STUDIA MATHEMATICA 226 (3) (2015)

Products of Lipschitz-free spaces and applications

by

PEDRO LEVIT KAUFMANN (São José dos Campos)

Abstract. We show that, given a Banach space X, the Lipschitz-free space over X, denoted by $\mathcal{F}(X)$, is isomorphic to $(\sum_{n=1}^{\infty} \mathcal{F}(X))_{\ell_1}$. Some applications are presented, including a nonlinear version of Pełczyński's decomposition method for Lipschitz-free spaces and the identification up to isomorphism between $\mathcal{F}(\mathbb{R}^n)$ and the Lipschitz-free space over any compact metric space which is locally bi-Lipschitz embeddable into $\mathbb{R}^n$ and which contains a subset that is Lipschitz equivalent to the unit ball of $\mathbb{R}^n$. We also show that $\mathcal{F}(M)$ is isomorphic to $\mathcal{F}(c_0)$ for all separable metric spaces M which are absolute Lipschitz retracts and contain a subset which is Lipschitz equivalent to the unit ball of c_0. This class includes all $C(K)$ spaces with K infinite compact metric (Dutrieux and Ferenczi (2006) already proved that $\mathcal{F}(C(K))$ is isomorphic to $\mathcal{F}(c_0)$ for those K using a different method).

1. Introduction. Let $(M, d, 0)$ be a pointed metric space (that is, a distinguished point 0 in M, called a *base point*, is chosen), and consider the Banach space $\mathrm{Lip}_0(M)$ of all real-valued Lipschitz functions on M which vanish at 0, equipped with the norm

$$\|f\|_{\mathrm{Lip}} := \inf_{x,y \in M,\, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

On the closed unit ball of $\mathrm{Lip}_0(M)$, the topology of pointwise convergence is compact, so $\mathrm{Lip}_0(M)$ admits a canonical predual, which is called the *Lipschitz-free space over* M and denoted by $\mathcal{F}(M)$. This space is the closure in $\mathrm{Lip}_0(M)^*$ of $\mathrm{span}\{\delta_x : x \in M\}$, where δ_x is the evaluation functional defined by $\delta_x(f) = f(x)$. It is readily verified that $\delta : x \mapsto \delta_x$ is an isometry from M into $\mathcal{F}(M)$. Given $0' \in M$, it is clear that $T : \mathrm{Lip}_0(M) \to \mathrm{Lip}_{0'}(M)$ defined by $T(f) := f - f(0')$ is a weak*-to-weak* continuous isometric isomorphism, thus the choice of different base points yields isometrically isomorphic Lipschitz-free spaces. We refer to [15] for a study of Lipschitz function spaces,

2010 *Mathematics Subject Classification*: Primary 46B20; Secondary 46T99.
Key words and phrases: Lipschitz-free spaces, geometry of Banach spaces, spaces of Lipschitz functions.

DOI: 10.4064/sm226-3-2

and to [15] and [5] for an introduction to Lipschitz-free spaces and their basic properties.

One of the main properties of Lipschitz-free spaces is that they permit one to interpret Lipschitz maps between metric spaces from the linear point of view:

PROPOSITION 1.1. *Let M and N be pointed metric spaces, let δ^M and δ^N be the isometries that assign to each $x \in M$ (respectively, $x \in N$) the corresponding evaluation functional δ_x^M on $\mathcal{F}(M)$ (respectively, δ_x^N on $\mathcal{F}(N)$), and suppose that $L : M \to N$ is a Lipschitz function such that $L(0_M) = 0_N$. Then there is a unique linear map $\hat{L} : \mathcal{F}(M) \to \mathcal{F}(N)$ such that $\hat{L} \circ \delta^M = \delta^N \circ L$, that is, the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "L"] \arrow[d, "\delta^M"'] & N \arrow[d, "\delta^N"] \\ \mathcal{F}(M) \arrow[r, dashed, "\hat{L}"] & \mathcal{F}(N) \end{tikzcd}$$

Moreover, $\|\hat{L}\| = \|L\|_{\mathrm{Lip}}$.

In particular, if M and N are *Lipschitz equivalent* (that is, there is a bi-Lipschitz bijection between M and N) then $\mathcal{F}(M)$ and $\mathcal{F}(N)$ are isomorphic. The converse is not true, even if M and N are assumed to be Banach spaces: if K is an infinite compact metric space, then $\mathcal{F}(C(K))$ is isomorphic to $\mathcal{F}(c_0)$, even though $C(K)$ is not Lipschitz equivalent to c_0 in general (recall that if $C(K)$ is uniformly homeomorphic to c_0, then it is isomorphic to c_0; see [8]). This first counterexample for the Banach space case was presented by Dutrieux and Ferenczi [3].

Despite the simplicity of the definition of Lipschitz-free spaces, many fundamental questions about their structure remain unanswered. Godard [4] characterized the metric spaces M such that $\mathcal{F}(M)$ is isometrically isomorphic to a subspace of L^1 as exactly those that are isometrically embeddable into $\mathbb{R}$-trees (that is, connected graphs with no cycles, with the graph distance); on the other hand, Naor and Schechtman [13] have shown that $\mathcal{F}(\mathbb{Z}^2)$ (thus also $\mathcal{F}(\mathbb{R}^2)$) is not isomorphic to any subspace of L^1. This prompts the natural question of characterizing the metric spaces M such that $\mathcal{F}(M)$ is (nonisometrically) isomorphic to L^1. Godefroy and Kalton [5] showed that a Banach space X has the bounded approximation property if and only if $\mathcal{F}(X)$ does, and recently Hájek and Pernecká [7] proved that $\mathcal{F}(\mathbb{R}^n)$ admits a Schauder basis, refining a result from [10], and raised the natural (and still unanswered) question of whether $\mathcal{F}(F)$ admits a Schauder basis for any given closed subset $F \subset \mathbb{R}^n$. Nor is it known to this author whether $\mathcal{F}(\mathbb{R}^n)$ is isomorphic to $\mathcal{F}(\mathbb{R}^m)$ for distinct $m, n \geq 2$.

In this context, we continue the exploration of what could be considered basic properties of Lipschitz-free spaces and their relation to the underlying metric spaces. We will show, for instance, that for any given Banach space X, $\mathcal{F}(X)$ is isomorphic to $(\sum_{n=1}^{\infty} \mathcal{F}(X))_{\ell_1}$ (Theorem 3.1). This provides in particular a kind of nonlinear version of Pełczyński's decomposition method (Corollary 3.2), which in turn can be used to obtain the above mentioned example by Dutrieux and Ferenczi of non-Lipschitz equivalent Banach spaces sharing the same Lipschitz-free space. In fact, we show that $\mathcal{F}(M)$ is isomorphic to $\mathcal{F}(c_0)$ for a wider class of metric spaces (Corollary 3.4). We will also show that, for compact metric spaces M which are locally bi-Lipschitz embeddable in $\mathbb{R}^n$, $\mathcal{F}(M)$ admits a complemented copy in $\mathcal{F}(\mathbb{R}^n)$; when moreover the euclidean ball $B_{\mathbb{R}^n}$ is bi-Lipschitz embeddable in M, $\mathcal{F}(M)$ and $\mathcal{F}(\mathbb{R}^n)$ are actually isomorphic (Theorem 3.7). The class of metric spaces satisfying both properties includes all n-dimensional compact Riemannian manifolds. We also show that, as a consequence of the construction in the proof of Theorem 3.1, the Lipschitz-free spaces over any Banach space and over its unit ball are isomorphic; this provides in particular a partial answer to Hájek and Pernecká's aforementioned question (see the remark after Corollary 3.5).

1.1. Notation. We say that two metric spaces M and N are *C-Lipschitz equivalent*, for some constant $C > 0$, if there is a bi-Lipschitz onto map $\varphi : M \to N$ satisfying $\|\varphi\|_{\mathrm{Lip}} \|\varphi^{-1}\|_{\mathrm{Lip}} \leq C$. Hence M and N are Lipschitz equivalent if they are C-Lipschitz equivalent for some $C > 0$; in that case we also write $M \overset{L}{\sim} N$. Given two Banach spaces X and Y, we write $X \cong Y$ when X and Y are isometrically isomorphic, $X \overset{c}{\hookrightarrow} Y$ when there is a complemented (isomorphic) copy of X in Y, and $X \simeq Y$ when X and Y are isomorphic. If X and Y are isomorphic, the *Banach–Mazur distance* between X and Y is defined by

$$d_{\mathrm{BM}}(X, Y) := \inf\{\|T\| \cdot \|T^{-1}\| : T \text{ is an isomorphism from } X \text{ onto } Y\}.$$

$\|T\| \cdot \|T^{-1}\|$ is called the (*linear*) *distortion* of T. When $d_{\mathrm{BM}}(X, Y) \leq C$ for some $C > 0$, we say that X *is isomorphic to* Y *with distortion bounded by* C.

$\mathrm{Ext}_0(F, M)$ denotes the set of linear extension operators for Lipschitz functions, and $\mathrm{Ext}_0^{\mathrm{pt}}(F, M)$ is the set of pointwise-to-pointwise continuous elements of $\mathrm{Ext}_0(F, M)$ (see Subsection 2.1).

1.2. Structure of this work. In Section 2, we present some background results on linear extension operators for Lipschitz functions and some ways to decompose the Lipschitz-free space over a metric space using metric quotients. In Section 3 we show that, for every Banach space X, $\mathcal{F}(X) \simeq (\sum_{n=1}^{\infty} \mathcal{F}(X))_{\ell_1}$, and derive some consequences. In Section 4 we show that, for every Banach space X, $d_{\mathrm{BM}}(\mathcal{F}(X), \mathcal{F}(X) \oplus_1 \mathcal{F}(X)) \leq 4$.

2. Linear extensions of Lipschitz functions and the Lipschitz-free space over metric quotients

2.1. Linear extensions of Lipschitz functions. Given a pointed metric space $(M, d, 0)$ and a subset F containing 0, let $\mathrm{Ext}_0(F, M)$ denote the set of all extensions $E : \mathrm{Lip}_0(F) \to \mathrm{Lip}_0(M)$ which are linear and continuous (E being an extension means that $E(f)|_F = f$ for all $f \in \mathrm{Lip}_0(F)$). It is immediate to see that if we choose another base point $0'$ contained in F, for each element $E \in \mathrm{Ext}_0(F, M)$ there is a corresponding $E' \in \mathrm{Ext}_{0'}(F, M)$, defined by $E'(f) := E(f - f(0')) + f(0')$, which satisfies $\|E'\| = \|E\|$, so generally it is not important which base point is chosen. Recall that there are always continuous but not necessarily linear extensions from $\mathrm{Lip}_0(F)$ to $\mathrm{Lip}_0(M)$; for example the infimum convolution

$$E(f)(x) := \inf_{y \in F} \{f(y) + \|f\|_{\mathrm{Lip}} d(x, y)\}$$

is such an extension, and it is an isometry, although in most cases it fails to be linear. It is possible, though, to have $\mathrm{Ext}_0(F, M) = \emptyset$; Brudnyi and Brudnyi [2, Theorem 2.18] provide an example of a two-dimensional Riemannian manifold M, equipped with its geodesic metric, which admits a subset F satisfying that condition.

We will be particularly interested in the subset $\mathrm{Ext}_0^{\mathrm{pt}}(F, M)$ of $\mathrm{Ext}_0(F, M)$ consisting of the pointwise-to-pointwise continuous elements. The fact that on bounded sets of $\mathrm{Lip}_0(F)$ the weak* and the pointwise topologies coincide implies that any element of $\mathrm{Ext}_0(F, M)$ is weak*-to-weak* continuous if and only if it belongs to $\mathrm{Ext}_0^{\mathrm{pt}}(F, M)$. Therefore, any $E \in \mathrm{Ext}_0^{\mathrm{pt}}(F, M)$ admits a preadjoint $P : \mathcal{F}(M) \to \mathcal{F}(F)$, which is a (continuous) canonical projection, in the sense that $P(\mu) = \mu|_F$ for all finitely supported $\mu \in \mathcal{F}(M)$. In particular, $\mathcal{F}(F)$ is complemented in $\mathcal{F}(M)$. Conversely, given a continuous projection $P : \mathcal{F}(M) \to \mathcal{F}(F)$ such that $P(\mu) = \mu|_F$ for all finitely supported $\mu \in \mathcal{F}(M)$, we have $P^* \in \mathrm{Ext}_0^{\mathrm{pt}}(F, M)$.

Even when M is a Banach space and F is a closed linear subspace, we might not get this complementability condition. Consider, for example, c_0 and let X be a subspace of c_0 which fails to have the bounded approximation property. As mentioned in the introduction, a Banach space Y has the bounded approximation property if and only if $\mathcal{F}(Y)$ does. Since this property is inherited by complemented subspaces, $\mathcal{F}(X)$ cannot be isomorphic to a complemented subspace of $\mathcal{F}(c_0)$. One can still ask whether or not $\mathrm{Ext}_0(X, c_0)$ is empty.

On the other hand, we have the following positive example:

Proposition 2.1 (Lee and Naor [12]). *There exists $C > 0$ such that, for each $n \in \mathbb{N}$ and each subset F of $\mathbb{R}^n$ containing 0, there exists E in $\mathrm{Ext}_0^{\mathrm{pt}}(F, \mathbb{R}^n)$ satisfying $\|E\| \le C\sqrt{n}$.*

Actually, the fact that the extension operator E constructed by Lee and Naor is pointwise-to-pointwise continuous was pointed out by Lancien and Pernecká [10, Proposition 2.3], who used it to study approximation properties of free spaces over subsets of finite-dimensional Banach spaces.

2.2. Metric quotients and Lipschitz-free spaces. We turn our attention to a special kind of metric quotient. Given a pointed metric space $(M, d, 0)$ and a subset F of M containing 0, let $\sim$ be the equivalence relation which collapses $\overline{F}$ to a point (that is, the equivalence classes are either singletons or $\overline{F}$). We define the *metric quotient* of M by F, denoted by M/F, as the pointed metric space $(M/\sim, \tilde{d}, [0])$, where $\tilde{d}$ is defined by

$$\tilde{d}([x],[y]) = \min\{d(x,y), d(x,F) + d(y,F)\}. \tag{2.1}$$

The space $\mathrm{Lip}_{[0]}(M/F)$ can be interpreted as the closed linear subspace of $\mathrm{Lip}_0(M)$ consisting of all of its functions which are null in F. Depending on how F is placed in M, we can have the following decomposition for $\mathcal{F}(M)$:

LEMMA 2.2. *Let* $(M, d, 0)$ *be a pointed metric space and* F *be a subset containing* 0, *and suppose that there exists* $E \in \mathrm{Ext}_0^{\mathrm{pt}}(F, M)$. *Then*

$$\mathcal{F}(M) \simeq \mathcal{F}(F) \oplus_1 \mathcal{F}(M/F),$$

with distortion bounded by $(\|E\| + 1)^2$.

Proof. Define $\Phi : \mathrm{Lip}_0(F) \oplus_\infty \mathrm{Lip}_0(M/F) \to \mathrm{Lip}_0(M)$ by $\Phi(f, g) \doteq E(f) + g$. It is straightforward that Φ is an onto isomorphism with $\|\Phi\| \le \|E\| + 1$, that Φ is pointwise-to-pointwise continuous and that its inverse $\Phi^{-1} : h \mapsto (h|_F, h - E(h|_F))$ has norm also bounded by $\|E\|+1$. It follows that Φ is the adjoint of an isomorphism Ψ between $\mathcal{F}(M)$ and $\mathcal{F}(F) \oplus_1 \mathcal{F}(M/F)$ satisfying the desired distortion bound. ∎

3. Products of Lipschitz-free spaces. In this section we will show that $\mathcal{F}(X) \simeq (\sum_{n=1}^\infty \mathcal{F}(X))_{\ell_1}$ for any Banach space X, and derive some consequences. To this end we will use the following construction by Kalton [9]. Let $(M, d, 0)$ be a pointed metric space, denote by B_r the closed ball centered at 0 and with radius $r > 0$ and consider, for each $k \in \mathbb{Z}$, the linear operator $T_k : \mathcal{F}(M) \to \mathcal{F}(B_{2^{k+1}} \setminus B_{2^{k-1}})$ defined by

$$T_k\delta_x := \begin{cases} 0 & \text{if } x \in B_{2^{k-1}}, \\ (\log_2 d(x,0) - k + 1)\delta_x & \text{if } x \in B_{2^k} \setminus B_{2^{k-1}}, \\ (k + 1 - \log_2 d(x,0))\delta_x & \text{if } x \in B_{2^{k+1}} \setminus B_{2^k}, \\ 0 & \text{if } x \notin B_{2^{k+1}}. \end{cases}$$

Lemma 4.2 from [9] says that for each $\gamma \in \mathcal{F}(M)$ we have $\gamma = \sum_{k\in\mathbb{Z}} T_k\gamma$

unconditionally and

$$\sum_{k\in\mathbb{Z}} \|T_k\gamma\|_{\mathcal{F}} \le 72\|\gamma\|_{\mathcal{F}}. \tag{3.1}$$

Lemma 4.1 of [9] states that, given $r_1,\dots,r_n,s_1,\dots,s_n\in\mathbb{Z}$ with $r_1<s_1<r_2<\dots<s_n$ and $\gamma_k\in\mathcal{F}(B_{2^{s_k}}\setminus B_{2^{r_k}})$, if $\theta:=\min_{k=1,\dots,n-1}\{r_{k+1}-s_k\}$, then

$$\|\gamma_1+\dots+\gamma_n\|_{\mathcal{F}} \ge \frac{2^\theta-1}{2^\theta+1}\sum_{k=1}^n \|\gamma_k\|_{\mathcal{F}}. \tag{3.2}$$

THEOREM 3.1. *Let X be a Banach space. Then*

$$\mathcal{F}(X)\simeq\Big(\sum_{n=1}^\infty \mathcal{F}(X)\Big)_{\ell_1}.$$

Proof. Note that $S:(\sum_{k\in\mathbb{Z}}\mathcal{F}(B_{2^{k+1}}\setminus B_{2^{k-1}}))_{\ell_1}\ni(\gamma_k)\mapsto\sum_{k\in\mathbb{Z}}\gamma_k\in\mathcal{F}(X)$ is linear, continuous and onto, and from (3.1) we infer that $T:\mathcal{F}(X)\ni\gamma\mapsto(T_k\gamma)\in(\sum_{k\in\mathbb{Z}}\mathcal{F}(B_{2^{k+1}}\setminus B_{2^{k-1}}))_{\ell_1}$ is a well defined one-to-one continuous linear operator. Thus $T\circ S$ is a continuous projection from $(\sum_{k\in\mathbb{Z}}\mathcal{F}(B_{2^{k+1}}\setminus B_{2^{k-1}}))_{\ell_1}$ onto the isomorphic copy $T(\mathcal{F}(X))$ of $\mathcal{F}(X)$.

Denote $M:=\bigcup_{k\in\mathbb{Z}}(B_{2^{2k+1}}\setminus B_{2^{2k}})$, and consider $E\in\mathrm{Ext}_0(M\cup\{0\},X)$ which extends each element of $\mathrm{Lip}_0(M\cup\{0\})$ linearly on each radial segment $[2^{2k-1},2^{2k}]x$, $k\in\mathbb{Z}$, $x\in S_X$. One readily verifies that E is indeed bounded (with $\|E\|\le 6$) after writing down the expression for $E(f)(x)$, where $f\in\mathrm{Lip}_0(M\cup\{0\})$ and $x\in[2^{2k-1},2^{2k}]S_X$, which reads

$$E(f)(x)=\frac{\|x\|-2^{2k-1}}{2^{2k-1}}f\Big(2^{2k}\frac{x}{\|x\|}\Big)+\frac{2^{2k}-\|x\|}{2^{2k-1}}f\Big(2^{2k-1}\frac{x}{\|x\|}\Big).$$

Clearly E is also pointwise-to-pointwise continuous, thus it is the adjoint of some bounded projection $P:\mathcal{F}(X)\to\mathcal{F}(M\cup\{0\})$ satisfying $P(\mu)=\mu|_M$ for all finitely supported $\mu\in\mathcal{F}(X)$. Note that $\mathcal{F}(M\cup\{0\})\cong\mathcal{F}(M)$, since $0\in\overline{M}$. Now by (3.2), the natural identification $\mathrm{Id}:\mathcal{F}(M)\to(\sum_{k\in\mathbb{Z}}\mathcal{F}(B_{2^{2k+1}}\setminus B_{2^{2k}}))_{\ell_1}$ is an isomorphism. So there is a complemented copy of $(\sum_{k\in\mathbb{Z}}\mathcal{F}(B_{2^{2k+1}}\setminus B_{2^{2k}}))_{\ell_1}$ in $\mathcal{F}(X)$.

Note that, by Proposition 1.1, rescalings of any metric space give rise to isometrically isomorphic Lipschitz-free spaces. Thus all spaces $\mathcal{F}(B_{2^{2k+1}}\setminus B_{2^{2k}})$, $k\in\mathbb{Z}$, are isometrically isomorphic to $\mathcal{F}(B_2\setminus B_1)$ and all spaces $\mathcal{F}(B_{2^{k+1}}\setminus B_{2^{k-1}})$, $k\in\mathbb{Z}$, are isometrically isomorphic to $\mathcal{F}(B_4\setminus B_1)$, which in turn is isomorphic to $\mathcal{F}(B_2\setminus B_1)$. It follows that

$$\mathcal{F}(X)\overset{c}{\hookrightarrow}\Big(\sum_{j=1}^\infty\mathcal{F}(B_2\setminus B_1)\Big)_{\ell_1}\quad\text{and}\quad\Big(\sum_{j=1}^\infty\mathcal{F}(B_2\setminus B_1)\Big)_{\ell_1}\overset{c}{\hookrightarrow}\mathcal{F}(X).$$

Since $(\sum_{j=1}^\infty\mathcal{F}(B_2\setminus B_1))_{\ell_1}$ is isomorphic to its ℓ_1-sum, by the standard

Pełczyński decomposition method (see [14]) we have

$$\mathcal{F}(X) \simeq \Big(\sum_{j=1}^{\infty} \mathcal{F}(B_2 \setminus B_1)\Big)_{\ell_1}, \tag{3.3}$$

and the conclusion follows immediately. ∎

Recall that a subset F of a metric space M is called a *Lipschitz retract* of M if there is a Lipschitz map from M onto F which coincides with the identity on F; in case such a map exists, it is called a *Lipschitz retraction.* As a direct consequence of Theorem 3.1 and Proposition 1.1, we get the following nonlinear version of Pełczyński's decomposition method for Lipschitz-free spaces.

COROLLARY 3.2. *Let X be a Banach space and M be a metric space, and suppose that X and M admit Lipschitz retracts N_1 and N_2, respectively, such that X is Lipschitz equivalent to N_2 and M is Lipschitz equivalent to N_1. Then $\mathcal{F}(X) \simeq \mathcal{F}(M)$.*

Proof. $\mathcal{F}(X)$ is isomorphic to $\mathcal{F}(N_2)$, which in turn is a complemented subspace of $\mathcal{F}(M)$. Analogously, $\mathcal{F}(M)$ is isomorphic to a complemented subspace of $\mathcal{F}(X)$. The conclusion follows by applying the standard Pełczyński decomposition method. ∎

COROLLARY 3.3. *Let X be a Banach space. Then*

$$\mathcal{F}(X) \simeq \mathcal{F}(B_1).$$

Proof. Since B_1 is a Lipschitz retract of X, it follows that $\mathcal{F}(X)$ contains a complemented copy of $\mathcal{F}(B_1)$. In the proof of Theorem 3.1 we have shown that $\mathcal{F}(X)$ is isomorphic to $(\sum_{k\in\mathbb{Z}} \mathcal{F}(B_{2^{2k+1}} \setminus B_{2^{2k}}))_{\ell_1}$, which is clearly isomorphic to $(\sum_{k<0} \mathcal{F}(B_{2^{2k+1}} \setminus B_{2^{2k}}))_{\ell_1}$ since all summands are isometrically isomorphic. Let $N := \bigcup_{k<0}(B_{2^{2k+1}} \setminus B_{2^{2k}})$. Again by (3.2), $(\sum_{k<0} \mathcal{F}(B_{2^{2k+1}} \setminus B_{2^{2k}}))_{\ell_1}$ is isomorphic to $\mathcal{F}(N)$, which is complemented in $\mathcal{F}(B_1)$ since there is a pointwise-to-pointwise continuous element in $\mathrm{Ext}_0(N\cup\{0\}, B_1)$. The conclusion follows by an application of Pełczyński's decomposition method. ∎

A metric space is said to be an *absolute Lipschitz retract* if it is a Lipschitz retract of every metric space containing it. Given any metric space M, the space $C_u(M)$ of real-valued bounded and uniformly continuous functions on M, equipped with the uniform norm, is an example of a Banach space which is an absolute Lipschitz retract (see e.g. [1, Theorem 1.6]). This class includes all $C(K)$ spaces for K a compact metric space, in particular it includes c_0. Since all separable metric spaces are bi-Lipschitz embeddable in c_0 (see [1, Theorem 7.11]), we obtain the following class of metric spaces M with $\mathcal{F}(M) \simeq \mathcal{F}(c_0)$:

Corollary 3.4. *Let M be a separable metric space containing a Lipschitz retract which is Lipschitz equivalent to the unit ball of c_0, and suppose that M is an absolute Lipschitz retract. Then $\mathcal{F}(M) \simeq \mathcal{F}(c_0)$. In particular, if K is an infinite compact metric space, then $\mathcal{F}(C(K)) \simeq \mathcal{F}(c_0)$.*

Proof. It is straightforward by Proposition 1.1 and Corollary 3.3 that there is a complemented copy of $\mathcal{F}(c_0)$ in $\mathcal{F}(M)$. The space M is Lipschitz equivalent to some subset F of c_0, and F is an absolute Lipschitz retract since this property is preserved by Lipschitz equivalences. Thus F is a Lipschitz retract of c_0, and again by Proposition 1.1 this implies that $\mathcal{F}(F)$ (and thus $\mathcal{F}(M)$) admits a complemented copy in $\mathcal{F}(c_0)$. The conclusion follows from Theorem 3.1 and an application of Pełczyński's decomposition method. ∎

Corollary 3.5. *Let F be a subset of $\mathbb{R}^n$ with nonempty interior. Then $\mathcal{F}(F) \simeq \mathcal{F}(\mathbb{R}^n)$.*

Proof. By Proposition 2.1, there is a complemented copy of $\mathcal{F}(F)$ in $\mathcal{F}(\mathbb{R}^n)$. Taking any closed ball $B \subset F$, it is easy to see that there is a Lipschitz retraction from F onto B; thus by Proposition 1.1 and Corollary 3.3 there is also a complemented copy of $\mathcal{F}(\mathbb{R}^n)$ in $\mathcal{F}(F)$. The result follows from Theorem 3.1 and an application of Pełczyński's decomposition method. ∎

Remark. As mentioned in the introduction, Hájek and Pernecká [7] have shown that $\mathcal{F}(\mathbb{R}^n)$ admits a Schauder basis, and raised the natural question of whether the same holds true for $\mathcal{F}(F)$, where F is any closed subset of $\mathbb{R}^n$. Note that, by Corollary 3.5, the problem is reduced to the case where F has empty interior.

In order to study Lipschitz-free spaces of locally euclidean metric spaces, alongside the corollaries of Theorem 3.1, the following result becomes handy:

Theorem 3.6 (Lang and Plaut [11]). *Let M be a compact metric space such that each point of M admits a neighborhood which is bi-Lipschitz embeddable in $\mathbb{R}^n$. Then M is bi-Lipschitz embeddable in $\mathbb{R}^n$.*

Theorem 3.7. *Let M be a compact metric space such that each $x \in M$ admits a neighborhood which is bi-Lipschitz embeddable in $\mathbb{R}^n$. Then there is a complemented copy of $\mathcal{F}(M)$ in $\mathcal{F}(\mathbb{R}^n)$.*

If moreover the unit ball of $\mathbb{R}^n$ is bi-Lipschitz embeddable into M, then $\mathcal{F}(M) \simeq \mathcal{F}(\mathbb{R}^n)$. In particular, the Lipschitz-free space over any n-dimensional compact Riemannian manifold equipped with its geodesic metric is isomorphic to $\mathcal{F}(\mathbb{R}^n)$.

Proof. The first part follows directly from Lang and Plaut's result and the fact that the Lipschitz-free space over any subset of $\mathbb{R}^n$ admits a complemented copy in $\mathcal{F}(\mathbb{R}^n)$.

For the second part, note that the closed unit ball of $\mathbb{R}^n$ is an absolute Lipschitz retract, and recall that that property is preserved by Lipschitz equivalences. The result then follows from Corollary 3.5, Theorem 3.1 and an application of Pełczyński's decomposition method. ∎

REMARK. Note that the compactness condition in Theorem 3.7 is necessary, even if we have uniformity of the embeddings into $\mathbb{R}^n$. For example, $\mathbb{Z} \times \mathbb{R}$ is locally isometric to line segments, but $\mathcal{F}(\mathbb{Z} \times \mathbb{R})$ is not isomorphic to a subspace of $\mathcal{F}(\mathbb{R}) \cong L^1$, by Naor and Schechtman's result mentioned in the introduction.

4. $\mathcal{F}(X) \simeq \mathcal{F}(X)^2$ with low distortion. Let X be a Banach space. By Theorem 3.1, $\mathcal{F}(X) \simeq \mathcal{F}(X)^2$. In this section we will show that we have the uniform bound $d_{\mathrm{BM}}(\mathcal{F}(X), \mathcal{F}(X) \oplus_1 \mathcal{F}(X)) \leq 4$; we will do this via an elementary construction based on the metric properties of X.

We start by recalling some definitions and results on quotient metric spaces which are of a more general kind than the ones presented in Section 2. For details, we refer to Weaver's book [15]. Let (M, d) be a complete metric space, and let $\sim$ be an equivalence relation on M. The element of $M/{\sim}$ containing $x \in M$ will be denoted by either $\tilde{x}$ or $[x]_\sim$. Define a pseudometric $\tilde{d}$ on $M/{\sim}$ by

$$\tilde{d}(\tilde{x}, \tilde{y}) := \inf \sum_{j=1}^{n} d(x_j, y_j), \tag{4.1}$$

where the infimum is taken over all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n, y_1, \ldots, y_n$ satisfying $x \sim x_1$, $y_j \sim x_{j+1}$ $(j = 1, \ldots, n-1)$, $y_n \sim y$. This pseudometric can be roughly interpreted in the following way: it is the length of the shortest discrete path from x to y when we are allowed to teleport between equivalent elements. An equivalent way to define $\tilde{d}$, which will be useful for further constructions, is the following:

$$\tilde{d}(\tilde{x}, \tilde{y}) = \sup |f(x) - f(y)|, \tag{4.2}$$

where the supremum is taken over all 1-Lipschitz $f : M \to \mathbb{R}$ which are constant on each $\tilde{z} \in \tilde{M}$.

On M we define yet another equivalence relation $\approx$ which identifies all $x, y \in M$ satisfying $\tilde{d}(\tilde{x}, \tilde{y}) = 0$, and on $M/{\approx}$ we define the metric $\tilde{\tilde{d}}(\tilde{\tilde{x}}, \tilde{\tilde{y}}) = \tilde{d}(\tilde{x}, \tilde{y})$. We define $M_\sim$, the *metric quotient* (or just *quotient*) *of* M *with respect to* $\sim$, as the completion of $M/{\approx}$. Note that, for a given complete metric space $(M, d, 0)$ and an equivalence relation $\sim$ on M, by (4.2) there is a canonical isometric isomorphism between $\mathrm{Lip}_{\tilde{\tilde{0}}}(M_\sim)$ and the closed subspace of $\mathrm{Lip}_0(M)$ consisting of all functions that are constant in each class $\tilde{x} \in M/{\sim}$.

We recall some definitions concerning path metric spaces. Let (M,d) be a pseudometric space, and let $\varphi : I \to M$ be a curve (that is, I is an interval and φ is continuous). The *length* of φ is $\ell(\varphi) := \sup\{\sum_{j=1}^{n} d(\varphi(x_{j-1}), \varphi(x_j))\}$, where the supremum is taken over $n \in \mathbb{N}$ and $x_j \in I$, $x_0 < \cdots < x_n$. (M,d) is said to be a *path metric space* if d is a metric and $d(x,y) = \inf\{\ell(\varphi) : \varphi$ is a curve in M having endpoints x and $y\}$. A *minimizing geodesic* in a path metric space is any curve $\varphi : I \to M$ such that $d(\varphi(t), \varphi(s)) = |t-s|$ for all $t, s \in I$; (M,d) is said to be *geodesic* if any two elements of M are joined by a minimizing geodesic.

Proposition 4.1. *Let (M,d) be a path metric space. Then each metric quotient of M is a path metric space.*

Proof. Fix an equivalence relation $\sim$ on M. Let $x, y \in M$, and for each $k \in \mathbb{N}$ consider pairs $(x_1^k, y_1^k), \ldots, (x_{n_k}^k, y_{n_k}^k)$ of elements of M such that

$$x \sim x_1^k, \quad y_j^k \sim x_{j+1}^k \quad (j = 1, \ldots, n_k - 1), \quad y_{n_k}^k \sim y$$

and $\sum_{j=1}^{n_k} d(x_j^k, y_j^k) \xrightarrow{k} \tilde{d}(\tilde{x}, \tilde{y})$. Since (M,d) is a path metric space, there exist, for each $k \in \mathbb{N}$ and $j = 1, \ldots, n_k$, curves φ_j^k with endpoints x_j^k and y_j^k, respectively, and such that

$$\sum_{j=1}^{n_k} \ell(\varphi_j^k) < \tilde{d}(\tilde{x}, \tilde{y}) + 1/k.$$

Concatenating these we get a curve $\tilde{\varphi}^k$ in $M/\!\sim$ with endpoints $\tilde{x}$ and $\tilde{y}$ satisfying $\ell(\tilde{\varphi}^k) < \tilde{d}(\tilde{x}, \tilde{y}) + 1/k$. Since for any curve $\tilde{\varphi}$ in $M/\!\sim$ with endpoints $\tilde{x}$ and $\tilde{y}$ we have $\tilde{d}(\tilde{x}, \tilde{y}) \le \ell(\tilde{\varphi})$, it follows that

$$\tilde{d}(\tilde{x}, \tilde{y}) = \inf\{\ell(\tilde{\varphi}) : \tilde{\varphi} \text{ is a curve in } M/\!\sim \text{ having endpoints } \tilde{x} \text{ and } \tilde{y}\},$$

and then clearly the same holds for $M/\!\approx$ and thus for $M_\sim$. ∎

Remark. In Proposition 4.1 we cannot substitute *path metric space* by *geodesic metric space*:

Proposition 4.2. *There is a geodesic metric space M which admits a metric quotient that is not a geodesic path metric space.*

Proof. Let e_j be the standard unit vectors of ℓ_1 and consider the metric subspace of ℓ_1 defined by $M := \bigcup_{j=1}^{\infty} [0,1] e_j$. Let $F := \bigcup_{j=1}^{\infty} \{e_j\}$ and suppose that $\sim$ is the equivalence relation which collapses F to a point. Note that, in this case, $M/\!\sim\, = M_\sim$, the $M_\sim$-distance between $\tilde{0}$ and $\tilde{e}_1 = F$ is 1, and there are minimizing geodesics with endpoints $\tilde{0}$ and $\tilde{e}_1$ going through each segment $[0,1]e_j$.

Let

$$F_j := \left[\frac{1}{4} + \frac{1}{2^{2+j}}, \frac{3}{4} - \frac{1}{2^{2+j}}\right] e_j$$

be interpreted as subsets of M, and consider on M the equivalence relation $\equiv$ that collapses each F_j to a point, and the respective quotient metric space $(M/{\equiv}, d)$ (again, in this case we have $M_{\equiv} = M/{\equiv}$). Then $d([\tilde{0}]_{\equiv}, [\tilde{e}_1]_{\equiv}) = 1/2$ and there are curves φ_j in $M/{\equiv}$ with endpoints $[\tilde{0}]_{\equiv}$ and $[\tilde{e}_1]_{\equiv}$ with $\ell(\varphi_j) \xrightarrow{j} 1/2$, even though there is no minimizing geodesic in $M/{\equiv}$ with endpoints $[\tilde{0}]_{\equiv}$ and $[\tilde{e}_1]_{\equiv}$. ■

LEMMA 4.3. *Let M be a path metric space, N a metric space, $f : M \to N$ and $C > 0$. Then f is C-Lipschitz if and only if it is locally C-Lipschitz.*

Proof. To prove the nontrivial implication, fix $\delta > 0$, let $x, y \in M$ and let $\varphi : I \to M$ be a curve with endpoints x and y satisfying

$$\ell(\varphi) < d_M(x, y) + \delta.$$

For each $t \in I$, by hypothesis there exists $\epsilon_t > 0$ such that $f|_{\varphi(]t-\epsilon_t, t+\epsilon_t[)}$ is C-Lipschitz. Since I is compact, there are $t_1 < \cdots < t_n$ such that $\bigcup_{j=1}^{\infty}]t_j - \epsilon_{t_j}, t_j + \epsilon_{t_j}[\supset I$. We can then easily find φ-consecutive points $z_1, \ldots, z_m$ in $\varphi(I)$ satisfying

$$\begin{aligned} d_N(f(x), f(y)) &\le d_N(f(x), f(z_1)) + d_N(f(z_1), f(z_2)) + \cdots + d_N(f(z_m), f(y)) \\ &\le C(d_M(x, z_1) + d_M(z_1, z_2) + \cdots + d_M(z_m, y)) \\ &\le C(d_M(x, y) + \delta). \end{aligned}$$

Since δ was arbitrary, the conclusion follows. ■

Let $(X, \|\cdot\|)$ be a Banach space. We now construct a pair X_L and X_R of metric quotients of X which have properties useful for studying products of $\mathrm{Lip}_0(X)$ (see Proposition 4.6). Let $\alpha, \beta : [0, \infty) \to [0, \infty)$ be the continuous functions defined in each $[2^m, 2^{m+1}]$, $m \in \mathbb{Z}$, by

$$\alpha(t) := \begin{cases} t - 2^{m-1} & \text{if } 2^m \le t \le 2^{m-1} + 2^m, \\ 2^m & \text{if } 2^{m-1} + 2^m \le t \le 2^{m+1}, \end{cases}$$

$$\beta(t) := \begin{cases} 2^{m-1} & \text{if } 2^m \le t \le 2^{m-1} + 2^m, \\ t - 2^m & \text{if } 2^{m-1} + 2^m \le t \le 2^{m+1}. \end{cases}$$

Consider the equivalence relations $\sim_L$ and $\sim_R$ on X defined by

$$x \sim_L y \iff x = y \text{ or } (x = \lambda y \text{ with } \lambda > 0, \text{ and } \alpha \text{ is constant in } [\|x\|, \|y\|]),$$
$$x \sim_R y \iff x = y \text{ or } (x = \lambda y \text{ with } \lambda > 0, \text{ and } \beta \text{ is constant in } [\|x\|, \|y\|]),$$

and denote by $X_L = (X_L, d_L)$ and $X_R = (X_R, d_R)$ the corresponding quotient metric spaces.

To prove the next lemma we use Hopf–Rinow's Theorem which states that in a complete and locally compact path metric space, each pair of points are joined by a minimizing geodesic (see e.g. [6]).

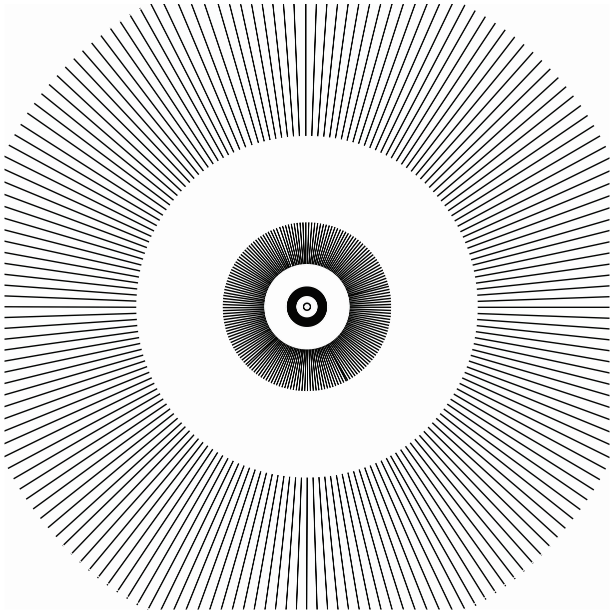

Fig. 1. This is what X_L looks like. The represented radial segments are collapsed to points. X_R looks the same, up to a factor two rescaling.

LEMMA 4.4. *X_L and X_R are geodesic.*

Proof. For any $x, y \in X$, the metric space $(\mathrm{span}_X\{x,y\}/{\sim_R}, d_R)$ satisfies the assumptions of Hopf–Rinow's Theorem, thus there is a minimizing geodesic γ in $(\mathrm{span}_X\{x,y\}/{\sim_R}, d_R)$ (thus also in X_R) with endpoints $\tilde{x}$ and $\tilde{y}$. The same argument holds for X_L. ■

LEMMA 4.5. *There exist onto bi-Lipschitz mappings $L : X \to X_L$ and $R : X \to X_R$ with $\|L\|_{\mathrm{Lip}} \le 1$, $\|L^{-1}\|_{\mathrm{Lip}} \le 4/3$, $\|R\|_{\mathrm{Lip}} \le 3/2$ and $\|R^{-1}\|_{\mathrm{Lip}} \le 1$.*

Proof. Denote $C_m := B_{2^{m+1}} \setminus B_{2^m}, N \in \mathbb{N}$, and for each $x \in X \setminus \{0\}$ let $m_x \in \mathbb{Z}$ be such that $\|x\| \in C_{m_x}$. Define a bicontinuous mapping $R : X \setminus \{0\} \to X_R \setminus \{\tilde{0}\}$ by

$$R(x) := \left(\left(\frac{1}{2} + \frac{2^{m_x}}{\|x\|}\right)x\right)^{\sim_R}.$$

What R does is to squeeze each crown C_m to the thinner crown $R(C_m) = (B_{2^{m+1}} \setminus B_{2^m+2^{m-1}})^{\sim_R}$. For $x \in X \setminus \{0\}$, let V_x be a neighborhood of x such that, for each $y \in V_x$, $\|x-y\| \le 2^{m_x-1}$ and $d_R(R(x), R(y)) \le 2^{m_x-2}$. This implies that, for any $y \in V_x$, we have $|m_x - m_y| \le 1$, the line segment with endpoints x and y intersects at most two crowns C_m and a minimizing geodesic with endpoints $R(x)$ and $R(y)$ intersects at most two crowns $R(C_m)$.

We shall show that

$$\|x-y\| \le d_R(R(x),R(y)) \le \tfrac{3}{2}\|x-y\|, \qquad x\in X\setminus\{0\},\, y\in V_x.$$

The fact that X and X_R are geodesic will allow us then to assert, by Lemma 4.3, that the above inequality holds without the restriction $y \in V_x$, and thus that R is bi-Lipschitz, $\|R\|_{\mathrm{Lip}} \le 3/2$ and $\|R^{-1}\|_{\mathrm{Lip}} \le 1$.

Indeed, let $x, y \in X \setminus B_m$ with $\|x-y\| \le 2^{m-1}$ and $d_R(R(x),R(y)) \le 2^{m-2}$. Assume without loss of generality that $\|x\| \le \|y\|$. Then one of the following conditions is true:

1. $m_x = m_y$, and $d_R(R(x),R(y)) = \|R(x)-R(y)\|$;
2. $m_x < m_y$;
3. $m_x = m_y$, and there is a minimizing geodesic with endpoints $R(x)$ and $R(y)$ passing through $R(C_{m_x-1})$.

If (1) is true, then $1/2 + 2^{m_x}/\|y\| \le 1/2 + 2^{m_x}/\|x\|$, and

$$\left(\frac{1}{2}+\frac{2^{m_x}}{\|y\|}\right)\|x-y\| \le \|R(x)-R(y)\| \le \left(\frac{1}{2}+\frac{2^{m_x}}{\|x\|}\right)\|x-y\|, \tag{4.3}$$

thus

$$\|x-y\| \le d_R(R(x),R(y)) \le \tfrac{3}{2}\|x-y\|. \tag{4.4}$$

If (2) is true, suppose that $\|x\| < 2^{m_x+1}$ (if $\|x\| = 2^{m_x+1}$, then x and y satisfy (1)) and let z be the intersection point of the line segment $[x,y]$ and $S_{2^{m_x+1}}$. Then the pairs x,z and z,y satisfy (1), and by (4.4) we have

$$\begin{aligned} d_R(R(x),R(y)) &\le d_R(R(x),R(z)) + d_R(R(z),R(y)) \\ &\le \tfrac{3}{2}(\|x-z\|+\|z-y\|) = \tfrac{3}{2}\|x-y\|. \end{aligned}$$

Similarly, let $\tilde{z}$ be the intersection of $R(S_{2^{m_x+1}})$ with a minimizing geodesic with endpoints $R(x)$ and $R(y)$. Then we have $d_R(R(x),\tilde{z}) = \|R(x)-\tilde{z}\|$ and $d_R(\tilde{z},R(y)) = \|\tilde{z}-R(y)\|$, and thus by (4.4),

$$\begin{aligned} \|x-y\| \le \|x-R^{-1}(\tilde{z})\| + \|R^{-1}(\tilde{z})-y\| &\le d_R(R(x),\tilde{z}) + d_R(\tilde{z},R(y)) \\ &= d_R(R(x),R(y)). \end{aligned}$$

For the remaining case (3), we can obtain the desired inequalities by taking a convenient point on a minimizing geodesic with endpoints $R(x)$ and $R(y)$ and reducing the problem to case (2).

The Lipschitz equivalence between X and X_L is given by the mapping $L : X\setminus\{0\} \to X_L \setminus \{\tilde{0}\}$ defined by

$$L(x) := \left(\left(\frac{1}{2}+\frac{2^{m_x-1}}{\|x\|}\right)x\right)^{\sim_L},$$

which squeezes each C_m to the thinner crown $L(C_m) = (B_{2^m+2^{m-1}} \setminus B_{2^m})^{\sim_L}$. To show this and obtain the Lipschitz constants, simply follow the same steps taken for R. The only difference will appear when getting to (4.3), which will read

$$\left(\frac{1}{2}+\frac{2^{m_x-1}}{\|y\|}\right)\|x-y\| \le \|L(x)-L(y)\| \le \left(\frac{1}{2}+\frac{2^{m_x-1}}{\|x\|}\right)\|x-y\|,$$

and thus (4.4) will read

$$\tfrac{3}{4}\|x-y\| \le d_L(L(x),L(y)) \le \|x-y\|.$$

Following analogous steps, we get the conclusion. ∎

We are now ready to prove the main result of this section:

Proposition 4.6. *Let X be a Banach space. Then*

$$d_{\mathrm{BM}}(\mathcal{F}(X),\mathcal{F}(X)\oplus_1\mathcal{F}(X)) \le 4.$$

Proof. Recall that, by (4.2), $\mathrm{Lip}_0(X_L)\cong Y_L$ and $\mathrm{Lip}_0(X_R)\cong Y_R$, where Y_L and Y_R are the closed subspaces of $\mathrm{Lip}_0(X)$ defined by

$$Y_L := \{f\in \mathrm{Lip}_0(X) : f \text{ is constant in each equivalence class of } X_L\},$$
$$Y_R := \{f\in \mathrm{Lip}_0(X) : f \text{ is constant in each equivalence class of } X_R\}.$$

Let $\Phi : Y_L\oplus_\infty Y_R \to \mathrm{Lip}_0(X)$ be defined by $\Phi(f,g) := f+g$. Then Φ is linear with norm $\|\Phi\|\le 2$. Moreover, Φ admits an inverse defined by

$$(\Phi^{-1}h)(x) = \left(\frac{\alpha(\|h(x)\|)}{\|h(x)\|}h(x), \frac{\beta(\|h(x)\|)}{\|h(x)\|}h(x)\right).$$

Since $x\mapsto \frac{\alpha(\|x\|)}{\|x\|}x$ and $x\mapsto\frac{\beta(\|x\|)}{\|x\|}x$ are 1-Lipschitz, it follows that $\|\Phi^{-1}\|\le 1$. Now Lemma 4.5 yields an isomorphism Ψ from $\mathrm{Lip}_0(X)\oplus_\infty \mathrm{Lip}_0(X)$ onto $Y_L\oplus_\infty Y_R$ satisfying $\|\Psi\|\cdot\|\Psi^{-1}\|\le \frac{4}{3}\cdot\frac{3}{2}=2$. Then $\Phi\circ\Psi$ is an isomorphism from $\mathrm{Lip}_0(X)\oplus_\infty\mathrm{Lip}_0(X)$ onto $\mathrm{Lip}_0(X)$ satisfying $\|\Phi\circ\Psi\|\cdot\|(\Phi\circ\Psi)^{-1}\|\le 4$. Since Φ and Ψ are pointwise-to-pointwise continuous, $\Phi\circ\Psi$ induces an isomorphism $T:\mathcal{F}(X)\to\mathcal{F}(X)\oplus_1\mathcal{F}(X)$ satisfying $T^*=\Phi\circ\Psi$ and $\|T\|\cdot\|T^{-1}\|\le 4$. ∎

Acknowledgements. This research was supported by CAPES, grant BEX 10057/12-9.

I am grateful for having been warmly welcomed by the Équipe d'Analyse Fonctionnelle during my postdoctoral period at the IMJ. I am especially grateful to Gilles Godefroy for many fruitful discussions which led to several improvements of this work. I thank Florent Baudier for driving my attention to [11]. I thank Bruno Braga, Marek Cúth and Michal Doucha for reading my original manuscript and pointing out flaws to be corrected. I thank the referee for relevant comments that helped improving and better formatting this work. The illustration was made by Gabriela Kaufmann Sacchetto.

References

[1] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Vol. 1, Amer. Math. Soc. Colloq. Publ. 48, Amer. Math. Soc., Providence, 2000.

[2] A. Brudnyi and Y. Brudnyi, *Metric spaces with linear extensions preserving Lipschitz condition*, Amer. J. Math. 129 (2007), 217–314.

[3] Y. Dutrieux and V. Ferenczi, *The Lipschitz free Banach spaces of $C(K)$-spaces*, Proc. Amer. Math. Soc. 134 (2006), 1039–1044.

[4] A. Godard, *Tree metrics and their Lipschitz-free spaces*, Proc. Amer. Math. Soc. 138 (2010), 4311–4320.

[5] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, Studia Math. 159 (2003), 121–141.

[6] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Springer, 2001.

[7] P. Hájek and E. Pernecká, *On Schauder bases in Lipschitz-free spaces*, J. Math. Anal. Appl. 416 (2014), 629–646.

[8] W. B. Johnson, J. Lindenstrauss and G. Schechtman, *Banach spaces determined by their uniform structures*, Geom. Funct. Anal. 6 (1996), 430–470.

[9] N. J. Kalton, *Spaces of Lipschitz and Hölder functions and their applications*, Collect. Math. 55 (2004), 171–217.

[10] G. Lancien and E. Pernecká, *Approximation properties and Schauder decompositions in Lipschitz-free spaces*, J. Funct. Anal. 264 (2013), 2323–2334.

[11] U. Lang and C. Plaut, *Bilipschitz embeddings of metric spaces into space forms*, Geom. Dedicata 87 (2001), 285–307.

[12] J. R. Lee and A. Naor, *Extending Lipschitz functions via random metric partitions*, Invent. Math. 160 (2005), 59–95.

[13] A. Naor and G. Schechtman, *Planar earthmover is not in L^1*, SIAM J. Comput. 37 (2007), 804–826.

[14] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. 19 (1960), 209–228.

[15] N. Weaver, *Lipschitz Algebras*, World Sci., River Edge, NJ, 1999.

Pedro Levit Kaufmann
Instituto de Ciência e Tecnologia
Universidade Federal de São Paulo
Campus São José dos Campos – Parque Tecnológico
Avenida Doutor Altino Bondensan, 500
12247-016 São José dos Campos/SP, Brazil
E-mail: plkaufmann@unifesp.br

Received April 21, 2014
Revised version March 10, 2015 (7936)