \mathbb{Z}$) by Schultz [21] (see also Vinsonhaler [24]). It originates from Problem 45 in Fuchs [9], that asks for a characterization of the rings A for which $A \cong \mathrm{End}_{\mathbb{Z}} A$ (as rings). We answer Schultz's question, thus contributing a large class of rings for Fuchs' Problem 45 which are not E-rings. Let R be a commutative ring with an element $p \in R$ such that the additive group R^+ is p-torsion-free and p-reduced (equivalently p is not a zero-divisor and $\bigcap_{n\in\omega} p^n R = 0$). As explained in the introduction we assume that either $|R| < 2^{\aleph_0}$ or R^+ is free (see Definition 1.1).

The main tool is an interesting connection between λ-calculus (used in theoretical computer science) and algebra. It seems reasonable to divide the work into two parts; in this paper we work in $V = L$ (Gödel's universe) where stronger combinatorial methods make the final arguments more transparent. The proof based entirely on ordinary set theory (the axioms of ZFC) will appear in a subsequent paper [12]. However the general strategy will be the same, but the combinatorial arguments will utilize a prediction principle that holds under ZFC.

1. Introduction to generalized $E(R)$-algebras. Let $\mathbb{S}$ be a countable, multiplicatively closed subset of a commutative ring R with 1. An

2000 *Mathematics Subject Classification*: Primary 20K20, 20K30; Secondary 16S60, 16W20.

Key words and phrases: E-rings, endomorphism rings.

Supported by the project No. I-706-54.6/2001 of the German-Israeli Foundation for Scientific Research & Development.

GbSh867 in Shelah's archive.

R-module M is $\mathbb{S}$-*reduced* if $\bigcap_{s\in\mathbb{S}} sM = 0$ and it is $\mathbb{S}$-*torsion-free* if $sm = 0$, $m \in M$, $s \in \mathbb{S}$, implies $m = 0$. Suppose that R (as an R-module) is $\mathbb{S}$-reduced and $\mathbb{S}$-torsion-free. Then R is called an $\mathbb{S}$-*ring* (see [16]). In order to avoid zero-divisors as in the case of $\mathbb{Z}$-adic completion $\prod_p J_p$ of $\mathbb{Z}$ we also assume that $\mathbb{S}$ is *cyclically generated*, i.e. $\mathbb{S} = \langle p \rangle := \{p^n : n \in \omega\}$ for some $p \in R$. We will concentrate on $\mathbb{S}$-cotorsion-free modules. An $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced R-module M is $\mathbb{S}$-*cotorsion-free* if $\mathrm{Hom}(\widehat{R}, M) = 0$, where $\widehat{R}$ denotes the $\mathbb{S}$-completion of R. A submodule $U \subseteq M$ is $\mathbb{S}$-*pure* (we also write $U \subseteq_* M$) if $sM \cap U \subseteq sU$ for all $s \in \mathbb{S}$. Note that R, being $\mathbb{S}$-reduced, is Hausdorff in the $\mathbb{S}$-topology. In the proof of Step Lemma 5.3 we will also use the following condition on the additive group R^+ of R which implies that R is $\mathbb{S}$-cotorsion-free.

Definition 1.1. An R-module M is $\Sigma\mathbb{S}$-*incomplete* if for any sequence $0 \neq m_n \in M$ $(n \in \omega)$ there are $a_n \in \{0, 1\}$ with $\sum_{n\in\omega} p^n a_n m_n \notin M$. If $M = R^+$ we say that R is $\Sigma\mathbb{S}$-*incomplete*.

All $\mathbb{S}$-rings of size $< 2^{\aleph_0}$ are $\Sigma\mathbb{S}$-incomplete, as shown in [11]. Thus it follows easily that any $\mathbb{S}$-ring which is a direct sum of $\mathbb{S}$-invariant subgroups of size $< 2^{\aleph_0}$ is $\Sigma\mathbb{S}$-incomplete as well. So we deduce from [11]

Corollary 1.2. *If an $\mathbb{S}$-ring R is a direct sum of $\mathbb{S}$-invariant subgroups of size $< 2^{\aleph_0}$, then R is $\Sigma\mathbb{S}$-incomplete. In particular, if $\mathbb{S}$ generates the ordinary p-adic topology* (*i.e. for $1 \in R$ there is $p \in \langle 1 \rangle$ and $\mathbb{S} = \langle p \rangle$*) *and the additive group R^+ is free, then R is $\Sigma\mathbb{S}$-incomplete.*

We recall the main definition.

Definition 1.3. If A is an R-algebra, then $\delta : A \to \mathrm{End}_R A$ denotes the homomorphism which takes any $a \in A$ to the R-endomorphism $a\delta = a_r$ which is multiplication by a on the right. If this homomorphism is an isomorphism, then A is called an $E(R)$-*algebra* and ${}_RA$ is called an $E(R)$-*module*. By ${}_RA$ we denote the R-module structure of an R-algebra A.

$E(R)$-algebras can also be defined dually, assuming that the homomorphism

$$\mathrm{End}_R A \to A \quad (\varphi \mapsto 1\varphi)$$

is an isomorphism. It is easy to see that $E(R)$-algebras are necessarily commutative.

For any $\mathbb{S}$-ring (with $\mathbb{S}$ cyclically generated) that is $\Sigma\mathbb{S}$-incomplete we will construct non-commutative R-algebras with $\mathrm{End}_R A \cong A$. Hence these As are generalized $E(R)$-algebras but not $E(R)$-algebras. If $R = \mathbb{Z}$ and ${}_RA$ is an abelian group, then we do not mention the ring $\mathbb{Z}$: e.g. $E(\mathbb{Z})$-modules are just $\mathbb{E}$-groups. The existence of generalized $\mathbb{E}$-rings answers a problem in [21, 24].

If κ is a cardinal, then let $\kappa^{\mathrm{o}} = \{\alpha : \mathrm{cf}(\alpha) = \omega,\ \alpha \in \kappa\}$. We will only need the existence of a non-reflecting subset $E \subseteq \kappa^{\mathrm{o}}$ for some regular uncountable, not weakly compact cardinal κ such that the diamond principle $\diamondsuit_\kappa E$ holds. It is well known (see e.g. Eklof–Mekler [6]) that $\diamondsuit_\kappa E$ holds for all non-reflecting subsets E of regular uncountable, not weakly compact cardinals κ in Gödel's universe (V = L). We indicate our (weaker) set-theoretic hypothesis (which also holds in other universes) as $\diamondsuit_\kappa E$ in our following main result.

THEOREM 1.4. *Let R be a $\Sigma\mathbb{S}$-incomplete $\mathbb{S}$-ring for some cyclically generated $\mathbb{S}$. If $\kappa > |R|$ is a regular, uncountable cardinal and $E \subseteq \kappa^{\mathrm{o}}$ a non-reflecting subset with $\diamondsuit_\kappa E$, then there is an $\mathbb{S}$-cotorsion-free, non-commutative R-algebra A of cardinality $|A| = \kappa$ with $\mathrm{End}_R A \cong A$. Moreover any subset of cardinality $< \kappa$ is contained in an R-monoid-algebra of cardinality $< \kappa$.*

A similar result without the set-theoretic assumption will be shown in [12]:

THEOREM 1.5. *Let R be a $\Sigma\mathbb{S}$-incomplete $\mathbb{S}$-ring for some cyclically generated $\mathbb{S}$. For any cardinal $\kappa = \mu^+$ with $|R| \leq \mu^{\aleph_0} = \mu$ there is an $\mathbb{S}$-cotorsion-free, non-commutative R-algebra A of cardinality $|A| = \kappa$ with $\mathrm{End}_R A \cong A$.*

It seems particularly interesting to note that the R-monoid A comes from (classical) λ-calculus taking into account that elements of an $E(R)$-algebra A are at the same time endomorphisms of A, thus the same phenomenon appears as known in computer science and studied intensively in logic in the thirties of the last century. The problems concerning the semantics of computer science were solved four decades later by Scott [22, 23]. We will describe the construction of the underlying monoid M explicitly. Since this paper should be readable for algebraists with only basic background in model theory, we will also elaborate the needed details coming from model theory. The basic knowledge on model theory is in [20], for example. In Sections 4 and 5 the monoid M will be completed and become the algebra A.

2. Model theory of bodies and skeletons via λ-calculus

2.1. *Discussion.* We begin by defining terms for a skeleton and will establish a connection with λ-calculus. Let R be any commutative $\mathbb{S}$-ring with $\mathbb{S} = \langle p \rangle$. ($\Sigma\mathbb{S}$-incompleteness will be added in Section 5.)

By definition of generalized $E(R)$-algebras A, endomorphisms of ${}_RA$ must be considered as members of A. Hence they act on ${}_RA$ as endomorphisms while they are elements of ${}_RA$ at the same time. Thus we will introduce the classical definitions from λ-calculus over an infinite set X of *free* variables and an infinite set Y of *bound* variables to represent those maps. First note that we can restrict ourselves to unary, linear functions because endomorphisms are of this kind. (The general argument to reduce λ-calculus to unary functions was given by Schönfinkel; see [1, p. 6].) What are the typical terms of our final objects, the bodies? If x_1 and x_2 are members of the generalized $E(R)$-algebras A and $a, b \in R$, then also polynomials like $\sigma_n(x_1, x_2) = ax_1^n + bx_2^3$ belong to the algebra A, so there are legitimate functions $p_n(y) = \lambda y.\sigma_n(y, x_2)$ on A taking $y \mapsto \sigma_n(y, x_2)$ and A must be closed under such "generalized polynomials". This observation will be described in Definition 4.2 and taken care of in Proposition 2.20 and in our Main Lemma 6.2. A first description of these generalized polynomials will also be the starting point for our construction and we begin with its basic settings.

2.2. *The notion of terms.* Let τ be a vocabulary with no predicates; thus τ is a collection of function symbols with an arity function $\tau \to \omega$ defining the places of function symbols. Moreover, let X be an infinite set of free variables. Then unspecified (τ, X)-terms (briefly called "terms") are defined inductively as the closure of the atomic terms under these function symbols (only), that is:

(i) Atomic terms are the 0-place functions: the individual constants (in our case 1) and members x from X.
(ii) The closure: If $\sigma_0, \dots, \sigma_{n-1}$ are terms and F is an n-place function symbol from τ, then $F(\sigma_0, \dots, \sigma_{n-1})$ is a term.

We also define the (usual) length $l(\sigma)$ of a term σ inductively: Let $l(\sigma) = 0$ if σ is atomic and $l(\sigma) = k + 1$ if σ derives from (ii) with $k = \max\{l(\sigma_i) : i < n\}$.

If σ is an unspecified (τ, X)-term, then we define (also by induction on $l(\sigma)$) a finite subset $\mathrm{FV}(\sigma) \subset X$ of *free variables* of σ:

(a) If σ is an individual constant, then $\mathrm{FV}(\sigma) = \emptyset$, and if $\sigma \in X$, then $\mathrm{FV}(\sigma) = \{\sigma\}$.
(b) If $\sigma = F(\sigma_0, \dots, \sigma_{n-1})$ is defined as in (ii), then

$$\mathrm{FV}(\sigma) = \bigcup_{i<n} \mathrm{FV}(\sigma_i).$$

We fix some further notation. Let $\overline{x} = \langle x_0, \dots, x_{n-1} \rangle$ be a finite sequence of members x_i from X without repetitions and $\mathrm{Im}(\overline{x}) = \{x_i : i < n\}$ (and similarly $\overline{x}'$). Then we define the (specified) (τ, X)-terms: A (τ, X)-term is a pair $\boldsymbol{\sigma} = (\sigma, \overline{x})$ with σ an unspecified (τ, X)-term and $\overline{x}$ a finite sequence from X with $\mathrm{FV}(\sigma) \subseteq \mathrm{Im}(\overline{x})$. If there is no danger of confusion, then we will also write $\boldsymbol{\sigma} = \sigma(\overline{x})$. If $t(\tau, X)$ is the set of all (τ, X)-terms, then $t(\tau) := \{\sigma : \boldsymbol{\sigma} \in t(\tau, X)\}$ is the set of all unspecified (τ, X)-terms. Furthermore observe that for $\overline{x} \subseteq \overline{x}'$ (as maps) with $(\sigma, \overline{x})$ a (τ, X)-term also $(\sigma, \overline{x}')$ is a (τ, X)-term. For (τ, X)-terms we can define a natural substitution: if $(\sigma, \overline{x}) \in t(\tau, X)$ with $\overline{x} = \langle x_0, \dots, x_{n-1}\rangle$ and $\sigma_0, \dots, \sigma_{n-1} \in t(\tau)$, then substitution is defined by

$$\mathrm{Sub}\,^{\langle x_0,\dots,x_{n-1}\rangle}_{\langle \sigma_0,\dots,\sigma_{n-1}\rangle}(\sigma, \overline{x}) := \sigma(\sigma_0, \dots, \sigma_{n-1}),$$

replacing every occurrence of x_i by σ_i. If we replace (if necessary) free variables of the σ_i, we can find a sequence $\overline{x}'$ with $(\sigma(\sigma_0, \dots, \sigma_{n-1}), \overline{x}') \in t(\tau, X)$. This is a good place for two standard notations: let $\overline{b} = \langle b_0, \dots, b_{n'-1}\rangle$ be a finite sequence of elements without repetition from a set B. If $n = n'$ and if $(\sigma, \overline{x})$ is as above, we say that $\overline{b}$ is a *sequence from* B (*suitable*) *for* $\overline{x}$ and write $\sigma(\overline{b}) = \mathrm{Sub}\frac{\overline{x}}{\overline{b}}(\sigma, \overline{x})$.

A free variable $x \in X$ is a *dummy variable* of $(\sigma, \overline{x})$ if $x \in \mathrm{Im}(\overline{x}) \setminus \mathrm{FV}(\sigma)$, and we say that $(\sigma, \overline{x})$ is X-*reduced* if it has no dummy variables, i.e. $\mathrm{FV}(\sigma) = \mathrm{Im}(\overline{x})$. Trivially, for any $(\sigma, \overline{x})$ we get a natural X-reduced term by removing those entries of $\overline{x}$ that correspond to dummy variables. In this case $\overline{x} = \langle x_0, \dots, x_{n-1}\rangle$ becomes $\overline{x}' = \langle x_{i_0}, \dots, x_{i_{t-1}}\rangle$ for some $0 \le i_0 < i_1 < \dots < i_{t-1} \le n-1$ and we can use substitution to replace $\overline{x}'$ by the more natural sequence $\overline{x}'' = \langle x_0, \dots, x_{t-1}\rangle$: if $\sigma' := \mathrm{Sub}\,^{\langle x_{i_0},\dots,x_{i_{t-1}}\rangle}_{\langle x_0,\dots,x_{t-1}\rangle}(\sigma, \overline{x}')$, then $(\sigma, \overline{x}') = (\sigma', \overline{x}'')$ (by an axiom below).

2.3. *The vocabulary of a skeleton and its laws.* Let Y be an infinite set of so-called *bound* variables (used as variables for function symbols) and (as before) let $\overline{y} = \langle y_0, \dots, y_{n-1}\rangle$ be a finite sequence of elements from Y without repetitions. Also in this particular case of the *vocabulary* τ^{sk} *of a skeleton* the collection τ^{sk} will consist of an individual constant 1, of variables and of function symbols (only), defined inductively as τ^{sk}_k $(k \in \omega)$; moreover, let $\tau^{\mathrm{sk}}_{<k} := \bigcup_{m<k} \tau^{\mathrm{sk}}_m$ for $k \le \omega$, $\tau^{\mathrm{sk}}_{\le k} := \bigcup_{m \le k} \tau^{\mathrm{sk}}_m$ for $k < \omega$ and $\tau^{\mathrm{sk}} := \tau^{\mathrm{sk}}_{<\omega}$.

- (i) (Step $k = 0$) The vocabulary τ^{sk}_0 consists of an individual constant 1, free variables $x \in X$ and bound variables $y \in Y$. Moreover, we need a particular "2-arity word product" function symbol $F_\odot$ such that $F_\odot(x_0, x_1) = x_0 x_1$ is *concatenation.*
- (ii) (Step $k = m+1$) Suppose that $\tau^{\mathrm{sk}}_{\le m}$ is defined and $(\sigma, \overline{x}) \in t(\tau^{\mathrm{sk}}_{\le m}, X)$ is a specified term of length k with $\overline{x} = \langle x_0, \dots, x_{n-1}\rangle$ and $\overline{y} = \langle y_0, \dots, y_{n-1}\rangle$ suitable for $\overline{x}$. Then $F_{\sigma(y_0,\dots,y_{n-1})}$ is an n-place function symbol belonging to τ^{sk}_k (but not to $\tau^{\mathrm{sk}}_{<k}$).

For the collection of terms of the skeletons we will write $t(\tau^{\mathrm{sk}}_{<k}, X)$ (where $k \le \omega$ is as above) and $t(\tau^{\mathrm{sk}}, X) := t(\tau^{\mathrm{sk}}_{<\omega}, X)$. Its members $(\sigma, \overline{x})$ will also be called (*generalized*) *monomials* (because they are expressed as products).

We now define inductively

(2.1) *The theory of skeletons, i.e. the axioms $T^{\rm sk}_{<k}$ for $\tau^{\rm sk}_{<k}$ $(k \le \omega)$.* In the following let $\overline{x} = \langle x_0, \dots, x_n \rangle$, put $T^{\rm sk}_{<k} := \bigcup_{m<k} T^{\rm sk}_m$ and $T^{\rm sk} := T^{\rm sk}_{<\omega}$.

(i) (Step $k = 0$) If $x \in X$, then $1x = x1 = x$ and $1 \cdot 1 = 1$ belong to $T^{\rm sk}_0$.

(ii) (Step $k = m+1$) $T^{\rm sk}_k$ comprises the following laws:

(a) If $(\sigma, \overline{x}) \in t(\tau^{\rm sk}, X)$, $x_0 \in {\rm FV}(\sigma)$ and $F_{\sigma(y_0,\dots,y_n)}$ is a function symbol in $\tau^{\rm sk}_{\le k} \setminus \tau^{\rm sk}_{<k}$ related to the term $(\sigma, \overline{x})$, then

$$xF_{\sigma(y_0,\dots,y_n)}(x_1, \dots, x_n) = \sigma(x, x_1, \dots, x_n).$$

(b) If $(\sigma, \overline{x}), (\sigma, \overline{x}')$ are $(\tau^{\rm sk}_{\le k}, X)$-terms with $\overline{x} \subseteq \overline{x}' := \langle x_0, \dots, x_{n'} \rangle$, then

$$F_{\sigma(y_0,\dots,y_n)}(x_1, \dots, x_n) = F_{\sigma(y_0,\dots,y_{n'})}(x_1, \dots, x_{n'}).$$

(c) If π is an injective map $\{1, \dots, n\} \to \omega \setminus \{0\}$ and $\sigma'(x_0, x_1, \dots, x_n) := \sigma(x_0, x_{\pi(1)}, \dots, x_{\pi(n)})$, then

$$F_{\sigma(y_0,\dots,y_n)}(x_1, \dots, x_n) = F_{\sigma'(y_0,\dots,y_n)}(x_{\pi(1)}, \dots, x_{\pi(n)}).$$

(d) If $(\sigma_i, \overline{x}) \in \tau^{\rm sk}_{\le m}$ for $i = 1, 2$ and $T^{\rm sk}_{\le m} \vdash (\sigma_1, \overline{x}) = (\sigma_2, \overline{x})$, then

$$F_{\sigma_1(y_0,\dots,y_n)}(x_1, \dots, x_n) = F_{\sigma_2(y_0,\dots,y_n)}(x_1, \dots, x_n).$$

REMARK 2.1.

(i) Recall that $T \vdash (\sigma, \overline{x})$ means that $(\sigma, \overline{x})$ follows from the axioms T. For convenience (as for free groups) we denote the empty product by 1.

(ii) Using the notion of λ-calculus for the law (ii)(a), the unary function $F_{\sigma(y_0,\dots,y_n)}(x_1, \dots, x_n)$ is $\lambda y_0.\sigma(y_0, x_1, \dots, x_n)$ and it acts as

$$x\lambda y_0.\sigma(y_0, x_1, \dots, x_n) = \sigma(x, x_1, \dots, x_n).$$

The axioms in $T^{\rm sk}_{<k}$ $(k \le \omega)$ are equations; thus we have an immediate important application from varieties.

OBSERVATION 2.2. *The theories $T^{\rm sk}_{<k}$ $(k \le \omega)$ are varieties with vocabulary $\tau^{\rm sk}_{<k}$. A model M of $T^{\rm sk}_{<k}$ is an algebra satisfying the axioms of $T^{\rm sk}_{<k}$ and there are models generated freely by any given set.*

Proof. See Grätzer [17, p. 167] or Bergman [2, Chapter 8]. ∎

We immediately derive one of our central definitions.

DEFINITION 2.3. Let $T^{\rm sk} := T^{\rm sk}_{<\omega}$ and $\tau^{\rm sk} = \tau^{\rm sk}_{<\omega}$ be as in Observation 2.2. Any $T^{\rm sk}$-model (an algebra satisfying $T^{\rm sk}$) is called a *skeleton*, and two skeletons are called *isomorphic* if they are isomorphic as $T^{\rm sk}$-models; see e.g. [2, p. 262] or [20, p. 5].

For applications it is useful to recall the following

Definition 2.4 ((Free) generators of a $T^{\mathrm{sk}}_{<k}$-skeleton).

(i) *The $T^{\mathrm{sk}}_{<k}$-model M is* generated *by a set $B \subseteq M$ if for any $m \in M$ there are $(\sigma, \overline{x}) \in t(\tau^{\mathrm{sk}}_{<k}, X)$ and a sequence $\overline{b}$ in B suitable for $\overline{x}$ with $F_{\sigma(\overline{y})}\overline{b} = m$.*

(ii) *The $T^{\mathrm{sk}}_{<k}$-model M is* freely generated *by $B \subseteq M$ (B is a basis of M) if B generates M and if for any $\overline{b}$ and $(\sigma, \overline{x}), (\sigma', \overline{x}) \in t(\tau^{\mathrm{sk}}_{<k}, X)$ with $\overline{b}$ from B suitable for $\overline{x}$, and $F_{\sigma(\overline{y})}\overline{b} = F_{\sigma'(\overline{y})}\overline{b}$, it follows from $T^{\mathrm{sk}}_{<k}$ that $(\sigma, \overline{x}) = (\sigma', \overline{x})$.*

For any set B we will construct a skeleton $\mathbb{B}$ freely generated by B. For this we need

2.4. *Reduction of terms.* Freeness can easily be checked by the usual rewriting process (as in group theory). Thus we define for each term $(\sigma, \overline{x}) \in t(\tau^{\mathrm{sk}}_{<k}, X)$ its reduced form $\mathrm{red}(\sigma, \overline{x}) := (\sigma^{\mathrm{r}}, \overline{x}^{\mathrm{r}}) \in t(\tau^{\mathrm{sk}}_{<k}, X)$. Inductively we apply the axioms (2.1) (in particular (ii)(a) which connect formulas with function symbols) to shorten the length of a term; note that by the axioms (2.1) terms remain the same; we arrive at an essentially unique reduced term. We first consider the reduction of unspecified terms and find σ^r from σ:

(2.2) *The reduction of terms.*

(i) If $\sigma = x \in X$, then $\sigma^{\mathrm{r}} = x$, and if $\sigma = 1$, then $\sigma^{\mathrm{r}} = 1$.

(ii) If $\sigma = \sigma'\sigma''$ and σ', σ'' are reduced, but σ'' does not have the form $F_{\sigma_0(y_0,\dots,y_t)}(\sigma_1, \dots, \sigma_t)$, then $\sigma^{\mathrm{r}} = \sigma$ is reduced.

(iii) Suppose that σ' and σ_i $(i \leq t)$ are reduced and that $\sigma = F_{\sigma'(y_0,\dots,y_t)}(\sigma_1, \dots, \sigma_t)$, with $(\sigma', \langle x_0, \dots x_t\rangle) \in t(\tau^{\mathrm{sk}}_{<k}, X)$ the corresponding specified term. First we get rid of dummy variables: let $u := \{i \in \{1, \dots, t\} : x_i \in \mathrm{FV}(\sigma')\}$; say $u = \{1 \leq i_1 < \dots < i_{|u|} \leq n\}$ and $\overline{x}' = \langle x_i : i \in u\rangle$. Then $\sigma^{\mathrm{r}} = F_{\sigma'(y_0, y_{i_1}, \dots, y_{i_{|u|}})}\langle \sigma_i : i \in u\rangle$. See below for a normalization.

(iv) If $\sigma = \sigma'\sigma''$ and σ', σ'' are reduced terms, but σ'' is a unary function of the form $F_{\sigma_0(y_0,\dots,y_t)}(\sigma_1, \dots, \sigma_t)$, then $\sigma^{\mathrm{r}} = \sigma_0(\sigma', \sigma_1, \dots, \sigma_t)$.

We are ready for

Definition 2.5. An unspecified term $\sigma \in t(\tau^{\mathrm{sk}}_{<k})$ is *reduced* if $\sigma^{\mathrm{r}} = \sigma$. A term $(\sigma, \overline{x}) \in t(\tau^{\mathrm{sk}}_{<k}, X)$ is *reduced* if $\sigma^{\mathrm{r}} = \sigma$ and $(\sigma, \overline{x})$ has no dummy variables, i.e. $\mathrm{FV}(\sigma) = \mathrm{Im}(\overline{x})$. Thus $\mathrm{red}(\sigma, \overline{x}) = (\sigma, \overline{x})$. Moreover, $(\sigma, \overline{x})$ is *normalized* if the free variables of σ are enumerated as $\langle x_0, \dots, x_{n-1}\rangle$.

It is now easy to extend the reduction inductively to $t(\tau^{\mathrm{sk}}_{<k}, X)$. Let $\mathrm{red}(x, \overline{x}) = (x, \langle x\rangle)$ and $\mathrm{red}(1, \overline{x}) = (1, \emptyset)$. In (ii) we first ensure (by free substitution) that the free variables of σ' and σ'' are disjoint and then order their union. We want to normalize $(\sigma^{\mathrm{r}}, \overline{x}')$ in (iii): the sequence $\overline{x}'$ is of

the form $\langle x_{i_1}, \ldots, x_{i_{|u|}}\rangle$ for some $1 \le i_1 < \cdots < i_{|u|} \le n$. We will replace it by the more natural sequence $\overline{x}'' = \langle x_1, \ldots, x_{|u|}\rangle$ and use the substitution $\sigma'' = \mathrm{Sub}^{\langle x_{i_1},\ldots,x_{i_{|u|}}\rangle}_{\langle x_1,\ldots,x_{|u|}\rangle}(\sigma^{\mathrm{r}}, \overline{x}')$; thus $T^{\mathrm{sk}}_{<k}$ implies $(\sigma^{\mathrm{r}}, \overline{x}') = (\sigma'', \overline{x}'')$ by (2.1)(ii)(b). In (iv) we order the union of the free variables $\mathrm{FV}(\sigma_i)$ $(i \le t)$ after making them pairwise disjoint by free substitutions.

Thus we have a definition and a consequence of the last considerations.

Definition-Observation 2.6. *Every term* $(\sigma, \overline{x})$ *can be reduced to a* (*normalized*) *reduced term* $\mathrm{red}(\sigma, \overline{x})$ *with* $T^{\mathrm{sk}}_{<k} \vdash \mathrm{red}(\sigma, \overline{x}) = (\sigma, \overline{x})$. *Let* $t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k}, X)$ *be the family of reduced terms from* $t(\tau^{\mathrm{sk}}_{<k}, X)$; *moreover let* $t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k}) = \{\sigma : \boldsymbol{\sigma} \in t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k}, X)\}$.

Thus we consider only elements from $t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k}, X)$ (so in particular function symbols F_σ have attached reduced terms $(\sigma, \overline{x})$). We want to discuss how much reduced terms can differ if they represent the same element of a free skeleton. We first give the definition which describes this.

Definition 2.7. Using induction, we say when two reduced elements $\sigma_1, \sigma_2 \in t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k})$ are *essentially equal*; we will write $\sigma_1 \doteq \sigma_2$.

(i) If σ_1 is atomic, then σ_2 is atomic and $\sigma_1 = \sigma_2$.
(ii) If $\sigma_1 = \sigma_1'\sigma_1''$ such that σ_1', σ_1'' are reduced, then $\sigma_2 = \sigma_2'\sigma_2''$ and $\sigma_1' \doteq \sigma_2', \sigma_1'' \doteq \sigma_2''$.
(iii) If $\sigma_i = F_{\sigma_i'(y_0,\ldots,y_{m_i})}(\sigma_1^i, \ldots, \sigma_{m_i}^i)$ for $i \le 2$, then $m_1 = m_2$ and there is a permutation π of $\{1, \ldots, m_1\}$ with $\sigma_j^1 \doteq \sigma^2_{\pi(j)}$ for all $j \le m_1$ and also $\sigma_1' \doteq \sigma_2'$.

Observation 2.8.

(i) *The relation* $\doteq$ *is an equivalence relation on* $t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k}, X)$.
(ii) *If* F *is an* n*-place function symbol in* $\tau^{\mathrm{sk}}_{<k}$ *and* $\sigma_i \doteq \sigma_i' \in t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k}, X)$ *for* $i < n$, *then* $F(\sigma_0, \ldots, \sigma_{n-1})^{\mathrm{r}} \doteq F(\sigma_0', \ldots, \sigma_{n-1}')^{\mathrm{r}}$.

Proof. This is immediate by induction using (2.1). ∎

Using normalization of $\overline{x}$ and $\overline{b}$ from Definition-Observation 2.6 we can deduce

Proposition 2.9. *If* M *is a* $T^{\mathrm{sk}}_{<k}$*-model and* $(\sigma, \overline{x}) \in t(\tau^{\mathrm{sk}}_{<k}, X)$ *with* $\mathrm{red}(\sigma, \overline{x}) = (\sigma^{\mathrm{r}}, \overline{x}^{\mathrm{r}})$, *then* $M \vdash \sigma(\overline{b}) = \sigma^{\mathrm{r}}(\overline{b}^{\mathrm{r}})$ *for any sequence* $\overline{b}$ *from* M *suitable for* $\overline{x}$ *with a sequence* $(\overline{b}^{\mathrm{r}})$ *obtained by normalization.*

Note that reduction of terms is defined for each $k \le \omega$, thus formally it depends on k. Moreover, $(\tau^{\mathrm{sk}}_{<h}, X) \subseteq (\tau^{\mathrm{sk}}_{<k}, X)$ for all $h \le k \le \omega$. Next we show that the reduction of terms in $(\tau^{\mathrm{sk}}_{<h}, X)$ is the same even if it takes place in $(\tau^{\mathrm{sk}}_{<k}, X)$, i.e. $t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<h}, X) = t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k}, X) \cap t(\tau^{\mathrm{sk}}_{<h}, X)$. A similar argument holds for freeness.

PROPOSITION 2.10. *Let $h \le k \le \omega$ and $(\sigma, \overline{x}) \in t(\tau^{\mathrm{sk}}_{<h}, X)$.*

(i) $\mathrm{red}(\sigma, \overline{x})$ *with respect to $t(\tau^{\mathrm{sk}}_{<h}, X)$ is the same as with respect to $t(\tau^{\mathrm{sk}}_{<k}, X)$.*
(ii) *If M_z is freely generated by B with respect to $t(\tau^{\mathrm{sk}}_{<z}, X)$ for $z \in \{h, k\}$, then there is an embedding $\iota : M_h \to M_k$ with $\iota\restriction B = \mathrm{id}_B$.*
(iii) *If $\sigma \in t(\tau^{\mathrm{sk}}_{<k})$, then $T^{\mathrm{sk}}_{<k} \vdash \sigma = \sigma^{\mathrm{r}}$.*

Proof. (i) follows because reduction of elements from $t(\tau^{\mathrm{sk}}_{<h}, X)$ only uses terms from $t(\tau^{\mathrm{sk}}_{<h}, X)$. In (ii) we can extend the identity $\mathrm{id} : B \to B$ naturally by induction to $M_h \to M_k$, and (iii) follows from Definition-Observation 2.6. ∎

The skeleton has the following important property.

COROLLARY 2.11. *For $\sigma_1, \sigma_2 \in \tau^{\mathrm{sk}}$ the following are equivalent*:

(i) $T^{\mathrm{sk}} \vdash \sigma_1 = \sigma_2$.
(ii) $\sigma_1^{\mathrm{r}} \doteq \sigma_2^{\mathrm{r}}$.

Proof. (ii)→(i). From Definition-Observation 2.6 it follows that $T^{\mathrm{sk}} \vdash \sigma_1 = \sigma_1^{\mathrm{r}}, \sigma_2 = \sigma_2^{\mathrm{r}}$ and Observation 2.8 gives $T^{\mathrm{sk}} \vdash \sigma_1^{\mathrm{r}} = \sigma_2^{\mathrm{r}}$, thus $T^{\mathrm{sk}} \vdash \sigma_1 = \sigma_2$.

(i)→(ii). From (i) it follows that $T^{\mathrm{sk}} \vdash \sigma_1^{\mathrm{r}} = \sigma_2^{\mathrm{r}}$. Thus $\sigma_1^{\mathrm{r}} \doteq \sigma_2^{\mathrm{r}}$ by Definition 2.7 and Observation 2.8. ∎

2.5. *The skeleton freely generated by X.* Next we construct and discuss free skeletons based on reduced terms. We will use the infinite set X of free variables to construct a skeleton M_X which is freely generated by a set which corresponds by a canonical bijection to X.

By Observation 2.8(i) we have an equivalence relation $\doteq$ on $\tau^{\mathrm{sk}} := \tau^{\mathrm{sk}}_{<\omega}$ with equivalence classes $[\sigma]$ for any $\sigma \in \tau^{\mathrm{sk}}$. Let

$$M_X = \{[\sigma] : \sigma \in t(\tau^{\mathrm{sk}})\}.$$

The equivalence classes $[\sigma]$ of atoms σ are singletons by Definition 2.7(i). If $B = \{[x] : x \in X\}$, then $\iota : X \to B$ $(x \mapsto [x])$ is a bijection and we see that M_X is a skeleton with basis $B \subseteq M_X$. Moreover, $[\ ]$ is compatible with the application of function symbols:

If $F = F_\sigma \in \tau^{\mathrm{sk}}$ with $(\sigma, \overline{x}) \in t(\tau^{\mathrm{sk}}, X)$ and $\overline{x} = \langle x_0, \ldots, x_{n-1}\rangle$ is an n-place function symbol and $\sigma_i \doteq \sigma_i'$ $(i < n)$, then $F(\sigma_0, \ldots, \sigma_{n-1})^{\mathrm{r}} \doteq F(\sigma_0', \ldots, \sigma_{n-1}')^{\mathrm{r}}$ by Observation 2.8(ii), thus

$$F([\sigma_0], \ldots, [\sigma_{n-1}]) = [F(\sigma_0, \ldots, \sigma_{n-1})] \quad (\sigma_i \in t^{\mathrm{r}}(\tau^{\mathrm{sk}}))$$

is well defined, as follows from Observation 2.8(ii).

We have the following

Theorem 2.12. *If X is an infinite set (of free variables) and M_X is defined as above, then the following hold*:

(a) $M_X = \{[\sigma] : \sigma \in \mathrm{t}^{\mathrm{r}}(\tau^{\mathrm{sk}})\}$.
(b) *M_X is a skeleton with n-place functions*

$$[F] : (M_X)^n \to M_X, \qquad ([\sigma_0], \ldots, [\sigma_{n-1}]) \mapsto [F(\sigma_0, \ldots, \sigma_{n-1})],$$

for each n-place function symbol $F = F_{\sigma(y_0,\ldots,y_{n-1})}$ for $(\sigma, \overline{x}) \in t^{\mathrm{r}}(\tau^{\mathrm{sk}}, X)$ with $\mathrm{FV}(\sigma) = \{x_0, \ldots, x_{n-1}\}$.

(c) *M_X is freely generated by $B = \{[x] : x \in X\}$, called the free skeleton over X. Using ι above we identify B and X.*

Proof. The axioms (2.1) are satisfied, e.g. the crucial condition (ii)(a) follows by definition of $[F]$. ∎

Remark 2.13. In the construction of the free skeleton M_X we also used an infinite set Y of bound variables. However, it follows by induction that another infinite set Y' of bound variables leads to an isomorphic copy of M_X. Thus we do not mention Y in Theorem 2.12.

Lemma 2.14. *Let B be a subset of the $T^{\mathrm{sk}}_{<k}$-model M for $k \le \omega$. Then B is a basis if and only if the following two conditions hold*:

(i) *If $c \in M$, then there are $(\sigma, \overline{x}) \in t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k}, X)$ and a sequence $\overline{b}$ for $\overline{x}$ from B such that $\sigma(\overline{b}) = c$.*
(ii) *If $(\sigma, \overline{x}), (\sigma', \overline{x}') \in t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k}, X)$ with $\overline{x} = \langle x_0, \ldots, x_n\rangle$, $\overline{x}' = \langle x_0, \ldots, x_{n'}\rangle$ and $\overline{b}, \overline{b}'$ are suitable sequences for $\overline{x}$, $\overline{x}'$, respectively from B, then $\sigma(\overline{b}) = \sigma'(\overline{b}')$ implies $n = n'$ and there is a permutation π of $\{0, \ldots, n\}$ such that $\sigma(\overline{x}) = \sigma'(x_{\pi(0)}, \ldots, x_{\pi(n)})$ and $b'_i = b_{\pi(i)}$ for all $i \le n$.*

Proof. If B is a basis of M, then by Definition 2.4 the two conditions of the lemma hold; see Proposition 2.10(i) & (iii) for (i). Conversely, suppose that (i) and (ii) hold. It is easy to extend inductively a bijection $B \to X$ to an isomorphism between M and the free skeleton M_X as in Proposition 2.10. Thus B is a basis. ∎

2.6. *The vocabulary of bodies and their laws.* Recall that R is an $\mathbb{S}$-ring of size $< \kappa$ with $\mathbb{S} = \langle p\rangle \subseteq R$ as explained in the introduction. Also recall that $\tau^{\mathrm{sk}}_{<k}$ is the vocabulary of skeletons from the last section, so in particular $\tau^{\mathrm{sk}} = \tau^{\mathrm{sk}}_{<\omega}$ with similar notations for the axioms $T^{\mathrm{sk}}_{<k}$.

We now extend the vocabulary $\tau^{\mathrm{sk}}_{<k}$ of skeletons to the vocabulary $\tau^{\mathrm{bd}}_{<k}$ of bodies: Let $\tau^{\mathrm{bd}}_{<k}$ comprise all function symbols from $\tau^{\mathrm{sk}}_{<k}$ (so $\tau^{\mathrm{sk}}_{<k} \subseteq \tau^{\mathrm{bd}}_{<k}$) and choose additional function symbols:

(0) An individual constant 0 (for 0 of an R-module).

(1) Let F_+ be a binary function symbol (in charge of addition in R-modules). Thus we will write $F_+(y_0, y_1) = y_0 + y_1$, as usual.
(2) For each $a \in R$ let F_a be a unary function symbol (for scalar multiplication by a on the left). Thus we will write $F_a(y) = ay$, also as usual.

Repeated application of (1) and (2) leads to finite sums like $\sum_{i=1}^n a_i y_i$ and we will write $\tau^{\mathrm{bd}} := \tau^{\mathrm{bd}}_{<\omega}$ and call this the *vocabulary of the bodies*. Again terms can be written as $(\sigma, \overline{x})$ with $\sigma \in \tau^{\mathrm{bd}}$ (or in $\tau^{\mathrm{bd}}_{<k}$ as for skeletons) with $\mathrm{FV}(\sigma) = \mathrm{Im}(\overline{x})$ for reduced terms. The collection of terms of the bodies will be $t(\tau^{\mathrm{bd}}_{<k}, X)$ (where $k \le \omega$ is as above). Its members $(\sigma, \overline{x})$ will also be called (*generalized*) *polynomials*, because we will show (Lemma 2.18) that they can be expressed as linear combinations of generalized monomials (terms from $t(\tau^{\mathrm{sk}}_{<k}, X)$ or from $t^{\mathrm{r}}(\tau^{\mathrm{sk}}_{<k}, X)$, respectively).

As in the case of skeletons we now derive the axioms of the bodies in order to see that they build a variety as well.

(2.3) *The theory T^{bd}_k of bodies for $k \le \omega$.*

(i) $T^{\mathrm{sk}}_k \subseteq T^{\mathrm{bd}}_k$.
(ii) Linearity: If $F \in \tau^{\mathrm{sk}}_{<k}$ is an n-place function symbol, $a_i \in R$ $(i \le t)$ and $1 \le l \le n$, then

$$F\Big(x_1, \ldots, x_{l-1}, \sum_{i=1}^t a_i x_{li}, x_{l+1}, \ldots, x_n\Big) = \sum_{i=1}^t a_i F(x_1, \ldots, x_{l-1}, x_{li}, x_{l+1}, \ldots, x_n).$$

(iii) The usual module laws: Let $a, b \in R$ and $w, y, z \in M$ (with M a T^{bd}_k-model). Then
  (a) $0 + y = y$, $z + y = y + z$, $w + (y + z) = (w + y) + z$.
  (b) $1y = y$, $a(by) = (ab)y$, $a(z + y) = az + ay$, $(a + b)y = ay + by$, $y + (-1)y = 0$.

Observation 2.15. *The theories $T^{\mathrm{bd}}_{<k}$ $(k \le \omega)$ are varieties with vocabulary $\tau^{\mathrm{bd}}_{<k}$. A model M of $T^{\mathrm{bd}}_{<k}$ is an algebra satisfying the axioms of $T^{\mathrm{bd}}_{<k}$ and there are models freely generated by any given set.*

Proof. See Grätzer [17, p. 198, Theorem 3] or Bergman [2, Chapter 8]. ∎

Definition 2.16. Let $T^{\mathrm{bd}} := T^{\mathrm{bd}}_{<\omega}$ and $\tau^{\mathrm{bd}} := \tau^{\mathrm{bd}}_{<\omega}$ be as in Observation 2.15. Any T^{bd}-model (an algebra satisfying T^{bd}) is called a *body*, and two bodies are *isomorphic* if they are isomorphic as T^{bd}-models; see e.g. [2, p. 262] or [20].

Observation 2.17. *Any (generalized) $E(A)$-algebra is a body.*

Proof. Generalized $E(A)$-algebras satisfy $\mathrm{End}_R A = A$. Thus any function symbol F_σ of τ^{bd} can be interpreted on A as a function and the axioms (2.1) and (2.3) hold. ■

But note that only free bodies come from skeletons (see Section 3.2).

2.7. *Linearity of unary body functions from $t(\tau^{\mathrm{bd}}_{<k}, X)$.* We will first show that terms in $t(\tau^{\mathrm{bd}}_{<k}, X)$ are linear combinations of terms in $t(\tau^{\mathrm{sk}}_{<k}, X)$, thus every polynomial (in $t(\tau^{\mathrm{bd}}_{<k}, X)$) is a linear combination of monomials (in $t(\tau^{\mathrm{sk}}_{<k}, X)$).

We will show the following

LEMMA 2.18. *Let $\overline{x} = \langle x_0, \ldots, x_{m-1}\rangle$ and $(\sigma, \overline{x}) \in t(\tau^{\mathrm{bd}}_{<k}, X)$. Then there is $\sum_{l<t} a_l\sigma_l(\overline{x})$ with*

(i) $(\sigma_l, \overline{x}) \in t(\tau^{\mathrm{sk}}_{<k}, X)$ *for* $l < t$,
(ii) $a_l \in R$ *for* $l < t$,
(iii) $T^{\mathrm{bd}}_{<k} \vdash \sigma = \sum_{l<t} a_l\sigma_l$.

Proof. (We will now suppress the index '$< k$'.) We let $(\sigma, \overline{x}) \in t(\tau^{\mathrm{bd}}, X)$ and prove the lemma by induction on the length of σ. If σ is atomic, then $(\sigma, \overline{x})$ is a monomial and there is nothing to show.

If F is an m-place function symbol from τ^{bd} and $\sigma = F(\sigma_0, \ldots, \sigma_{m-1})$ with $\mathrm{FV}(\sigma_l) \subseteq \mathrm{FV}(\sigma)$, then by induction hypothesis for σ_l there are polynomials $\sigma_l = \sum_{i<t_l} a_{li}\sigma_{li}(\overline{x})$ with $a_{li} \in R$ and σ_{li} monomials (terms in $t(\tau^{\mathrm{sk}}, X)$). We substitute these sums into F and apply axiom (2.3)(ii) (the linearity) for functions in the theory of bodies. Thus also σ is as required.

If $\sigma = F_+(\sigma_1, \sigma_2) = \sigma_1 + \sigma_2$ comes from (1) and if $\sigma = a\sigma_1$ comes from (2) the linearity follows from (2.3)(iii). Thus the lemma is shown. ■

LEMMA 2.19. *If $\overline{x} = \langle x_0, \ldots, x_m\rangle$ and $(\sigma, \overline{x})$ is a monomial (a term in $t(\tau^{\mathrm{sk}}, X)$), then $\sigma(\sum_{l<t} x_{0l}a_l, x_1, \ldots, x_m) = \sum_{l<t} a_l\sigma(x_{0l}, x_1, \ldots, x_m)$.*

Proof. This is an easy induction on the length of σ:

If $\sigma = 1$ and $\sigma = x_0$, then the claim holds trivially.

If $\sigma = F(\sigma_0, \ldots, \sigma_m)$, then the claim follows from axiom (2.3)(ii) of T^{bd}. Similarly, if $\sigma = F_+(\sigma_1, \sigma_2)$ and $\sigma = F_a(\sigma_1)$, then the linearity follows by definition of these functions and induction hypothesis. ■

Recall the notion from λ-calculus in Remark 2.1(ii).

PROPOSITION 2.20 (Weak completeness of bodies). *Let M be a body and $(\sigma, \overline{x})$ a polynomial (a term in $t(\tau^{\mathrm{bd}}, X)$) with $\overline{x} = \langle x_0, \ldots, x_m\rangle$ and $d_1, \ldots, d_m \in M$. Then there is $(\sigma', \langle x_1, \ldots, x_n\rangle) \in t(\tau^{\mathrm{bd}}, X)$ and the following hold:*

(i) *M is an R-module.*

(ii) *The unary function* $\lambda.z\sigma(z,d_1,\dots,d_m) : M \to M$ *defined by* $z \mapsto \sigma(z,d_1,\dots,d_m)$ *is the* R*-endomorphism* $\lambda y.y\sigma'(z,d_1,\dots,d_m) \in \mathrm{End}_R(M)$ *given by* $y \mapsto y\sigma'(d_1,\dots,d_m)$.

REMARK. We will show here that there is a function symbol $(\sigma',\overline{x}) \in t(\tau^{\mathrm{bd}},X)$ such that $\sigma(d,d_1,\dots,d_m) = d\sigma'(d_1,\dots,d_m)$ for all $d \in M$; see axiom (2.1)(ii)(a) of the skeletons.

Proof of Proposition 2.20. By axioms (2.3) of T^{bd} it is clear that M is an R-module. It remains to show (ii). Let $(\sigma,\overline{x}') \in t(\tau^{\mathrm{bd}},X)$ with $\overline{x}' = \langle x_0,\dots,x_m\rangle$ and $\overline{x} = \langle x_1,\dots,x_m\rangle$. By Lemma 2.18 there are monomials $(\sigma_l,\overline{x}') \in t(\tau^{\mathrm{sk}},X)$ and $a_l \in R$ such that

$$\sigma = \sum_{l<t} a_l\sigma_l.$$

From the construction of τ^{sk} we also have function symbols $F_{\sigma_l(y_0,\dots,y_m)}$ satisfying axioms (2.1) and (2.3). Thus

$$\sigma_l(\overline{x}') = x_0F_{\sigma_l(y_0,\dots,y_m)}(\overline{x})$$

and we put

$$\sigma'(\overline{x}) = \sum_{l<t} a_lF_{\sigma_l(y_0,\dots,y_m)}(\overline{x}).$$

For (ii) it remains to show $\sigma(d,d_1,\dots,d_m) = d\sigma'(d_1,\dots,d_m)$ for all $d \in M$, which will follow from $\sigma(x_0,\overline{x}) = x_0\sigma'(\overline{x})$. We use the three displayed formulas and calculate

$$\begin{aligned}\sigma(x_0,\overline{x}) &= \sum_{l<t} a_l\sigma_l(x_0,\overline{x}) = \sum_{l<t} a_l(x_0F_{\sigma_l(y_0,\dots,y_m)}(\overline{x}))\\ &= x_0\Big(\sum_{l<t} a_lF_{\sigma_l(y_0,\dots,y_m)}(\overline{x})\Big) = x_0\sigma'(\overline{x}).\end{aligned}$$

Hence (ii) follows. ∎

3. From the skeleton to the body

3.1. *The monoid structure of skeletons.* Recall from Theorem 2.12 that the skeleton on an infinite set X of free variables is the set $M_X = \{[\sigma] : \sigma \in t^{\mathrm{r}}(\tau^{\mathrm{sk}})\}$ with n-place functions

$$[F] : (M_X)^n \to M_X, \qquad ([\sigma_0],\dots,[\sigma_{n-1}]) \mapsto [F(\sigma_0,\dots,\sigma_{n-1})],$$

for each n-place function symbol $F = F_{\sigma(y_0,\dots,y_{n-1})}$ with $(\sigma,\overline{x}) \in t^{\mathrm{r}}(\tau^{\mathrm{sk}})$ and $\mathrm{FV}(\sigma) = \{x_0,\dots,x_{n-1}\}$. For simplicity we will also write Latin letters for the members of M_X, e.g. $m = [\sigma] \in M_X$. The set M_X has a distinguished element $1 = [1]$ and $m1 = 1m = m$ holds for all $m \in M_X$ (thus M_X is an applicative structure with 1). In order to turn M_X into a monoid, we first represent M_X as a submonoid of $\mathrm{Mono}(M_X)$, the injective maps on M_X, say $\iota : M_X \to \mathrm{Mono}(M_X)$.

Let $a = [\sigma] \in M_X$ and $\sigma' \in t(\tau^{\mathrm{bd}})$. We will use induction. If $a = [1]$ then $[\sigma']a = \sigma'$; if $a = [x]$, then $[\sigma']a = \sigma' x$; and if $a = [F_{\sigma(y_0,\dots,y_{n-1})}\langle x_1,\dots,x_{n-1}\rangle]$ is a unary function as above, then $[\sigma']a = \sigma(\sigma', x_1,\dots,x_{n-1})$ (so $a\iota = [\lambda y.y\sigma]$). Thus $a\iota$ maps any $m = [\sigma'] \in M_X$ to $m(a\iota) = m(\lambda y.y\sigma) = [m\sigma] \in M_X$, which can be represented by a reduced element using (2.2). If $a \neq b \in M_X$, then $1(a\iota) = a \neq b = 1(b\iota)$, thus $\iota : M_X \to \mathrm{Mono}(M_X) \subseteq M_X$ is an embedding. We define multiplication of elements $a, b \in M_X$ as composition of functions $(a\iota)(b\iota) = (ab)\iota$. This is to say that from $a = [\sigma]$, $b = [\sigma']$ we get the product as the equivalence class of $\lambda y.((y\sigma')\sigma)$. We will write $a \cdot b = ab$ and will often suppress the map ι. From the definition of $\mathrm{Mono}(M_X)$ it follows that also M_X is a monoid. Note furthermore that $[x][x'] \neq [x'][x]$ for any free variables $x, x' \in X$. We get

Observation 3.1. *The free skeleton* $(M_X, \cdot, 1)$ *with composition of functions as product is a non-commutative* (*associative*) *monoid with multiplication defined as above by the action on* M_X: *If* $[\sigma], [\sigma'] \in M_X$, *then* $[\sigma'] \cdot [\sigma] = [\lambda y.((y\sigma')\sigma)]$.

3.2. *Free bodies from skeletons.* Finally, we will associate with any skeleton M its (canonical) body $\mathbb{B}_R M$. Let $\mathbb{B}_R M$ be the R-monoid algebra RM of the monoid M. Moreover, any n-place function $F : M^n \to M$ extends uniquely by linearity to $F : \mathbb{B}_R M^n \to \mathbb{B}_R M$. We deduce

Lemma 3.2. *If R is a commutative ring as above and M is a skeleton, then the R-monoid algebra $\mathbb{B}_R M$ of the monoid M is a body. If the skeleton M_X is freely generated by X, then also $\mathbb{B}_R M$ is freely generated by X as a body. Moreover* ${}_R\mathbb{B}_R M_X = \bigoplus_{m\in M} mR$.

Proof. It is easy to see that $\mathbb{B}_R M$ (with the linear n-place functions) is a body. We first claim that X, viewed as $\{[x] : x \in X\} \subseteq \mathbb{B}_R M_X$, is a basis. Indeed, apply Lemma 2.18 to the R-monoid $\mathbb{B}_R M_X$: Any $(\sigma, \overline{x}) \in t(\tau^{\mathrm{bd}}, X)$ can be written as a polynomial $\sigma = \sum_l \sigma_l a_l$ with monomial $(\sigma_l, \overline{x}) \in t(\tau^{\mathrm{sk}}, X)$. Moreover, any σ_l is viewed as an element of $\mathrm{Mono}(M_X)$, so axiom (2.1)(ii)(a) applies and σ_l becomes a product of elements from X. Thus X generates $\mathbb{B}_R M_X$. The monomials of the skeleton M extend uniquely by linearity to polynomials of the free R-module ${}_R\mathbb{B}_R M_X = \bigoplus_{m\in M} mR$ from its basis M. ■

We will also need the notion of an extension of bodies.

Definition 3.3. Let $\mathbb{B}$ and $\mathbb{B}'$ be two bodies. Then $\mathbb{B} \leq \mathbb{B}'$ ($\mathbb{B}'$ *extends* $\mathbb{B}$) if and only if $\mathbb{B} \subseteq \mathbb{B}'$ as R-algebras and if $(\sigma, \overline{x}) \in t(\tau^{\mathrm{bd}}, X)$ and F_σ is a function symbol with corresponding unary R-linear function F of $\mathbb{B}'$, then its natural restriction to $\mathbb{B}$ is the function for $\mathbb{B}$ corresponding to F_σ.

EXAMPLE 3.4. Let $X \subseteq X'$ be sets of free variables and $\mathbb{B}, \mathbb{B}'$ be the free bodies generated by the free skeletons obtained from X and X', respectively. Then $\mathbb{B} \leq \mathbb{B}'$. In this case we say that $\mathbb{B}'$ is *free over* $\mathbb{B}$.

4. The technical tools for the main construction. The endomorphism ring $\mathrm{End}_R \mathbb{B}_R M_X$ of the R-module ${}_R\mathbb{B}_R M_X$ has natural elements as endomorphisms: these are the (generalized) polynomials interpreted by the terms in $\sigma \in t(\tau^{\mathrm{bd}}, X)$ acting by scalar multiplication on $\mathbb{B}_R M_X$ as shown in Proposition 2.20(ii). The closure under these polynomials is dictated by the properties of $E(R)$-algebras. Thus we would spoil our aim to construct generalized $E(R)$-algebras if we "lose these R-linear maps" on the way.

DEFINITION 4.1. Let $(\sigma, \overline{x}) \in t(\tau^{\mathrm{bd}}, X)$ with $\overline{x} = \langle x_0, \ldots, x_n \rangle$. If $\mathbb{B}$ is a body and $\overline{d} = \langle d_1, \ldots, d_n \rangle$ with $d_1, \ldots, d_n \in \mathbb{B}$, then we call $s_{\overline{d}}(y) = \lambda y.\sigma(y, \overline{d})$ the (*generalized*) *polynomial over* $\mathbb{B}$ *with coefficients* $\overline{d}$.

Note that $\sigma_{\overline{d}}(y)$ is a sum of products of elements d_i and y. Here we must achieve (full) completeness of the final body, thus showing that any endomorphism is represented. By a prediction principle we kill all endomorphisms that are not represented by $t(\tau^{\mathrm{bd}}, X)$—thus the resulting structure will be complete: Any R-endomorphism of an extended body $\mathbb{B}_R M_X$ will be represented by a polynomial $q(x)$ over $\mathbb{B}_R M_X$, so $\mathbb{B}_R M_X$ is complete or equivalently an $E(R)$-algebra.

The fact that $\mathbb{B}_R M_X$ is not just the R-linear closure (or A-linear closure for some algebra A) makes this final task, to get rid of undesired endomorphisms, harder than in the case of realizing algebras as endomorphism algebras (where the closure is not that floppy).

DEFINITION 4.2. Let $\mathbb{B}$ be a body and $G = {}_R\mathbb{B}$. Then $\varphi \in \mathrm{End}_R G$ is said to be *represented* (*by* $q(y)$) if there is a generalized polynomial $q(y)$ with coefficients in $\mathbb{B}$ such that $g\varphi = q(g)$ for all $g \in G$.

If all elements from $\mathrm{End}_R G$ are represented, then $\mathbb{B}$ is a generalized $E(R)$-algebra.

As for other algebraic structures, we have

LEMMA 4.3. *Let R be an $\mathbb{S}$-ring as above and $\mathbb{B}$ be a body generated by B. Then B is a basis of $\mathbb{B}$ if one of the following equivalent conditions holds*:

- (i) *If $B' = \mathbb{B}_R M_X$ is the body generated by the free skeleton M_X and $X \to B$ is a bijection, then this map extends to an isomorphism $\mathbb{B}' \to \mathbb{B}$.*
- (ii) *B is independent in $\mathbb{B}$, i.e. if $(\sigma_1, \overline{x}), (\sigma_2, \overline{x}) \in t^{\mathrm{r}}(\tau^{\mathrm{bd}}, X)$ and the sequence $\overline{b}$ from B is suitable for $\overline{x}$ such that $\sigma_1(\overline{b}) = \sigma_2(\overline{b})$, then $T^{\mathrm{bd}} \vdash \sigma_1(\overline{x}) = \sigma_2(\overline{x})$.*

(iii) *For all bodies* H *and maps* $\varphi : B \to H$ *there is an extension* $\overline{\varphi} : \mathbb{B}_R M_X \to H$ *as* T^{bd}*-homomorphism.*

Proof. The proof is well known for varieties (see Grätzer [17, p. 198, Theorem 3] or Bergman [2, Chapter 8]), so it follows from Observation 2.15. ∎

Freeness Proposition 4.4. *Let* R *be an* $\mathbb{S}$*-ring as above and* $X \subseteq X'$ *be sets of variables and* $\mathbb{B}_R M_X \subseteq \mathbb{B}_R M_{X'}$ *the corresponding free bodies. If* $u \in \mathbb{B} := \mathbb{B}_R M_X$ *and* $v \in X' \setminus X$, *then* $w := u + v \in \mathbb{B}' := \mathbb{B}_R M_{X'}$ *is free over* $\mathbb{B}$, *i.e. there is a basis* X'' *of* $\mathbb{B}'$ *with* $w \in X'' \supseteq X$.

Proof. We will use Lemma 4.3(iii) to show that the set $X'' := (X' \setminus \{v\}) \cup \{w\}$ is a basis of $\mathbb{B}'$. First note that X'' also generates $\mathbb{B}'$, thus $\mathbb{B}' = \mathbb{B}_R M_{X''}$.

Given $\varphi : X'' \to H$ for a body H, we must extend this map to $\overline{\varphi} : \mathbb{B}' \to H$. Let $\varphi' := \varphi \restriction (X' \setminus \{v\})$ and note that the set $X' \setminus \{v\} = X'' \setminus \{w\}$ is independent. Thus if $\mathbb{B}_0 := \mathbb{B}_R M_{X' \setminus \{v\}}$, then φ' extends to $\overline{\varphi'} : \mathbb{B}_0 \to H$ by freeness, and from $u \in \mathbb{B}_0$ follows the existence of $u\overline{\varphi'} \in H$. We now define $\overline{\varphi}$: if $\overline{\varphi} \restriction \mathbb{B}_0 := \overline{\varphi'}$, then $\overline{\varphi} \restriction (X' \setminus \{v\}) = \varphi' = \varphi \restriction (X' \setminus \{v\})$. Thus it remains to extend $\overline{\varphi'}$ to $\overline{\varphi} : \mathbb{B}' \to H$ in such a way that $w\overline{\varphi} = w\varphi$. If $w\varphi =: h \in H$, then we must have $h = w\overline{\varphi} = (u + v)\overline{\varphi} = u\overline{\varphi} + v\overline{\varphi}$. Hence put $v\overline{\varphi} := h - u\overline{\varphi} = h - u\overline{\varphi'}$. Now $\overline{\varphi} : \mathbb{B}' \to H$ exists, because X' is free, $\varphi' \subseteq \overline{\varphi}$ and $w\overline{\varphi} = (u + v)\overline{\varphi} = u\overline{\varphi} + h - u\overline{\varphi} = h = w\varphi$, thus $\varphi \subseteq \overline{\varphi}$ holds as required. ∎

The following corollaries (used several times for exchanging basis elements) are immediate consequences of the last proposition.

Corollary 4.5. *Let* X *be a basis for the body* $\mathbb{B}$, *let* $v \in X$ *and let* $\mathbb{B}'$ *be the subbody of* $\mathbb{B}$ *generated by* $X \setminus \{v\}$ *and* $w \in \mathbb{B}'$. *Then* $X' = X \setminus \{v\} \cup \{v + w\}$ *is another basis for* $\mathbb{B}$.

Corollary 4.6. *If* $X \subseteq X'$ *and* $\mathbb{B}_R M_X \subseteq \mathbb{B}_R M_{X'}$, *then any basis of* $\mathbb{B}_R M_X$ *extends to a basis of* $\mathbb{B}_R M_{X'}$.

Proof. If X'' is a basis of $\mathbb{B}_R M_X$, then it is left as an exercise to show that $(X' \setminus X) \cup X''$ is a basis of $\mathbb{B}_R M_{X'}$. ∎

The last corollaries have another implication.

Corollary 4.7. *Suppose that* $\mathbb{B}_\alpha$ $(\alpha \le \delta)$ *is an ascending*, *continuous chain of bodies such that* $\mathbb{B}_{\alpha+1}$ *is free over* $\mathbb{B}_\alpha$ *for all* $\alpha < \delta$. *Then* $\mathbb{B}_\delta$ *is free over* $\mathbb{B}_0$, *and if* $\mathbb{B}_0$ *is free*, *then* $\mathbb{B}_\delta$ *is free as well.*

The proof of the next lemma is also obvious. It follows by application of the distributive law in T^{bd} and collection of summands with p.

Lemma 4.8. *Let* $q(y)$ *be a generalized polynomial and* $r \in R$. *Then there is a polynomial* $q'(y)$ *such that* $q(y_1 + ry_2) = q(y_1) + rq'(y_1, y_2)$.

Proof. By Lemma 2.18 we can write $\sigma = \sum_{l<t} a_l\sigma_l$ with $(\sigma_l, \overline{x}) \in t(\tau^{\mathrm{sk}}, X)$ for any specified term $(\sigma, \overline{x}) \in t(\tau^{\mathrm{bd}}, X)$. Thus it is enough to show $q(y_1 + ry_2) = q(y_1) + rq'(y_1, y_2)$ for generalized monomials q, and this is obvious by iterated use of axiom (2.3)(ii). ■

LEMMA 4.9. *Let X_0, X_1, X_2 be pairwise disjoint infinite sets, $\mathbb{B}_0 := \mathbb{B}(X_0) \subseteq \mathbb{B} := \mathbb{B}(X_0 \cup X_1 \cup X_2)$ and $q(y), q_1(y), q_2(y)$ polynomials over $\mathbb{B}_0$ such that*

$$q(g + v_1 + v_2) = q_1(v_1) + q_2(v_2)$$

for some $g \in \mathbb{B}_0, v_1 \in X_1, v_2 \in X_2$. Then the following hold:

(i) *$q(y)$ is a linear polynomial in y, i.e. y appears at most once in every monomial.*
(ii) *$q_1(y) - q_2(y)$ does not depend on y.*

Proof. (i) Write $q(y) = \sum_{i=1}^{n} m_i(y)$ as a sum of minimal length of generalized R-monomials and suppose for contradiction that y appears n times in $m_1(y)$ with $n > 1$. Also let, without loss of generality, n be maximal for the chosen monomial $m_1(y)$.

By the distributive law the monomials of the polynomial $q(g + v_1 + v_2)$ include those monomials induced by $m_1(y)$ replacing all entries of the variable y by arbitrary choices of v_1 and v_2. Let m_1' be one of these monomials. If there are further such monomials m_i' $(i \le k)$ like m_1' coming from this substitution into monomials $m_i(y)$ of $q(y)$ with $\sum_{i=1}^{k} m_i' = 0$, then replacing all v_1s and v_2s by ys gives $\sum_{i=1}^{k} m_i(y) = 0$, contradicting the minimality of the above sum. Thus m_1' represents a true monomial (not canceled by others) of $q(g + v_1 + v_2)$, and as $n > 1$ we may also assume that v_1 and v_2 both appear in m_1'. This monomial does not exist on the right-hand side of the equation in the lemma—a contradiction. Thus (i) holds.

(ii) First substitute in the given equation $v_1 := y$, $v_2 := 0$ and $v_1 := 0$, $v_2 := y$, respectively. Thus $q(g+y) = q_1(y)+c$ and $q(g+y) = q_2(y)+c'$, where $c := q_2(0), c' := q_1(0) \in \mathbb{B}_0$. Subtraction now yields $0 = q_1(y)-q_2(y)+(c-c')$, thus $q_1(y) - q_2(y) = c' - c$ does not depend on y, as required. ■

In order to establish the Step Lemmas below, we next prepare some preliminary results. Let $X_\omega = \bigcup_{n\in\omega} X_n$ be the union of a strictly increasing sequence of infinite sets X_n of variables and fix a sequence $v_n \in X_n \setminus X_{n-1}$ of elements $(n \in \omega)$. Moreover, let $M_\alpha = M_{X_\alpha}$ be the skeleton and $\mathbb{B}_\alpha := \mathbb{B}(X_\alpha)$ be the body generated by X_α for $\alpha \le \omega$, respectively. Note that by our identification $\mathbb{B}(X_\alpha)$ is an R-algebra and, restricting to the module structure, $G_\alpha := {}_R\mathbb{B}(X_\alpha)$ is an R-module, which is free by Lemma 3.2. Recall that $\mathbb{S} = \{p^n : n \in \omega\}$ for some $p \in R$ (with $\bigcap_{n\in\omega} p^nR = 0$) generates the $\mathbb{S}$-topology on R-modules. Thus the $\mathbb{S}$-topology is Hausdorff on G_α and G_α is naturally an $\mathbb{S}$-pure R-submodule of its $\mathbb{S}$-completion $\widehat{G}_\alpha$; we write

$G_\alpha \subseteq_* \widehat{G}_\alpha$ and pick particular elements $w_n \in \widehat{G}_\omega$. If $a_i \in \{0,1\}$ and $l_n \in \mathbb{N}$ is increasing, then we define

$$w_n(v_n, l_n, a_n) := w_n := \sum_{k \geq n} p^{l_k - l_n} a_k v_k \in \widehat{G}_\omega \tag{4.1}$$

and easily check that

$$w_n - p^{l_{n+1} - l_n} w_{n+1} = a_n v_n \in G_n \quad \text{for all } n \in \omega. \tag{4.2}$$

Proposition 4.10. *Let $a_n \in \{0,1\}$ and $l_n \in \mathbb{N}$ be as above. If $X_{\omega+1} = X_\omega \setminus \{v_n : a_n = 1, n > 0\} \cup W$ with $W = \{w_n : n > 0\}$ and $\mathbb{B}_{\omega+1} := \mathbb{B}(X_{\omega+1})$, $G_{\omega+1} = {}_R\mathbb{B}_{\omega+1}$, then the following hold*:

(i) $G_\omega \subseteq_* G_{\omega+1} \subseteq_* \widehat{G}_\omega$.
(ii) $G_{\omega+1}/G_\omega$ *is p-divisible, thus an $\mathbb{S}^{-1}R$-module.*
(iii) $X_{\omega+1}$ *is a basis of the* (*free*) *skeleton* $M_{\omega+1} = M_{X_{\omega+1}}$.
(iv) *The R-algebra $\mathbb{B}_{\omega+1}$ is freely generated by the skeleton $M_{\omega+1}$, thus* $\mathbb{B}_{\omega+1} = RM_{\omega+1}$ *and* $G_{\omega+1} = \bigoplus_{m \in M_{\omega+1}} Rm$.
(v) $\mathbb{B}(X_{\omega+1})$ *is free over* $\mathbb{B}(X_n)$ (*as body*).

Proof. (i) Clearly $G_\omega \subseteq_* \widehat{G}_\omega$ and $G_{\omega+1} \subseteq \widehat{G}_\omega$. From $w_n, w_{n+1} \in X_{\omega+1}$, $a_n = 1$ and (4.2) it follows that $v_n \in G_{\omega+1}$. Hence $v_n \in G_{\omega+1}$ for all $n \in \omega$ and $G_\omega \subseteq \mathbb{B}(X_{\omega+1}) = G_{\omega+1}$ follows at once. Thus $G_{\omega+1}/G_\omega \subseteq \widehat{G}_\omega/G_\omega$ and purity ($G_{\omega+1} \subseteq_* \widehat{G}_\omega$) follows if $G_{\omega+1}/G_\omega$ is p-divisible. This is the content of part (ii).

(ii) By definition of the body $\mathbb{B}(X_{\omega+1})$, any element $g \in G_{\omega+1}$ is a sum of monomials in $X_{\omega+1}$. If w_n, w_{n+1} are involved in such a monomial, then we apply (4.2) and get $w_n = pw_{n+1} + a_n v_n$, which is $w_n \equiv pw_{n+1} \bmod G_\omega$. Let m be the largest index of the w_ns which contributes to g. We can remove all w_i of smaller index $i < m$ and also write $w_m \equiv pw_{m+1} \bmod G_\omega$. Thus $g + G_\omega$ is divisible by p and $G_{\omega+1}/G_\omega$ is an $\mathbb{S}^{-1}R$-module.

(iii) It is enough to show that $X_{\omega+1}$ is free, because $X_{\omega+1}$ generates $M_{\omega+1}$ by definition of the skeleton. First we claim that

$$X' = (X_\omega \setminus \{v_n\}) \cup \{w_n\} \text{ is free.}$$

We apply the characterization of a basis by Lemma 4.3(ii). Let $(\sigma_1, \overline{x}), (\sigma_2, \overline{x}) \in t^{\mathrm{r}}(\tau^{\mathrm{bd}}, X)$ ($\overline{x} = \langle x_1, \ldots, x_k \rangle$) be such that

$$\sigma_1(y_1, \ldots, y_k) = \sigma_2(y_1, \ldots, y_k)$$

for some $y_i \in X'$ and suppose that $y_1 = w_n$ (there is nothing to show if w_n does not appear among the y_i's, because they are free; otherwise we relabel the y_i's such that $y_1 = w_n$). Now we consider the above equation as an element in $\widehat{G}_\omega$ and note that the support $[y_i] \subseteq X_\omega$ of the elements y_i ($i > 1$) is finite, while w_n has infinite support $\{v_k : k > n\} \subseteq [w_n]$. Thus we project $\sigma_i(y_1, \ldots, y_k)$ onto a free summand from $[w_n] \setminus \bigcup_{1 < i \leq k} [y_i]$ and

y_1 can be replaced by a free variable v (over $y_2, \ldots, y_k$) and $\sigma_1(v, y_2, \ldots, y_k) = \sigma_2(v, y_2, \ldots, y_k)$ are the same. Hence the first claim follows.

By the first claim and induction it follows that

$$(X_\omega \setminus \{v_1, \ldots, v_n\}) \cup \{w_1, \ldots, w_n\} \text{ is free.} \tag{4.3}$$

Finally let $y_1, \ldots, y_m$ be any finite subset of $X_{\omega+1}$. We may assume that $y_1, \ldots, y_k \in W_n = \{w_i : i \le n\}$ and $y_{k+1}, \ldots, y_m \in X_{\omega+1} \setminus W$. Hence

$$y_1, \ldots, y_m \in W_n \cup X_\omega \setminus \{v_1, \ldots, v_n\},$$

which is free by (4.3), thus $X_{\omega+1}$ is free and (iii) follows.

(iv) is a consequence of (iii) and the definitions.

(v) We note that (by (iii)) the body $\mathbb{B}(X_{\omega+1})$ is freely generated by $X_{\omega+1}$ and also (using (4.2)) by the (free) set $X' = (X_{\omega+1} \setminus \{w_1, \ldots, w_n\}) \cup \{v_1, \ldots, v_n\}$. However, $X_n \subseteq X'$, which generates $\mathbb{B}(X_n)$, hence (v) also follows. ∎

Throughout the remaining part of this section and Section 5 we use the notations from Proposition 4.10. Moreover, we assume the following, where we view $\mathbb{B}_\omega$ as an R-algebra:

$$\text{Let } \varphi \in \operatorname{End}_R G_\omega \setminus \mathbb{B}_\omega, \text{ with } G_n\varphi \subseteq G_n \text{ for all } n \in \omega. \tag{4.4}$$

LEMMA 4.11. *Let φ be as in* (4.4). *If $w_0\varphi \in G_{\omega+1}$, then the following hold*:

- (i) *There exist $m \in \omega$ and a generalized polynomial $q_0(y)$ over $\mathbb{B}_\omega$ such that $w_0\varphi = q_0(w_m)$.*
- (ii) *There exists an $n_* > m$ such that q_0 is a polynomial over $\mathbb{B}_{n_*}$.*

Proof. (i) If $w_0\varphi \in G_{\omega+1}$, then there exists some $m \in \omega$ such that $w_0\varphi \in \mathbb{B}_R M_{X_\omega \cup \{w_0, \ldots, w_m\}}$. Using $w_i \equiv p w_{i+1} \bmod G_\omega$ it follows that

$$w_0\varphi \in \mathbb{B}_R M_{X_\omega \cup \{w_m\}},$$

and there is a generalized polynomial $q_0(y)$ over $\mathbb{B}_\omega$ such that $w_0\varphi = q(w_m)$.

(ii) The coefficients of q_0 are in some $\mathbb{B}_{n_*}$ for some $n_* > m$. ∎

5. The three Step (or Stop) Lemmas. We will use the notations from Proposition 4.10 and (4.4).

We begin with our first Step Lemma, which will stop φ becoming an endomorphism of our final module.

STEP LEMMA 5.1. *Let $\varphi \in \operatorname{End}_R G$ be an endomorphism as in* (4.4) *such that for all $n \in \omega$ there is $g_n \in X_{n+1} \setminus G_n$ with $g_n\varphi \notin \mathbb{B}_R M_{X_n \cup \{g_n\}}$, and let $G_{\omega+1}$ be defined with $w_n = w_n(g_n, l_n, 1)$ as in* (4.1) *for suitable elements $l_n \in \omega$. Then φ does not extend to an endomorphism in* $\operatorname{End}_R G_{\omega+1}$.

Proof. We define inductively an ascending sequence $l_n \in \omega$. If $C \subseteq G_\omega$ is a submodule, then the p-closure of C is defined by $\overline{C} = \bigcap_{n\in\omega}(p^n G_\omega + C)$.

It is the closure of C in the $\mathbb{S}$-adic topology, which is Hausdorff on G_ω (i.e. $\bigcap_{n\in\omega} p^n G_\omega = 0$). In particular $C = \overline{C}$ if C is a summand of G_ω (e.g. $C = 0$ is closed).

By hypothesis we have $g_n\varphi \notin \mathbb{B}_R M_{X_n\cup\{g_n\}}$ and $\mathbb{B}_R M_{X_n\cup\{g_n\}}$ is a summand of G_ω, so it is closed in the $\mathbb{S}$-topology. There is an $l \in \omega$ such that $g_n\varphi \notin \mathbb{B}_R M_{X_n\cup\{g_n\}} + p^l G_\omega$. If l_{n-1} is given, we may choose $l = l_n$ such that $l_n > 3l_{n-1}$. We will ensure (just below) that $G_\omega \subseteq_* G_{\omega+1}$ is $\mathbb{S}$-pure, thus $p^{l_n} G_{\omega+1} \cap G_\omega \subseteq p^{l_n} G_\omega$, and hence:

(5.1) There is a sequence $l_n \in \omega$ with $l_{n+1} > 3l_n$ and

$$g_n\varphi \notin \mathbb{B}_R M_{X_n\cup\{g_n\}} + p^{l_n} G_{\omega+1}.$$

Hence $G_{\omega+1}$ is well defined and Proposition 4.10 holds; in particular $G_\omega \subseteq_* G_{\omega+1}$ and G_ω is dense in $G_{\omega+1}$ ($\overline{G}_\omega = G_{\omega+1}$). Suppose for contradiction that $\varphi \in \operatorname{End}_R G_\omega$ extends to an endomorphism of $G_{\omega+1}$; this extension is unique, and we call it also $\varphi \in \operatorname{End}_R G_{\omega+1}$. In particular $w_0\varphi \in G_{\omega+1}$; by Lemma 4.11 there is a polynomial $q_0(y)$ with coefficients in $\mathbb{B}_{n_*}$ for some $n_* \in \omega$ and with $w_0\varphi = q_0(w_m)$ for some $m \in \omega$. We choose $n > \max\{n_*, m\}$ and use (4.1) to compute w_0:

$$w_0 = \sum_{i=1}^{n} p^{l_i - l_0} g_i + p^{l_{n+1}-l_0} w_{n+1}.$$

Application of φ gives

$$w_0\varphi \equiv \sum_{i=1}^{n-1} p^{l_i-l_0}(g_i\varphi) + p^{l_n-l_0}(g_n\varphi) \bmod p^{l_{n+1}-l_0} G_{\omega+1}.$$

If $i < n$, then $g_i \in G_n$ and $g_i\varphi \in G_n$ by the choice of φ. The last equality becomes $w_0\varphi \equiv p^{l_n-l_0}(g_n\varphi) \bmod p^{l_{n+1}-l_0} G_{\omega+1} + G_n$, hence

$$q_0(w_0) \equiv p^{l_n-l_0}(g_n\varphi) \bmod (p^{l_{n+1}-l_0} G_{\omega+1} + G_n).$$

Finally, we determine $p^{l_n-l_0}(g_n\varphi)$ in terms of $\mathbb{B}_R M_{X_n\cup\{g_n\}}$. From $w_0\varphi = q_0(w_m)$, $n > m$ and the definition of w_m in (4.2) we get

$$w_m = \sum_{i=m}^{n-1} p^{l_i-l_m} g_i + p^{l_n-l_m} g_n + p^{l_{n+1}-l_m} w_{n+1},$$

thus

$$q_0(w_m) \equiv q_0\Big(\sum_{i=m}^{n-1} p^{l_i-l_m} g_i + p^{l_n-l_m} g_n\Big) \bmod p^{l_{n+1}-l_m} G_{\omega+1},$$

and

$$\begin{aligned} p^{l_n - l_0}(g_n\varphi) &\equiv q_0(w_m) \\ &\equiv q_0\Big(\sum_{i=m}^{n-1} p^{l_i - l_m} g_i + p^{l_n - l_m} g_n\Big) \bmod (p^{l_{n+1}-l_m} G_{\omega+1} + G_n). \end{aligned}$$

Now we use (again) the fact that $g_i \in G_n$ for all $i < n$. The last equation reduces to $p^{l_n - l_0}(g_n\varphi) \in \mathbb{B}_R M_{X_n \cup \{g_n\}} + p^{l_{n+1}-l_m} G_{\omega+1}$, hence $g_n\varphi \in \mathbb{B}(G_n, g_n) + p^{l_{n+1}-l_m-l_n} G_{\omega+1}$. Note that $l_{n+1} > 3l_n$ by the choice of the l_ns, hence $l_{n+1} - l_m - l_n > l_n$, so we get a formula

$$g_n\varphi \in \mathbb{B}_R M_{X_n \cup \{g_n\}} + p^{l_n} G_{\omega+1}$$

that contradicts (5.1) and Step Lemma 5.1 follows. ∎

Step Lemma 5.2. *Let $\varphi \in \operatorname{End}_R G$ be an endomorphism as in* (4.4). *Moreover suppose there are elements $u_n, g_n \in X_{n+1} \setminus G_n$ (for each $n \in \omega$) with $u_n\varphi = q_n^1(u_n)$ and $g_n\varphi = q_n^2(g_n)$, where q_n^1, q_n^2 are polynomials over $\mathbb{B}_0$ such that*

$$q_n^1(y) - q_n^2(y) \notin \mathbb{B}_0, \ \textit{i.e. } y \textit{ appears in the difference.}$$

If $G_{\omega+1}$ is defined with $w_n = w_n(g_n + u_n, l_n, 1)$ as in (4.1) *for suitable elements $l_n \in \omega$, then φ does not extend to an endomorphism in* $\operatorname{End}_R G_{\omega+1}$.

Proof. Let $c_k := g_k + u_k$. The set $X' := (X_\omega \setminus \{u_k : k < \omega\}) \cup \{c_k : k < \omega\}$ is now a basis of $\mathbb{B}_\omega$ by Corollary 4.5, thus Proposition 4.10 applies and $G_{\omega+1}$ is well defined. By definition of w_n and (4.2) we have $p^{l_{n+1}-l_n} w_{n+1} + c_n = w_n$, and as in the proof of Step Lemma 5.1 we get

$$w_0\varphi = q_0(w_m) \ \Rightarrow \ \sum_{k \geq 0} p^{l_k - l_0}(c_k\varphi) = q_0\Big(\sum_{k \geq m} p^{l_k - l_m} c_k\Big)$$

for $n_*, m, q_0(y)$ as in Step Lemma 5.1. Furthermore,

$$c_k\varphi = (g_k + u_k)\varphi = g_k\varphi + u_k\varphi = q_k^1(g_k) + q_k^2(u_k).$$

Thus

$$\sum_{k \geq 0} p^{l_k - l_0}(q_k^1(g_k) + q_k^2(u_k)) = q_0\Big(\sum_{k \geq m} p^{l_k - l_m}(g_k + u_k)\Big),$$

where $q_k^1(g_k) \in {}_R\mathbb{B}_{X_0 \cup \{g_k\}}$ and $q_k^2(u_k) \in {}_R\mathbb{B}_{X_0 \cup \{u_k\}}$, and arguments similar to Lemma 4.9 apply: in every monomial of $q_0(y)$ the variable y appears at most once as there are no mixed monomials on the left-hand side, and the same holds for $q_k^1(y), q_k^2(y)$. Furthermore, the variable y does not appear in $q_k^1(y) - q_k^2(y)$, which contradicts our assumption on the q_k^is. ∎

The next lemma is the only place where we will use the fact that R is $\Sigma\mathbb{S}$-incomplete in order to find a sequence $a_n \in \{0, 1\}$ (see Definition 1.1). Recall that this condition follows by Corollary 1.2 if the $\mathbb{S}$-ring is a direct sum of $\mathbb{S}$-invariant subgroups of size $< 2^{\aleph_0}$. Hence it will be sufficient if R^+ is free and $\mathbb{S}$ defines the usual p-adic topology on R.

Step Lemma 5.3. *Let R be a $\Sigma\mathbb{S}$-incomplete $\mathbb{S}$-ring, let $\varphi \in \operatorname{End}_R G$ be an endomorphism as in* (4.4) *and let $q = q(y)$ be a polynomial in y with coefficients in $\mathbb{B}_0$ such that $g\varphi - q(g) \in G_0$ for all $g \in G$. Moreover, suppose that for all $n \in \omega$ there are elements*

$$g_n \in X_{n+1} \setminus G_n \quad \textit{such that} \quad g_n\varphi - q(g_n) \neq 0.$$

If $G_{\omega+1}$ is defined with $w_n = w_n(g_n\varphi - q(g_n), l_n, 1)$ as in (4.1) *for suitable elements $l_n \in \omega$, then φ does not extend to an endomorphism in* $\operatorname{End}_R G_{\omega+1}$.

Proof. Choose $g_n \in X_{n+1}$ as in the lemma, and put $h_n = g_n\varphi - q(g_n) \neq 0$. By assumption on φ and q it follows that $h_n \in G_0$. Let $w_n = w_n(g_n, n, a_n)$ be defined as in (4.1) for a suitable sequence of elements $a_n \in \{0,1\}$ and $l_n = n$ for all $n \in \omega$. We define again $G_{\omega+1}$ as in Proposition 4.10 using the new choice of elements w_n. Note that G_0 is a free R-module. By the assumption that R is $\Sigma\mathbb{S}$-incomplete there is a sequence $a_n \in \{0,1\}$ with $\sum_{k\in\omega} p^k a_k h_k \notin G_0$. However, $\sum_{k\in\omega} p^k a_k h_k \in \widehat{G}_0$ by the choice of h_n, hence $\sum_{k\in\omega} p^k a_k h_k \notin G_{\omega+1}$ by definition of $G_{\omega+1}$. Recall $w_0 = \sum_{k\in\omega} p^k a_k g_k$ and suppose that $w_0\varphi \in G_{\omega+1}$. We compute

$$w_0\varphi = \Big(\sum_{k\in\omega} p^k a_k g_k\Big)\varphi = \sum_{k\in\omega} p^k a_k (g_k\varphi)$$

and

$$\begin{aligned}\sum_{k\in\omega} p^k a_k h_k = \sum_{k\in\omega} p^k a_k (g_k\varphi - q(g_k)) &= \sum_{k\in\omega} p^k a_k g_k - \sum_{k\in\omega} q(g_k) p^k a_k \\ = w_0 - \sum_{k\in\omega} p^k a_k q(g_k).&\end{aligned}$$

From $\sum_{k\in\omega} p^k a_k h_k \notin G_{\omega+1}$ follows $\sum_{k\in\omega} p^k a_k q(g_k) \notin G_{\omega+1}$. However, by the definition of the bodies $\mathbb{B}_\alpha$, the map taking $g \mapsto q(g)$ for any $g \in G_{\omega+1}$ is an endomorphism of $G_{\omega+1}$, and also $w_0 = \sum_{k\in\omega} p^k a_k g_k \in G_{\omega+1}$, hence $\sum_{k\in\omega} p^k a_k q(g_k) \in G_{\omega+1}$ is a contradiction. We deduce $w_0\varphi \notin G_{\omega+1}$ and φ does not extend to an endomorphism of $G_{\omega+1}$. ∎

6. Constructing generalized $E(R)$-algebras

Lemma 6.1. *Let κ be a regular, uncountable cardinal and $\mathbb{B} = \bigcup_{\alpha\in\kappa} \mathbb{B}_\alpha$ a κ-filtration of bodies. Also let $G_\alpha = {}_R\mathbb{B}_\alpha$ and $G = {}_R\mathbb{B}$. Then the following holds for any $\varphi \in \operatorname{End}_R G$:*

(i) *If there is $g \in G$ such that $g\varphi \notin (\mathbb{B}_\alpha)_{\{g\}}$, then there is also $h \in G$ free over $\mathbb{B}_\alpha$ such that $h\varphi \notin (\mathbb{B}_\alpha)_{\{h\}}$.*

(ii) *If there are $g \in G$ and a polynomial $q(y)$ over $\mathbb{B}_\alpha$ such that $g\varphi - q(g) \notin (\mathbb{B}_\alpha)_{\{g\}}$, then there is also $h \in G$ free over $\mathbb{B}_\alpha$ such that $h\varphi - q(h) \notin (\mathbb{B}_\alpha)_{\{h\}}$.*

Proof. If $g \in G$ satisfies the requirements in (i) or (ii), respectively, then choose any element $h' \in G$ which is free over $\mathbb{B}_\alpha$. If h' also satisfies the conclusion of the lemma, then let $h = h'$ and the proof is finished. Otherwise let $h = h' + g$, which is also free over $\mathbb{B}_\alpha$ by Proposition 4.4. In this case $h'\varphi \in (\mathbb{B}_\alpha)_{\{h'\}}$ or $h'\varphi - q(h') \in (\mathbb{B}_\alpha)_{\{h'\}}$, respectively. It follows that $h\varphi \notin (\mathbb{B}_\alpha)_{\{h\}}$ or $h\varphi - q(h) \notin (\mathbb{B}_\alpha)_{\{h\}}$, respectively. ∎

The next lemma is based on results of the last section concerning the Step Lemmas and Lemma 6.1. We will construct first the κ-filtration of $\mathbb{B}_\alpha$s for applications using $\diamondsuit_\kappa E$ for some non-reflecting subset $E \subseteq \kappa^{\mathrm{o}}$. Recall that $\diamondsuit_\kappa E$ holds for all regular, uncountable, not weakly compact cardinals κ and non-reflecting subsets E in $V = L$.

Construction of a κ-filtration of free bodies. Let $\{\varphi_\varrho : \varrho \in E\}$ be the family of Jensen functions given by $\diamondsuit_\kappa E$. The body $\mathbb{B}$ and the R-module ${}_R\mathbb{B}$ will be constructed as a κ-filtration $\mathbb{B} = \bigcup_{\alpha\in\kappa} \mathbb{B}_\alpha$ of bodies. We choose

$$|\mathbb{B}_\alpha| = |\alpha| + |R| = |\mathbb{B}_{\alpha+1} \setminus \mathbb{B}_\alpha|$$

and fix for each $\alpha \in E$ a strictly increasing sequence

$$\alpha_n \in \alpha \setminus E \quad \text{with} \quad \sup_{n\in\omega} \alpha_n = \alpha.$$

This is possible, because E consists of limit ordinals cofinal to ω only and we can pick α_n as a successor ordinal. We will use the same Greek letter for a converging sequence and its limit, so the elements of the sequence only differ by the suffix.

As E is non-reflecting, we may also choose a strictly increasing, continuous sequence $\alpha_\nu,\ \nu \in \mathrm{cf}(\alpha)$, with

$$\sup_{\nu\in\mathrm{cf}(\alpha)} \alpha_\nu = \alpha \quad \text{and} \quad \alpha_\nu \in \alpha \setminus E$$

if $\mathrm{cf}(\alpha) > \omega$. This is crucial, because the body $\mathbb{B}_\alpha$ of the (continuous) κ-filtration of $\mathbb{B}$ must be free in order to proceed by a transfinite construction. This case does not occur for $\kappa = \aleph_1$.

Using Step Lemmas 5.1–5.3 inductively, we define the body structure on $\mathbb{B}_\nu$. We begin with $\mathbb{B}_0 = 0$, and by continuity of the ascending chain the construction reduces to an inductive step passing from $\mathbb{B}_\nu$ to $\mathbb{B}_{\nu+1}$. We will carry over our induction hypothesis of the filtration at each step. In particular, the following three conditions must hold:

(i) $\mathbb{B}_\nu$ is a free body.
(ii) If $\varrho \in \nu \setminus E$, then $\mathbb{B}_\nu$ is a free body over $\mathbb{B}_\varrho$.

(iii) If $\varrho \in E$ then let ϱ_n $(n \in \omega)$ be the given sequence with $\sup_{n\in\omega} \varrho_n = \varrho$. Suppose that the hypothesis of one of the three Step Lemmas holds for $G_n = {}_R\mathbb{B}_{\varrho_n}$ $(n \in \omega)$. We identify $\mathbb{B}_{\varrho+1}$ with $\mathbb{B}_{\omega+1}$ from the Step Lemmas (so φ_ϱ does not extend to an endomorphism of $\mathbb{B}_{\varrho+1}$).

Following these rules we step to $\nu + 1$: If the hypotheses of condition (iii) are violated, for instance, if $\nu \notin E$, we choose $\mathbb{B}_{\nu+1} := (\mathbb{B}_\alpha)_{\{v_\alpha\}}$ adding any new free variable v_ν to the body. However, next we must check that these conditions (i) to (iii) can be carried over to $\nu + 1$. If the hypotheses of condition (iii) are violated, this is obvious. In the other case the Step Lemmas are designed to guarantee our inductive requirements:

Condition (i) is the freeness of $\mathbb{B}_{\omega+1}$ in Proposition 4.10. Condition (ii) requires that $\mathbb{B}_{\nu+1}$ is a free body over $\mathbb{B}_\varrho$. However, $\mathbb{B}_\varrho \subseteq \mathbb{B}_{\nu_n}$ for a large enough $n \in \omega$. Hence (ii) follows from freeness of $\mathbb{B}_{\nu_n}$ over $\mathbb{B}_\varrho$ (inductively) and of $\mathbb{B}_{\nu+1}$ over $\mathbb{B}_{\nu_n}$ (by Proposition 4.10 and Corollary 4.7).

In the case of limit γ we have two possibilities: If $\mathrm{cf}(\gamma) = \omega$ then $\sup_{n\in\omega} \gamma_n = \gamma$, hence $\mathbb{B}_\gamma = \bigcup_{n\in\omega} \mathbb{B}_{\gamma_n}$ and $\mathbb{B}_\gamma$ is a free body with the help of (i) and (ii) by induction (see Corollary 4.7). If $\mathrm{cf}(\gamma) > \omega$, then by our set-theoretic assumption (E is non-reflecting) we have a limit $\sup_{\alpha\in\mathrm{cf}(\gamma)} \gamma_\alpha = \gamma$ of ordinals not in E. The union of the chain $\mathbb{B}_\gamma = \bigcup_{\alpha\in\mathrm{cf}(\gamma)} \mathbb{B}_{\gamma_\alpha}$ by (i) and (ii) is again a free body (see Corollary 4.7). Thus we proceed and obtain $\mathbb{B} = \bigcup_{\nu\in\kappa} \mathbb{B}_\nu$, which is a κ-filtration of free bodies. It remains to show

Main Lemma 6.2. *Assume $\diamondsuit_\kappa E$. Let κ be a regular, uncountable cardinal and $\mathbb{B} = \bigcup_{\alpha\in\kappa} \mathbb{B}_\alpha$ be the κ-filtration of bodies just constructed. Also let $G_\alpha = {}_R\mathbb{B}_\alpha$ and $G = {}_R\mathbb{B}$. Suppose that $\varphi \in \mathrm{End}_R G$ does not satisfy the following conditions* (i) *or* (ii) *for any $\alpha \in \kappa$ and any polynomial $q(y)$ over $\mathbb{B}_\alpha$:*

(i) *There is $g \in G$ such that $g\varphi \notin {}_R(\mathbb{B}_\beta)_{\{g\}}$.*
(ii) *There is $g \in G$ such that $g\varphi - q(g) \notin {}_R(\mathbb{B}_\beta)_{\{g\}}$.*

Then φ is represented in $\mathbb{B}$.

Proof. Suppose for contradiction that φ is not represented in $\mathbb{B}$. Let $E \subseteq \kappa^o$ be given from $\diamondsuit_\kappa E$, let $\{\varphi_\delta : \delta \in E\}$ be the family of Jensen functions and define a stationary subset $E'_\varphi = \{\delta \in E : \varphi{\restriction}G_\delta = \varphi_\delta\}$. Note that $C = \{\delta \in \kappa : G_\delta\varphi \subseteq G_\delta\}$ is a cub, thus $E_\varphi := E'_\varphi \cap C$ is also stationary.

As a consequence we see that there is $\delta \in E_\varphi$ satisfying one of the following conditions:

(i) For every $\alpha < \delta \in E_\varphi$ there is $g \in G_\delta$ such that $g\varphi \notin {}_R(\mathbb{B}_\alpha)_{\{g\}}$.
(ii) There is $\alpha < \delta \in E_\varphi$ such that $g\varphi \in {}_R(\mathbb{B}_\alpha)_{\{g\}}$ for all $g \in G_\delta$ (not case (i)) but for every $\alpha < \delta$ and every polynomial $q(y)$ over $\mathbb{B}_\alpha$ represented by an endomorphism of G there is $g \in G_\delta$ with $g\varphi - q(g) \notin G_\alpha$.

(iii) There is $\alpha < \delta \in E_\varphi$ such that $g\varphi \in {}_R(\mathbb{B}_\alpha)_{\{g\}}$ and there is a polynomial $q(y)$ over $\mathbb{B}_\alpha$ with $g\varphi - q(g) \in G_\alpha$ for all $g \in G_\delta$ (neither (i) nor (ii) holds), but φ is not represented by $\mathbb{B}$. Thus there are a sequence $\delta_n < \delta$ $(n \in \omega)$ with $\sup_{n\in\omega} \delta_n = \delta$ and $g_n \in G_{\delta_n}$ such that $g_n\varphi - q(g_n) \neq 0$ for all $n \in \omega$.

By Lemma 6.1 we may assume that the elements g existing by (i) and (ii), respectively, are free over $\mathbb{B}_\alpha$. Moreover, if $g \in G$, then by $\mathrm{cf}(\kappa) > \omega$ we can choose $\delta \in E_\varphi$ such that $g \in G_\delta$.

If (i) holds, then we can choose a proper ascending sequence $\delta_n \in E_\varphi$ with $\sup_{n\in\omega} \delta_n = \delta$ and elements $g_n \in G_{\delta_{n+1}}$ such that g_n is free over $\mathbb{B}_{\delta_n}$ and

$$g_n\varphi \notin {}_R(\mathbb{B}_{\delta_n})_{\{g_n\}} \quad \text{for all } n \in \omega.$$

We identify G_{δ_n} with G_n in Step Lemma 5.1 and note that (since $\delta_n \in E_\varphi$) $\varphi{\restriction}G_n$ is an endomorphism with $G_n\varphi \subseteq G_n$ which is predicted as a Jensen function. By construction of $G_{\delta+1}$ (as a copy of $G_{\omega+1}$ from Step Lemma 5.1), the endomorphism $\varphi{\restriction}G_\delta$ does not extend to $\mathrm{End}_R\, G_{\delta+1}$. However $\varphi \in \mathrm{End}\, G$, thus $G_{\delta+1}\varphi \subseteq G_\alpha$ for some $\alpha < \kappa$. Finally, note that $G_{\delta+1}$ is the $\mathbb{S}$-adic closure of G_δ in G because G_δ is $\mathbb{S}$-dense in $G_{\delta+1}$ and $G_{\delta+1}$ is a summand of G_α, hence $\mathbb{S}$-closed in G_α. We derive the contradiction that indeed $\varphi{\restriction}G_{\delta+1} \in \mathrm{End}_R\, G_{\delta+1}$. Hence case (i) is discarded.

Now we turn to case (ii). Suppose that (ii) holds (so condition (i) is not satisfied). In this case there is an ascending sequence $\delta_n \in E_\varphi$ with $\sup_{n\in\omega} \delta_n = \delta$ as above and there are free elements $g_n, u_n \in G_{\delta_{n+1}}$ (also free over $\mathbb{B}_{\delta_n}$) and polynomials q_n^1, q_n^2 over $\mathbb{B}_{\delta_0}$ such that $u_n\varphi = q_n^1(u_n) \neq g_n\varphi = q_n^2(g_n)$. Moreover, the polynomials $q_n^1(y) - q_n^2(y)$ are not constant over $\mathbb{B}_{\delta_0}$. Step Lemma 5.2 applies and we get a contradiction as in case (i). Thus also case (ii) is discarded.

Finally, suppose for contradiction that (iii) holds (so (i) and (ii) are not satisfied). There are $\alpha < \kappa$ and $q(y)$ a polynomial over $\mathbb{B}_\alpha$ such that $g\varphi - q(g) \in G_\alpha$ for all $g \in G$. The polynomial $q(y)$ is represented by an endomorphism of G_α. Moreover (from (iii)) we find $g_n \in G_{\delta_{n+1}}$ free over G_{δ_n} for a suitable sequence δ_n with $\sup_{n\in\omega} \delta_n = \delta$ such that $g_n\varphi - q(g_n) \neq 0$. We now apply Step Lemma 5.3; the argument from case (i) gives a final contradiction. Thus the Main Lemma holds. ∎

Proof of Main Theorem 1.4. Let $\mathbb{B}$ be the body over the $\mathbb{S}$-ring R constructed at the beginning of this section using E as in Theorem 1.4; moreover let $G = {}_R\mathbb{B}$. Thus $|\mathbb{B}| = \kappa$ and by the construction and Proposition 4.10 any subset of size $< \kappa$ is contained in an R-monoid-algebra of cardinality $< \kappa$; the algebra $\mathbb{B}$ is the union of a κ-filtration of free bodies $\mathbb{B}_\alpha$. By Main Lemma 6.2 every element $\varphi \in \mathrm{End}_R\, G$ is represented (by a polynomial $q(y)$ with coefficients from $\mathbb{B}$); see Definition 4.2. Thus $g\varphi = q(g)$ for all $g \in G$ and $\varphi = q(y) \in \mathbb{B}$. It follows that $\mathbb{B} = \mathrm{End}_R\, G$ is the R-endomorphism algebra of G.

Finally, recall that there is $\mathbb{B}_\alpha \subseteq \mathbb{B}$ which is an R-monoid-algebra over a non-commutative monoid from Observation 3.1. Thus $\mathbb{B}$ cannot be commutative either and Theorem 1.4 is shown. ∎

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Received 17 November 2005;
in revised form 13 June 2006