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Completely monotone functions of finite order and Agler's conditions

by

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Dedicated to our teacher Ameer Athavale on the occasion of his 60th birthday

Abstract. Motivated by some structural properties of Drury–Arveson d-shift, we investigate a class of functions consisting of polynomials and completely monotone functions defined on the semi-group $\mathbb{N}$ of non-negative integers, and its operator-theoretic counterpart which we refer to as the class of completely hypercontractive tuples of finite order. We obtain a Lévy–Khinchin type integral representation for the spherical generating tuples associated with such operator tuples and discuss its applications.

Contents

1. Introduction. A fruitful synthesis of harmonic analysis on semigroups and operator theory on Hilbert spaces allows one to explore natural connections between some special classes of functions on a semigroup and some special classes of operators on Hilbert spaces. In particular, the class of completely monotone functions gets naturally tied up with the class of contractive subnormal operators.

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Capitalizing on this association, in this paper, we propose to set up a general framework that would encompass the classes of operators such as contractive subnormals and m-isometries as special cases. Note that a non-constant polynomial does not belong to the class of completely monotone functions. We study a class of functions, referred to as completely monotone functions of finite order, that includes polynomials as well as completely monotone functions as special cases. We obtain an integral representation of such functions, study their properties and discuss the operator theory that gets naturally associated with these functions. A well known characterization for a subnormal contraction T, given by J. Agler [1], demands T to satisfy the following positivity conditions:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}T^{*k}T^k \geq 0 \quad \text{for all } n \geq 1.$$

A completely monotone function of order k gets coupled with an operator that satisfies the Agler conditions for all $n \geq k$, for some positive integer k. We carry out the analysis in the multi-variable set-up, especially the spherical case, and highlight some concrete examples of operator tuples in this class. The study of these operators is rewarding in the sense that it offers a unified treatment to a large class of operators that includes joint hypercontractions, p-isometries, and a wide range of examples not belonging to these classes as well.

The integral representations of various classes of functions, their properties and interesting interconnections among these classes have been thoroughly discussed in [11] and [30]. These books have played a decisive role in understanding the underlying function theory, to be associated with operator theory.

2. Prelude. The symbol $\mathbb{N}$ stands for the set of non-negative integers; $\mathbb{N}$ forms a semigroup under addition. Let $\mathbb{N}^m$ denote the cartesian product $\mathbb{N}\times\cdots\times\mathbb{N}$ (m times). Let $p \equiv (p_1,\ldots,p_m)$ and $n \equiv (n_1,\ldots,n_m)$ be in $\mathbb{N}^m$. We write $|p| := \sum_{i=1}^m p_i$ and $p \leq n$ if $p_i \leq n_i$ for $i = 1,\ldots,m$. For $n \in \mathbb{N}^m$, we let $n! := \prod_{i=1}^m n_i!$.

A real-valued map φ on $\mathbb{N}$ is said to be *positive definite* if

$$\sum_{i,j=1}^{n} c_i\bar{c}_j\varphi(s_i+s_j) \geq 0$$

for all $n \geq 1$, $\{s_1,\ldots,s_n\} \subseteq \mathbb{N}$ and $\{c_1,\ldots,c_n\} \subseteq \mathbb{C}$, the field of complex numbers. A real-valued map ψ on $\mathbb{N}$ is said to be *conditionally positive definite* if $\sum_{i,j=1}^n c_i\bar{c}_j\psi(s_i+s_j) \geq 0$ for all $n \geq 1$, all $\{s_1,\ldots,s_n\} \subseteq \mathbb{N}$ and all $\{c_1,\ldots,c_n\} \subseteq \mathbb{C}$ such that $\sum_{i=1}^n c_i = 0$. A real-valued map ψ on $\mathbb{N}$ is said to be *negative definite* if $-\psi$ is conditionally positive definite.

We find it convenient to include the following characterization of negative definite sequences for ready reference.

THEOREM 2.1 ([11, Theorem 2.6, Chapter 6]). *A function $\psi : \mathbb{N} \to \mathbb{R}$ is negative definite if and only if it has a representation of the form*

$$\psi(n) = a + bn - cn^2 + \int_{\mathbb{R}\setminus\{1\}} (1 - x^n - n(1-x))\, d\mu(x) \qquad (n \in \mathbb{N}), \tag{2.1}$$

where $a, b \in \mathbb{R}$, $c \geq 0$, and μ is a Radon measure on $\mathbb{R} \setminus \{1\}$ such that

$$\int_{0<|x-1|<1} (1-x)^2\, d\mu(x) < \infty, \qquad \int_{|x-1|\geq 1} |x|^n\, d\mu(x) < \infty \quad (n \in \mathbb{N}).$$

REMARK 2.2. The representing measure μ appearing in the integral representation of negative definite sequences will be referred to as the *Lévy measure* associated with ψ. Such a representing measure may not be unique (refer to [11, Chapter 6]).

For a real-valued map φ on $\mathbb{N}$, we define (backward and forward) difference operators ∇ and Δ as follows: $(\nabla\varphi)(n) = \varphi(n) - \varphi(n+1)$ and $(\Delta\varphi)(n) = \varphi(n+1) - \varphi(n)$. The operators ∇^n and Δ^n are inductively defined for all $n \geq 0$ through the relations $\nabla^0\varphi = \Delta^0\varphi = \varphi$, $\nabla^n\varphi = \nabla(\nabla^{n-1}\varphi)$ $(n \geq 1)$, $\Delta^n\varphi = \Delta(\Delta^{n-1}\varphi)$ $(n \geq 1)$. A non-negative map φ on $\mathbb{N}$ is said to be *completely monotone* if $(\nabla^k\varphi)(n) \geq 0$ for all $n \geq 0$ and $k \geq 1$. A real-valued map ψ on $\mathbb{N}$ is said to be *completely alternating* if $(\nabla^k\psi)(n) \leq 0$ for all $n \geq 0$ and $k \geq 1$. Completely monotone maps on $\mathbb{N}$ form an extreme subset of the set of positive definite functions on $\mathbb{N}$, while the completely alternating maps form an extreme subset of the set of negative definite functions on $\mathbb{N}$ (refer to [11]).

For a complex, infinite-dimensional, separable Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$.

An operator S in $B(\mathcal{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N in $B(\mathcal{K})$ such that $N\mathcal{H} \subseteq \mathcal{H}$ and $N|_{\mathcal{H}} = S$. A comprehensive account of the theory of subnormal operators can be found in [15]. We say that T is *hyponormal* if its self-commutator $T^*T - TT^*$ is positive. Every subnormal operator is hyponormal. An operator T in $B(\mathcal{H})$ will be referred to as a *contraction* or an *expansion* depending on whether $I - T^*T \geq 0$ or $I - T^*T \leq 0$.

As mentioned earlier, J. Agler [1, Theorem 3.1] proved that T in $B(\mathcal{H})$ is a subnormal contraction if and only if

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k}T^k \geq 0 \quad \text{for all } n \geq 1. \tag{2.2}$$

Note that Agler's result is an immediate consequence of Hausdorff's theorem on solution of the moment problem [11] and Lambert's characterization of subnormality [26] (see also the discussion following [1, Theorem 3.1]). It was observed in [5] that the condition (2.2) is equivalent to requiring, for every h in $\mathcal{H}$, the map $\varphi_h(n) = \|T^n h\|^2$ to be completely monotone on $\mathbb{N}$.

Completely hyperexpansive operators were introduced independently in [3] and [7]. A bounded linear operator T on $\mathcal{H}$ is said to be *completely hyperexpansive* if

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}T^{*k}T^k \leq 0 \quad \text{for all } n \geq 1. \tag{2.3}$$

It was observed in [7] that the condition (2.3) is equivalent to requiring, for every h in $\mathcal{H}$, the map $\psi_h(n) = \|T^n h\|^2$ to be completely alternating on $\mathbb{N}$. The symbiotic relationship between completely monotone and completely alternating maps carries over to subnormal contractions and completely hyperexpansive operators, and this theme was focussed upon in [7].

By a *commuting m-tuple* T on $\mathcal{H}$, we mean a tuple $(T_1, \ldots, T_m)$ of commuting bounded linear operators $T_1, \ldots, T_m$ on $\mathcal{H}$. For a commuting m-tuple T on $\mathcal{H}$, we interpret T^* to be $(T_1^*, \ldots, T_m^*)$, and T^p to be $T_1^{p_1} \cdots T_m^{p_m}$ for $p = (p_1, \ldots, p_m) \in \mathbb{N}^m$.

For the definitions and the basic theory of various spectra including the Taylor spectrum, the reader is referred to [17]. For a commuting m-tuple T, we reserve the symbols $\sigma(T)$ and $\sigma_{\mathrm{ap}}(T)$ for the Taylor spectrum and the approximate-point spectrum of T respectively. Let π denote the Calkin map from $B(\mathcal{H})$ into the Calkin algebra $B(\mathcal{H})/K(\mathcal{H})$, where $K(\mathcal{H})$ denotes the ideal of compact operators on $\mathcal{H}$.

Recall that a commuting m-tuple $T = (T_1, \ldots, T_m)$ on a Hilbert space $\mathcal{H}$ is said to be *joint subnormal* if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a commuting m-tuple $N = (N_1, \ldots, N_m)$ of normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that

$$N_i h = T_i h \quad \text{for every } h \in \mathcal{H} \text{ and } 1 \leq i \leq m.$$

Given a commuting m-tuple $T = (T_1, \ldots, T_m)$ on $\mathcal{H}$, we set

$$Q_T(X) := \sum_{i=1}^{m} T_i^* X T_i \ (X \in B(\mathcal{H})). \tag{2.4}$$

We refer to Q_T as the *spherical generating* 1*-tuple associated with* T. The operator Q_T^n is inductively defined for all $n \geq 0$ through the relations $Q_T^0(X) = X$ and $Q_T^n(X) = Q_T(Q_T^{n-1}(X))$ $(n \geq 1)$ for $X \in B(\mathcal{H})$. It is easy to see that $Q_T^n(I) = \sum_{|p|=n} \frac{n!}{p!} T^{*p} T^p$.

Definition 2.3. Let Q_T be as given in (2.4). For an integer $q \geq 0$, let

$$B_q(Q_T) := \sum_{s=0}^{q} (-1)^s \binom{q}{s} Q_T^s(I). \tag{2.5}$$

Fix an integer $q \geq 1$. If $B_q(Q_T) = 0$, then T is called as a *joint q-isometry.* We say that T is a *joint q-contraction* (resp. *joint q-expansion*) if

$$B_q(Q_T) \geq 0 \quad (\text{resp. } \leq 0).$$

We say that T is a *joint complete hypercontraction* (resp. *joint complete hyperexpansion*) if T is a joint q-contraction (resp. joint q-expansion) for all positive integers q. In all the definitions above, if $q = 1$ then we drop the prefix “1-”, and if $m = 1$ then we drop the word “joint”.

Remark 2.4. It is known that a joint q-isometry is a joint p-isometry for every integer $p \geq q$. This may be deduced from the identity

$$B_{n+1}(Q_T) = B_n(Q_T) - Q_T(B_n(Q_T)) \tag{2.6}$$

for any integer $n \geq 1$ (refer to [19]).

An m-variable weighted shift $T = (T_1, \ldots, T_m)$ with respect to an orthonormal basis $\{e_n\}_{n \in \mathbb{N}^m}$ of a Hilbert space $\mathcal{H}$ is defined by

$$T_i e_n := w_n^{(i)} e_{n+\epsilon_i} \quad (1 \leq i \leq m),$$

where ϵ_i is the m-tuple with 1 in the ith place and zeros elsewhere. We always assume that the weight multi-sequence of T consists of positive numbers and that T is commuting. Notice that T_i commutes with T_j if and only if $w_n^{(i)} w_{n+\epsilon_i}^{(j)} = w_n^{(j)} w_{n+\epsilon_j}^{(i)}$ for all $n \in \mathbb{N}^m$.

The multiplication m-tuple M_z on the Hardy space $H^2(\mathbb{B})$ of the unit ball (commonly known as the *Szegö m-shift*) provides an important example of a joint isometry and joint subnormal m-tuple, where $\mathbb{B}$ denotes the open unit ball in the m-dimensional hermitian space $\mathbb{C}^m$. The Szegö m-shift is an m-variable weighted shift with weight multi-sequence

$$\left\{ \sqrt{\frac{n_i + 1}{|n| + m}} : 1 \leq i \leq m,\, n \in \mathbb{N}^m \right\}.$$

Example 2.5. The *Drury–Arveson m-shift* is the operator m-tuple $M_{z,m}$ of multiplication by the coordinate functions $z_1, \ldots, z_m$ in the reproducing kernel Hilbert space associated with the positive definite kernel

$$\frac{1}{1 - z_1\overline{w}_1 - \cdots - z_m\overline{w}_m} \quad (z, w \in \mathbb{B}),$$

where $\overline{u}$ denotes the complex conjugate of the complex number u [4]. The multiplication m-tuple $M_{z,m}$ can also be realized as the weighted shift with weight multi-sequence

$$\left\{\sqrt{\frac{n_i+1}{|n|+1}} : 1 \leq i \leq m,\ n \in \mathbb{N}^m\right\}.$$

It is observed in [19] that the Drury–Arveson m-shift is a joint m-isometry: $B_m(Q_{M_{z,m}}) = 0$.

As a multivariable generalization of a result by Agler and Stankus [2], it was observed in [19] that for a joint q-isometry T, $B_{q-1}(Q_T) \leq 0$ if q is even, and $B_{q-1}(Q_T) \geq 0$ otherwise. We shall obtain the latter result as a special case of a more general fact about completely hypercontractive tuples of finite order (see Proposition 4.10 below).

The following connection between joint subnormal tuples and completely monotone functions is well-known [5]. We include it for the sake of completeness.

Proposition 2.6. *Let T be a joint subnormal and a joint contraction m-tuple on $\mathcal{H}$. Let Q_T be the spherical generating* 1*-tuple associated with T. Then for every $h \in \mathcal{H}$, the sequence $\{\langle Q_T^k(I)h, h\rangle\}_{k\in\mathbb{N}}$ is completely monotone.*

The connection between conditionally positive definite functions and joint subnormal tuples was first noticed in [34, Theorem 4.1]. The following simple observation connects the study of joint 3-isometries with the theory of conditionally positive definite functions.

Proposition 2.7. *Let T be a joint q-isometry on $\mathcal{H}$ and let Q_T be the spherical generating* 1*-tuple associated with T. If $q \leq 3$ then for every $h \in \mathcal{H}$, the sequence $\{\langle Q_T^k(I)h, h\rangle\}_{k\in\mathbb{N}}$ is conditionally positive definite.*

Proof. Without loss of generality, we may assume that T is a joint 3-isometry. Let $B_q(Q_T)$ be as given in (2.5). A straightforward inductive argument shows that $Q_T^n(I) = I + nA + n^2B$ for all integers $n \geq 1$, where

$$A = -B_1(Q_T) - \frac{1}{2}B_2(Q_T), \qquad B = \frac{1}{2}B_2(Q_T).$$

By [19, Proposition 2.3], B is positive. We now check that $\{\langle Q_T^k(I)h, h\rangle\}_{k\in\mathbb{N}}$ is conditionally positive definite. To see that, let $c_1, \ldots, c_n \in \mathbb{C}$ be such that $\sum_{i=0}^n c_i = 0$. Then

$$\begin{aligned}\sum_{j,k=0}^{n} c_j\overline{c}_k Q^{j+k}(I) &= \sum_{j,k=0}^{n} c_j\overline{c}_k(I + (j+k)A + (j+k)^2B)\\ &= 2\Big|\sum_{j=0}^{n} jc_j\Big|^2 B \geq 0.\end{aligned}$$

Note that this can also be deduced from Theorem 2.1. ∎

REMARK 2.8. Note that for any $h \in \mathcal{H}$, the Lévy measure in the integral representation (2.1) of the negative definite sequence $\{\langle -Q_T^k(I)h, h\rangle\}_{k\in\mathbb{N}}$ can be chosen to be the zero measure.

The present work is partly motivated by the work of Aleman [3], in which the idea of renorming of Hilbert norms is used for modelling Dirichlet-type operators. This interesting idea, which first appeared in the proof of [3, Theorem 2.5], can be efficiently used to get integral representations for various classes of hypercontractive tuples. As an illustration, we implement this idea in the proof of the following theorem. We shall prove a vast generalization of this result later (see Theorem 4.11).

THEOREM 2.9. *Let T be an m-tuple on $\mathcal{H}$ and let Q_T be the spherical generating* 1*-tuple associated with T. If $B_1(Q_T)$ is semibounded from below by a positive constant then the following statements are equivalent:*

(1) *T is a joint k-contraction for all $k \geq 2$.*
(2) *There exist a semispectral measure E on $[0,1]$ and a self-adjoint operator $A \in B(\mathcal{H})$ such that for every integer $n \geq 0$,*

$$Q_T^n(I) = I + nA + \frac{n^2}{2}E(\{1\}) + \int\limits_{[0,1)} \frac{n(1-x) - 1 + x^n}{(1-x)^2}\, dE(x).$$

(3) *For every $h \in \mathcal{H}$, the sequence $\{\langle Q^n(I)h, h\rangle_{n\geq 0}\}$ is conditionally positive definite with associated Lévy measure supported on $[0,1)$.*

If the integral representation in (2) *exists, then it is unique.*

REMARK 2.10. The integrand $\frac{n(1-x)-1+x^n}{(1-x)^2}$, $x \in [0,1)$, appearing in the integral representation of (2) above is non-negative and bounded.

Proof of Theorem 2.9. Suppose that $B_1(Q_T)$ is semibounded from below by a positive constant.

(1) implies (2): We introduce a new norm $\|\cdot\|_0$ on $\mathcal{H}$:

$$\|h\|_0^2 := \sum_{i=1}^m \langle T_i h, T_i h\rangle - \|h\|^2 \quad (h \in \mathcal{H}).$$

Let $\langle\cdot\,,\cdot\rangle_0$ denote the inner product induced by $\|\cdot\|_0$ and let $\mathcal{H}_0$ denote the inner product space $\mathcal{H}$ endowed with the inner product $\langle\cdot,\cdot\rangle_0$. Let T_0 denote the m-tuple on $\mathcal{H}_0$ whose action is the same as that of T. Since $B_1(Q_T)$ is invertible, the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. In particular, $\mathcal{H}_0$ is a Hilbert space and T_0 is a commuting m-tuple of bounded linear operators on $\mathcal{H}_0$.

By using the identity (2.6), it is easy to see that

$$\langle B_n(Q_{T_0})h, h\rangle_0 = -\langle B_{n+1}(Q_T)h, h\rangle \quad (h \in \mathcal{H},\ n \geq 1).$$

It follows that T_0 is a joint complete hyperexpansion m-tuple acting on $\mathcal{H}_0$. A straightforward adaptation of [20, proof of Theorem 4.2] yields a unique semispectral measure F on $[0,1]$ such that

$$\langle Q^n_{T_0}(I)h,h\rangle_0 = \|h\|_0^2 + \int\limits_{[0,1]} (1+x+\cdots+x^{n-1})\,\langle dF(x)h,h\rangle_0 \quad (h\in\mathcal{H},\, n\ge 1).$$

Since $\langle Q^n_{T_0}(I)h,h\rangle_0 = \langle Q^{n+1}_T(I)h,h\rangle - \langle Q^n_T(I)h,h\rangle$, we obtain

$$\begin{aligned}\langle Q^{n+1}_T(I)h,h\rangle &= \langle Q^n_T(I)h,h\rangle + \langle Q_T(I)h,h\rangle - \|h\|^2 \\ &\quad + \int\limits_{[0,1]} (1+x+\cdots+x^{n-1})\,\langle dE(x)h,h\rangle \quad (h\in\mathcal{H},\, n\ge 1),\end{aligned}$$

where $E(\cdot)$ is the semispectral measure $(Q_T(I)-I)F(\cdot)$ governed by

$$\langle E(\cdot)f,g\rangle = \langle F(\cdot)f,g\rangle_0 \quad (f,g\in\mathcal{H}).$$

This gives us the following representation for any integer $n\ge 1$:

$$Q^{n+1}_T(I) = Q^n_T(I) + (Q_T(I)-I) + \int\limits_{[0,1]} (1+x+\cdots+x^{n-1})\,dE(x).$$

Using this formula recursively, we obtain, for every integer $n\ge 1$,

$$\begin{aligned}Q^{n+1}_T(I) &= Q_T(I) + n(Q_T(I)-I) + \int\limits_{[0,1]} \Big(\sum_{j=1}^{n}\sum_{i=0}^{n-j} x^i\Big)\,dE(x) \\ &= Q_T(I) + n(Q_T(I)-I) + \frac{n(n+1)}{2}E(\{1\}) \\ &\quad + \int\limits_{[0,1)} \Big(\sum_{j=1}^{n}\frac{1-x^{n-j+1}}{1-x}\Big)\,dE(x).\end{aligned}$$

Note that

$$\sum_{j=1}^{n}\frac{1-x^{n-j+1}}{1-x} = \frac{n-x^n-\cdots-x}{1-x} = \frac{(n+1)(1-x)-1+x^{n+1}}{(1-x)^2}.$$

It follows that

$$\begin{aligned}Q^n_T(I) &= Q_T(I) + (n-1)(Q_T(I)-I) + \frac{(n-1)n}{2}E(\{1\}) \\ &\quad + \int\limits_{[0,1)} \frac{n(1-x)-1+x^n}{(1-x)^2}\,dE(x) \\ &= I + n(Q_T(I)-I-E(\{1\})/2) + \frac{n^2}{2}E(\{1\}) \\ &\quad + \int\limits_{[0,1)} \frac{n(1-x)-1+x^n}{(1-x)^2}\,dE(x)\end{aligned}$$

for every integer $n \geq 2$. Note that the last identities also hold for $n = 0, 1$. The desired representation holds with $A = Q_T(I) - I - E(\{1\})/2$.

(2) implies (3): This is a straightforward consequence of Theorem 2.1.

(3) implies (1): Since $\{-\langle Q_T^n(I)h, h\rangle\}_{n\in\mathbb{N}}$ is negative definite with associated Lévy measure supported on $[0,1)$, by Theorem 2.1, there exists a positive finite Borel measure μ_h on $[0,1)$ such that for every integer $n \geq 0$,

$$\langle Q_T^n(I)h, h\rangle = \|h\|^2 + an + bn^2 + \int\limits_{[0,1)} \frac{n(1-x) - 1 + x^n}{(1-x)^2}\, d\mu_h(x),$$

where the constants $a \in \mathbb{R}$ and $b \geq 0$ depend on h. Fix $q \geq 2$ and observe that

$$\langle B_q(Q_T)h, h\rangle = \sum_{s=0}^{q} (-1)^s \binom{q}{s} (\|h\|^2 + as + bs^2 + C_h(s)),$$

where

$$C_h(s) := \int\limits_{[0,1)} \frac{s(1-x) - 1 + x^s}{(1-x)^2}\, d\mu_h(x).$$

Since $\sum_{s=0}^{q}(-1)^s\binom{q}{s}p(s) = 0$ for any polynomial p of degree less than q by [18, Theorem 8.4], we obtain

$$\langle B_q(Q_T)h, h\rangle = \sum_{s=0}^{q} (-1)^s \binom{q}{s} (bs^2 + D_h(s)),$$

where

$$D_h(s) := \int\limits_{[0,1)} \frac{x^s}{(1-x)^2}\, d\mu_h(x).$$

It is clear that

$$\sum_{s=0}^{q} (-1)^s \binom{q}{s} bs^2 = \begin{cases} 2b \geq 0 & \text{if } q = 2, \\ 0 & \text{if } q \geq 3. \end{cases}$$

Finally note that $\sum_{s=0}^{q}(-1)^s\binom{q}{s}x^s = (1-x)^q$, and hence

$$\sum_{s=0}^{q} (-1)^s \binom{q}{s} D_h(s) = \int\limits_{[0,1)} (1-x)^{q-2}\, d\mu_h(x) \geq 0.$$

Uniqueness: We divide the verification into the following steps.

• Since $E(\{1\}) = Q_T^n(B_2(Q_T)) - \int_{[0,1)} x^n dE(x)$, the uniqueness of $E(\{1\})$ follows from Lebesgue's Dominated Convergence Theorem.

• The uniqueness of A follows by letting $n = 1$ in the integral representation in view of the uniqueness of $E(\{1\})$.

• To see the uniqueness of the semispectral measure $E(\cdot)$ on $[0,1)$, it suffices to check that $\int_{[0,1)} x^k \, dF(x) = 0$ for all integers $k \geq 0$ provided

$$I_n := \int\limits_{[0,1)} \frac{n(1-x) - 1 + x^n}{(1-x)^2} \, dF(x) = 0 \quad (n \geq 1), \tag{2.7}$$

where $F(\cdot)$ is a complex $B(\mathcal{H})$-valued measure (cf. [20, Lemma 4.1]). This is due to the determinacy of the Hausdorff moment problem. We verify this by induction on k. One may let $n = 2$ in (2.7) to see that $\int_{[0,1)} 1 \, dF(x) = 0$. Suppose that the induction hypothesis holds for $k = 0, \ldots, n-1$. By the induction hypothesis, we obtain

$$\begin{aligned} I_{n+2} &= \int\limits_{[0,1)} \frac{(n+2) - (1 + x + \cdots + x^{n+1})}{1-x} \, dF(x) \\ &= \sum_{i=1}^{n+1} \int\limits_{[0,1)} \frac{1-x^i}{1-x} \, dF(x) = \int\limits_{[0,1)} \frac{1-x^{n+1}}{1-x} \, dF(x) \\ &= \int\limits_{[0,1)} (1 + x + \cdots + x^n) \, dF(x) = \int\limits_{[0,1)} x^n \, dF(x). \end{aligned}$$

The desired conclusion now follows from (2.7). Alternatively, the uniqueness may also be derived from the uniqueness of $E(\{1\})$ and the following identity:

$$B_q(Q_T) = E(\{1\})\delta_{q,2} + \int\limits_{[0,1)} (1-x)^{q-2} dE(x), \quad q \geq 2.$$

This completes the proof of the theorem. ∎

3. Function theory. In Section 2, we dealt with completely monotone and conditionally positive definite sequences, and proved that these sequences occur naturally in the representations of q-isometries ($q \leq 3$) and joint k-contractions (for all $k \geq 2$). In this section, we define completely monotone sequences of finite order, which in some sense generalize both completely monotone and conditionally positive definite sequences. Further, we obtain an integral representation of these sequences. The operator-theoretic analogue of these sequences, studied in Section 4, is a crucial part of this paper.

3.1. Completely monotone sequences of finite order

DEFINITION 3.1. Let k be a non-negative integer. A map $\psi : \mathbb{N} \to (0, \infty)$ is said to be *completely monotone of order* k if $\nabla^k \psi$ is completely monotone. A completely monotone sequence of order 0 will be referred to as a *completely monotone sequence.*

The class of completely monotone sequences of order k will be denoted by $\mathcal{CM}_k$.

REMARK 3.2. Note that a completely monotone sequence $\psi : \mathbb{N} \to (0, \infty)$ of order 1 turns out to be completely monotone.

Here is an interesting subclass of completely monotone sequences of finite order.

EXAMPLE 3.3. Let k be a positive integer. Let δ_n be a completely monotone sequence and let p be a monic polynomial of degree at most k. Then $\tilde{\delta}_n := \delta_n + p(n)$ belongs to $\mathcal{CM}_k$. This follows from the basic fact that $\nabla^k p = 0$ but $\nabla^{k-1} p \neq 0$ for any polynomial p in n of degree $k-1$ (see [18, Theorem 8.4]).

REMARK 3.4. The sequence $\beta_n = n + \frac{1}{n+1}$ is completely monotone of order 2, which is not *convex* in the sense that β_{n+1}/β_n is non-increasing. Later we see that any convex sequence in $\mathcal{CM}_k$ is necessarily completely monotone (see Corollary 4.9 below).

We now give an example of a sequence in $\mathcal{CM}_k$ which is not a polynomial perturbation of a moment sequence. For simplicity, we illustrate this in case $k = 2$.

EXAMPLE 3.5. Consider the sequence

$$\psi(n) = n + 1 - H(n+1),$$

where $H(n)$ is nth partial sum of the harmonic series $\sum_{n=1}^{\infty} 1/n$. Note that $\nabla\psi(n) = \frac{1}{n+2} - 1$. Since $\frac{1}{n+2}$ is completely monotone, $\nabla^k \nabla\psi \geq 0$ for any $k \geq 1$. Thus $\psi \in \mathcal{CM}_2$.

We claim that ψ is not a polynomial perturbation of a completely monotone sequence. Assume the contrary, so that

$$\psi(n) = p(n) + \int_{[0,1]} x^n \, d\mu(x)$$

for a polynomial p and a positive Radon measure μ on $[0,1]$. Now applying ∇ two times on both sides, we obtain

$$\nabla^2\psi(n) = \nabla^2 p(n) + \int_{[0,1]} x^n (1-x)^2 \, d\mu(x) \quad \text{for every integer } n \geq 0.$$

Also,

$$\nabla^2\psi(n) = \frac{1}{n+2} - \frac{1}{n+3} = \int_{[0,1]} x^n (1-x) x \, dx \quad \text{for every } n \geq 0.$$

We thus obtain

$$\nabla^2 p(n) + \int_{[0,1]} x^n (1-x)^2 \, d\mu(x) = \int_{[0,1]} x^n (1-x) x \, dx \quad \text{for every } n \geq 0.$$

It may be deduced from Lebesgue's Dominated Convergence Theorem that $\nabla^2 p$ must be identically zero. Since a Stieltjes moment sequence (with a compactly supported representing measure) is determinate, the representing measures $(1-x)^2\, d\mu(x)$ and $(1-x)x\, dx$ of $\nabla^2\psi$ must coincide. In particular, $(1-x)\, d\mu(x) = x\, dx$. Thus for any $0 < \epsilon < 1$ and any positive integer l, we have

$$\begin{aligned}\psi(n) &\geq p(n) + \int\limits_{[0,1-\epsilon]} x^n\, d\mu(x) = p(n) + \int\limits_{[0,1-\epsilon]} \frac{x^n}{1-x} x\, dx \\ &= p(n) + \int\limits_{[0,1-\epsilon]} \sum_{j=0}^{\infty} x^{n+j+1}\, dx \geq p(n) + \int\limits_{[0,1-\epsilon]} \sum_{j=0}^{l} x^{n+j+1}\, dx \\ &= p(n) + \sum_{j=0}^{l} \frac{(1-\epsilon)^{n+j+2}}{n+j+2} \qquad \text{for every } n \geq 0.\end{aligned}$$

Letting ϵ tend to 0 on the right hand side, we obtain

$$\psi(n) \geq p(n) + \sum_{j=0}^{l} \frac{1}{n+j+2}$$

for any integer $l \geq 1$, which is impossible. Hence the claim is verified.

Let x be any real number. In keeping with the classical combinatorial theory, we define $(x)_0 = 1$, $(x)_1 = x$, and $(x)_k = x(x-1)\dots(x-k+1)$ for any integer $k \geq 2$.

We combine the solution of the Hausdorff moment problem with Newton's interpolation formula to characterize all completely monotone sequences of finite order.

THEOREM 3.6. *Let $\psi : \mathbb{N} \to (0,\infty)$ be given and let k be a positive integer. Then the following statements are equivalent:*

(1) *ψ is a completely monotone sequence of order k.*
(2) *There exist a polynomial p_k of degree $k-1$ and a positive Radon measure μ on $[0,1]$ such that for every $n \in \mathbb{N}$,*

$$\begin{aligned}\psi(n) = p_k(n) &+ (-1)^k \binom{n}{k} \mu(\{1\}) \\ &+ \int\limits_{[0,1)} \frac{1}{(1-x)^k}\left(x^n - \sum_{j=0}^{k-1} \frac{(x-1)^j}{j!}(n)_j\right) d\mu(x) \\ = p_k(n) &+ \int\limits_{[0,1]} \sum_{j=0}^{n-k}\left((-1)^k \frac{(x-1)^j}{(j+k)!}(n)_{j+k}\right) d\mu(x),\end{aligned}$$

where the integral in the last expression is absent if $n < k$.

If (2) *holds then the integral representation in* (2) *is unique in the sense that the coefficients of* p_k *and the measure* μ *are completely determined by* ψ.

Proof. (1) implies (2): Suppose that $\nabla^k\psi$ is completely monotone. Thus there exists a positive Radon measure μ on $[0,1]$ such that

$$\nabla^k\psi(n) = \int\limits_{[0,1]} x^n\,d\mu(x) \qquad (n\in\mathbb{N}). \tag{3.8}$$

The proof relies on Newton's Interpolation Formula,

$$\beta(n) = \sum_{j=0}^{n}\binom{n}{j}\Delta^j\beta(0) \qquad (n\in\mathbb{N}).$$

Note that Newton's Interpolation Formula is exactly Newton's Binomial Formula because $\beta(n) = (I+\Delta)^n\beta(0)$ for $n\in\mathbb{N}$.

We claim that (2) holds with the choices

$$p_k(n) = \sum_{j=0}^{k-1}\frac{\Delta^j\psi(0)}{j!}(n)_j$$

and probability measure μ as appearing in (3.8). To see that, first note that by (3.8),

$$\Delta^{k+l}\psi(0) = \int\limits_{[0,1]}(-1)^k(x-1)^l\,d\mu(x) \qquad (l\in\mathbb{N}).$$

It follows from Newton's Interpolation Formula that for every $n\in\mathbb{N}$,

$$\begin{aligned}\psi(n) &= p_k(n) + \sum_{j=k}^{n}\int\limits_{[0,1]}(-1)^k\frac{(x-1)^{j-k}}{j!}(n)_j\,d\mu(x)\\
&= p_k(n) + (-1)^k\binom{n}{k}\mu(\{1\})\\
&\quad+\int\limits_{[0,1)}\frac{1}{(1-x)^k}\sum_{j=k}^{n}\frac{(x-1)^j}{j!}(n)_j\,d\mu(x)\\
&= p_k(n) + (-1)^k\binom{n}{k}\mu(\{1\})\\
&\quad+\int\limits_{[0,1)}\frac{1}{(1-x)^k}\left(x^n-\sum_{j=0}^{k-1}\frac{(x-1)^j}{j!}(n)_j\right)d\mu(x).\end{aligned}$$

Both the integral representations in (2) follow immediately from this.

(2) implies (1): Recall that $\nabla^l p(n) = 0$ for any polynomial p in n of degree less than l. In particular, $\nabla^l p_k(n) = 0$ for any integer $l\ge k$. Also, $\nabla^k(-1)^k\binom{n}{k}\mu(\{1\})\ge 0$ and $\nabla^l(-1)^k\binom{n}{k}\mu(\{1\}) = 0$ for $l\ge k+1$. It now

suffices to check that $\nabla^l \gamma_k(n) \geq 0$ for all $l \geq k$, where

$$\gamma_k(n) := x^n - \sum_{j=0}^{k-1} \frac{(x-1)^j}{j!}(n)_j \quad (n \in \mathbb{N}).$$

Note that $\nabla^l x^n = x^n(1-x)^l$. Since $\sum_{j=0}^{k-1} \frac{(x-1)^j}{j!}(n)_j$ is a polynomial in n of degree less than k, the desired conclusion follows.

Uniqueness: It is easy to see that $p_k(n) = \sum_{j=0}^{k-1} \frac{\Delta^j \psi(0)}{j!}(n)_j$. Assume that

$$I_l := \int_{[0,1]} \sum_{j=0}^{l-k} \left((-1)^k \frac{(x-1)^j}{(j+k)!}(l)_{j+k} \right) d\mu(x) = 0 \quad (l \geq k), \tag{3.9}$$

where μ is a complex (or real) measure on $[0,1]$. In view of the Riesz Representation Theorem and the Weierstrass Approximation Theorem, it now suffices to check that $J_n := \int_{[0,1]} x^n \, d\mu(x) = 0$ for every integer $n \geq 0$ (cf. [20, Lemma 4.1]). We prove this by induction n. Note that $I_k = 0$ gives $J_0 = 0$. Now assume that $J_i = 0$ for $i = 1, \ldots, n-1$. It follows from (3.9) that $J_n = (-1)^k I_{n+k} = 0$. ∎

Corollary 3.7. *If ψ is a completely monotone sequence of order k such that $\nabla^p \psi = 0$ for some integer $p \geq 1$ then ψ is a polynomial of degree less than or equal to* $\min\{k, p-1\}$.

Proof. Suppose that $\nabla^p \psi = 0$ for some $p \geq 1$. We claim that the representing measure μ of ψ cannot have support on $[0,1)$ (see Theorem 3.6(2)). To see that, let $i := \max\{k, p\}$ and apply ∇^i on both sides of the integral representation of ψ to get

$$\begin{aligned}\nabla^i \psi(n) &= \nabla^i \left(p_k(n) + (-1)^k \binom{n}{k} \mu(\{1\}) \right) \\ &\quad + \int_{[0,1)} \frac{1}{(1-x)^k} \nabla^i \left(x^n - \sum_{j=0}^{k-1} \frac{(x-1)^j}{j!}(n)_j \right) d\mu(x) \\ &= \int_{[0,1)} x^n (1-x)^{i-k} \, d\mu(x),\end{aligned}$$

which is zero since $\nabla^i \psi(n) = 0$. Let $\epsilon > 0$ and note that

$$\int_{[0,1-\epsilon]} x^n (1-x)^{i-k} \, d\mu(x) = 0.$$

A simple application of the Stone–Weierstrass Theorem shows that $\mu|_{[0,1-\epsilon]}$ is identically zero. Since $\epsilon > 0$ is arbitrary, $\mu|_{[0,1)}$ is identically zero. It follows from the integral representation that ψ must be a polynomial of degree at most k. Also, since $\nabla^p \psi = 0$, ψ is a polynomial of degree less than p. The desired conclusion is now immediate. ∎

3.2. Completely monotone functions of finite order. The purpose of this section is two-fold. Firstly, the notion of completely monotone functions of finite order is a natural generalization of that of completely monotone functions, and hence from the function-theoretic viewpoint, it is of independent interest. Secondly, the tools from classical analysis, available in abundance, lead one to some subtle properties of this class. These properties carry over naturally to the operator level, and are otherwise difficult to discover.

DEFINITION 3.8. Let k be a non-negative integer. A C^∞ function $f : (0,\infty) \to (0,\infty)$ is said to be *completely monotone of order* k if $(-1)^l f^{(l)} \geq 0$ for all $l \geq k$, where $f^{(l)}$ denotes the lth derivative of f. A completely monotone function of order 0 will be referred to as a *completely monotone function.*

The class of completely monotone functions of order k will be denoted by $\mathcal{L}_k$.

REMARK 3.9. Note that f belongs to $\mathcal{L}_k$ if and only if $(-1)^k f^{(k)}$ is a completely monotone function.

It is clear from the definition that $\mathcal{L}_k$ is closed under addition. However, unlike the case $k = 1$, $\mathcal{L}_k$ may not be closed under pointwise multiplication. Note that x and e^{-x} belong to $\mathcal{L}_2$. However, the following formula reveals that $xe^{-x} \notin \mathcal{L}_k$ for any $k \geq 1$:

$$(xe^{-x})^{(n)} = (-1)^n e^{-x}(x-n) \quad \text{for any positive integer } n.$$

EXAMPLE 3.10. A polynomial of degree at most $k-1$ is a completely monotone function of order k, and a completely monotone function certainly belongs to $\mathcal{L}_1$. Further, if f is a perturbation of a completely monotone function by a polynomial of degree at most $k-1$, then f belongs to $\mathcal{L}_k$.

EXAMPLE 3.11. Consider the function $f(x) = \log(1+x)$ and let, for $x \in (0,\infty)$,

$$I(f)(x) = \int_0^x \log(1+y)\, dy.$$

Then $I(f)$ belongs to $\mathcal{L}_2$ but does not belong to $\mathcal{L}_1$. More generally, $I^{2k-1}(f)$ belongs to $\mathcal{L}_{2k}$ but does not belong to $\mathcal{L}_{2k-1}$ for any positive integer k.

REMARK 3.12. It is clear from the definition that $\mathcal{L}_k \subseteq \mathcal{L}_{k+1}$. However, we shall see that the classes $\mathcal{L}_k$ and $\mathcal{L}_{k+1}$ coincide if k is even. The examples above show that the containment is strict if k is odd. Thus for any integer $k \geq 1$, we have

$$\mathcal{L}_{2k-1} \subsetneq \mathcal{L}_{2k} = \mathcal{L}_{2k+1} \subsetneq \mathcal{L}_{2k+2}.$$

Capitalizing on the observation made in Remark 3.9 and using the integral representation of a completely monotone function, we get the following integral representation for a completely monotone function of finite order.

THEOREM 3.13. *Let k be a positive integer and $f:(0,\infty)\to\mathbb{R}$ be a C^∞ function such that $f^{(l)}(0+)<\infty$ for $0\le l\le k-1$. Then f is a completely monotone function of order k if and only if it admits the representation*

$$f(x)=p_k(x)+\int\limits_{(0,\infty)}\frac{e^{-tx}-q_{k-1}(tx)}{t^k}\,d\mu(t), \tag{3.10}$$

where μ is a measure on $[0,\infty)$, p_k is a polynomial of degree at most k with leading coefficient equal to $(-1)^k\mu(\{0\})/k!$, and q_{k-1} is the polynomial of degree $k-1$ given by

$$q_{k-1}(xt)=\sum_{n=0}^{k-1}(-1)^n\frac{(xt)^n}{n!},$$

the kth partial sum of the series expansion of e^{-xt}.

Proof. The proof relies on a result of Bernstein [30, Theorem 1.4], which characterizes completely monotone functions: *Every completely monotone function $f:(0,\infty)\to\mathbb{R}$ is the Laplace transform of a unique measure μ on $[0,\infty)$, that is, for all $\lambda>0$,*

$$f(\lambda)=\int\limits_{[0,\infty)}e^{-\lambda t}\,d\mu(t).$$

We use the fact that f belongs to $\mathcal{L}_k$ if and only if $(-1)^kf^{(k)}$ is a completely monotone function. Thus by using the integral representation of $(-1)^kf^{(k)}$ as ensured by Bernstein's Theorem, we shall obtain the integral representation of $f^{(k-1)}$, and then by finite backward induction, we obtain the representation for f. We remark that at every inductive step the integrands involved are non-negative or non-positive, and hence the Fubini–Tonelli Theorem justifies the change of order of integration.

We claim that for $m=1,\dots,k$,

$$f^{(k-m)}(x)=p_m(x)+(-1)^{k-m}\int\limits_{(0,\infty)}\frac{e^{-tx}-q_{m-1}(xt)}{t^m}\,d\mu(t).$$

Indeed, we have

$$\begin{aligned}f^{(k-1)}(x)&=\int_0^x f^{(k)}(y)\,dy+f^{(k-1)}(0+)\\&=(-1)^k\int_0^x(-1)^kf^{(k)}(y)\,dy+f^{(k-1)}(0+)\end{aligned}$$

$$= (-1)^k \int_0^x \int_{[0,\infty)} e^{-ty}\, d\mu(t)\, dy + f^{(k-1)}(0+)$$

$$= (-1)^k \int_0^x \Big[\mu(\{0\}) + \int_{(0,\infty)} e^{-ty}\, d\mu(t)\, dy\Big] + f^{(k-1)}(0+)$$

$$= p_1(x) + (-1)^{k-1} \int_{(0,\infty)} \frac{(e^{-tx} - 1)}{t}\, d\mu(t)$$

$$= p_1(x) + (-1)^{k-1} \int_{(0,\infty)} \frac{(e^{-tx} - q_0(xt))}{t}\, d\mu(t),$$

where $p_1(x) := f^{(k-1)}(0+) + (-1)^k x\mu(\{0\})$ and $q_0(xt) = 1$. Thus the claim holds for $m = 1$.

Now suppose that for some m, $1 < m < k$, we have

$$f^{(k-m)} = p_m(x) + (-1)^{k-m} \int_{(0,\infty)} \frac{e^{-tx} - q_{m-1}(xt)}{t^m}\, d\mu(t),$$

where $p_m(x)$ is a polynomial of degree m with leading coefficient $(-1)^k\mu\{0\}/m!$ and $q_{m-1}(xt)$ is the mth partial sum of e^{-xt}. Now

$$\begin{aligned} f^{(k-m-1)}(x) &= \int_0^x f^{(k-m)}(y)dy + f^{(k-m-1)}(0+) \\ &= \int_0^x \left(p_m(x) + (-1)^{k-m} \int_{(0,\infty)} \frac{e^{-ty} - q_{m-1}(yt)}{t^m}\, d\mu(t)\right) dy \\ &\quad + f^{(k-m-1)}(0+) \\ &= p_{m+1}(x) + (-1)^{k-m} \int_{(0,\infty)} \int_0^x \frac{e^{-ty} - q_{m-1}(yt)}{t^m}\, dy\, d\mu(t) \\ &= p_{m+1}(x) + (-1)^{k-m} \int_{(0,\infty)} \frac{-e^{-tx} + q_m(xt)}{t^{m+1}}\, d\mu(t) \\ &= p_{m+1}(x) + (-1)^{k-m-1} \int_{(0,\infty)} \frac{e^{-tx} - q_m(xt)}{t^{m+1}}\, d\mu(t). \end{aligned}$$

In the equations above, $p_{m+1}(x)$ is a polynomial of degree $m+1$ with leading coefficient $(-1)^k\mu(\{0\})/(m+1)!$ and $q_{m-1}(xt)$ is the $(m+1)$th partial sum of e^{-xt}. The desired integral representation is the case of $m = k$, and now follows by induction. Note that the leading coefficient of p_k is $(-1)^k\mu(\{0\})/k!$.

Conversely, if f has a representation as stated in the theorem, then

$$f^{(k)}(x) = (-1)^k \mu(\{0\}) + (-1)^k \int\limits_{(0,\infty)} t^k e^{-tx}\, d\mu(t).$$

It follows that $(-1)^k f^{(k)}$ is a completely monotone function, and hence f is completely monotone of order k. ∎

Remark 3.14. Observe that if k is odd, $e^{-tx} - q_{k-1}(tx) \leq 0$ for all $x \in (0,\infty)$. Also in this case, $(-1)^k \mu(\{0\})/k!$, the leading coefficient of p_k, is negative, unless $\mu(\{0\}) = 0$. Thus if k is odd, there does not exist a completely monotone function of order k which is not completely monotone of order $k-1$. We shall rederive this result by using a more general statement (see Proposition 3.17).

Corollary 3.15. *If $f : (0,\infty) \to (0,\infty)$ is a completely monotone function of order k such that $f^{(p)} = 0$ for some $p \geq 1$ then f is a polynomial of degree less than or equal to* $\min\{k, p-1\}$.

Proof. The proof is similar to the proof of Corollary 3.7, and hence we do not include it. ∎

Lemma 3.16. *Let $k \geq 2$ be an integer. Then there does not exist a C^∞ function $f : (0,\infty) \to (0,\infty)$ such that $f^{(k)}(x) \leq 0$ for all x and $f^{(k-1)}(x_0) < 0$ for some x_0.*

Proof. We begin with the case $k = 2$. Suppose such a function exists. Note that $f^{(2)}(x) \leq 0$ for all x implies that f' is a non-increasing function. Thus $f'(x) \leq f'(x_0) < 0$ for all $x > x_0$. Now consider the function

$$g(x) = f(x) - f(x_0) - f'(x_0)(x - x_0).$$

Note that $g(x_0) = 0$. Further $g'(x) = f'(x) - f'(x_0)$. Thus $g'(x) < 0$ if $x > x_0$. So g is decreasing in (x_0,∞). But as $g(x_0) = 0$, $g(x) \leq 0$ for all $x > x_0$. Hence

$$f(x) \leq f(x_0) + f'(x_0)(x - x_0).$$

Now as $f'(x_0) < 0$, there exists $x_1 > x_0$ such that $f(x_0) + f'(x_0)(x_1 - x_0) < 0$. Thus $f(x_1) < 0$, a contradiction.

A careful analysis of this argument reveals that if $f^{(2)}(x) \leq 0$ for all $x \in (0,\infty)$, then we get $x_0 \in (0,\infty)$ and $x_1 \in (x_0,\infty)$ such that $f'(x) \leq 0$ for all $x \in (x_0,\infty)$ and $f(x_1) < 0$. Now mimicking the argument for arbitrary k, we get an interval $I = (x_0,\infty)$ such that $f^{(k-1)}(x) \leq 0$ for all $x \in I$ and a point $x_1 \in I$ such that $f^{(k-2)}(x_1) < 0$. By repeating the argument for $f^{(k-1)}$ on I, we get an interval $J = (x_1,\infty)$ and a point $x_2 \in J$ such that $f^{(k-2)}(x) \leq 0$ for all $x \in J$ and $f^{(k-3)}(x_2) < 0$. Proceeding inductively yields the desired result. ∎

Proposition 3.17. *For an odd integer $k \geq 1$, if $f : (0,\infty) \to (0,\infty)$ belongs to $\mathcal{L}_k$, then f belongs to $\mathcal{L}_{k-1}$.*

Proof. If f is completely monotone of order k then $(-1)^k f^{(k)} \geq 0$. As k is odd, we have $f^{(k)} \leq 0$. Now Lemma 3.16 implies that $f^{(k-1)} \geq 0$, which in turn implies that f is completely monotone of order $k-1$. ■

Here is the discrete analog of Lemma 3.16: If $k \geq 2$ and $\{\psi(n)\}_{n\in\mathbb{N}}$ is a positive sequence such that $\Delta^k\psi(n) \leq 0$ for all $n \in \mathbb{N}$, then $\Delta^{k-1}\psi(n) \geq 0$ for all $n \in \mathbb{N}$. This follows simply by imitating the proof of Lemma 3.16 by replacing the derivative of a certain order by the difference operator of the respective order. The following result is now immediate.

Corollary 3.18. *For an odd integer $k \geq 1$, if ψ is a completely monotone sequence of order k, then ψ is a completely monotone of order $k-1$.*

We next provide a sufficient and necessary condition for a function in $\mathcal{L}_k$ to be completely monotone.

Proposition 3.19. *For an integer $k \geq 2$, let $f : (0,\infty) \to (0,\infty)$ be in $\mathcal{L}_k$. Then f is completely monotone if and only if $f' \leq 0$, where f' denotes the derivative of f.*

Proof. If $f \in \mathcal{L}_k$ then $(-1)^n f^{(n)} \geq 0$ for all $n \geq k$, and thus $(-1)^n f^{(n)} \geq 0$ for infinitely many values of n. So by [30, Proposition 1.9], the condition $f' \leq 0$ forces f to be completely monotone. ■

Here is the discrete analog of Proposition 3.19.

Corollary 3.20. *For an integer $k \geq 2$, let ψ be a completely monotone sequence of order k. Then ψ is completely monotone if and only if ψ is non-increasing.*

We conclude the section with a brief discussion of an interpolation problem. It is known that a completely monotone sequence $\{f(n)\}_{n\in\mathbb{N}}$ is an interpolation of a completely monotone function f on $[0,\infty)$ if and only if $\{f(n)\}_{n\in\mathbb{N}}$ is *minimal* in the sense that for any $\epsilon > 0$, the sequence $\{f(0)-\epsilon, f(1), f(2), \ldots\}$ is not completely monotone. For a detailed discussion of minimality, the reader is referred to [35, Chapter IV]. For the sake of completeness, we record that a completely monotone sequence $\{f(n)\}_{n\in\mathbb{N}}$ of order k is an interpolation of a completely monotone function $f : [0,\infty) \to (0,\infty)$ of order k if and only the representing measure μ appearing in (3.10) satisfies the condition $\mu(\{0\}) = 0$. Though this characterization provides a relation between completely monotone sequences of finite order and completely monotone functions of finite order, it does not have any bearing on the operator theory that we propose to discuss in the next section.

4. Operator theory. In the preceding sections, we developed the basic theory of completely monotone sequences of finite order as well as completely monotone functions of finite order. Though the function theory so developed has its own place, the motivation lies in the application of the theory for defining and studying a new class of operators, referred to below as completely hypercontractive tuples of finite order. The properties of sequences in $\mathcal{CM}_k$ get mirrored into the operator-theoretic properties of completely hypercontractive tuples of order k resulting into a rewarding fusion of the two theories. Association with function theory not only transforms function-theoretic statements into operator-theoretic ones, but also allows one to obtain new proofs of some known results in operator theory.

4.1. Completely hypercontractive tuples of finite order. Let T be a commuting m-tuple of operators $T_1, \ldots, T_m \in B(\mathcal{H})$, and let $Q_T(X) = \sum_{i=1}^m T_i^* X T_i$ $(X \in B(\mathcal{H}))$ be the spherical generating 1-tuple associated with T. Let $B_q(Q_T)$ be as defined in (2.5).

Definition 4.1. We say that T is a *joint complete hypercontraction of order* k if $B_q(Q_T) \geq 0$ for all $q \geq k$. A joint complete hypercontraction of order 1 will be referred to as a *joint complete hypercontraction.*

Remark 4.2. Note that T is a joint complete hypercontraction of order k if and only if for every $h \in \mathcal{H}$, the sequence $\{\langle Q_T^n(I)h, h\rangle\}_{n\in\mathbb{N}}$ is completely monotone of order k.

Example 4.3. Note that the direct sum of a joint k-isometry and a joint subnormal, joint contraction is a complete hypercontraction of order k.

Proposition 4.4. *Let T be a complete hypercontraction of order k. Then the following statements are equivalent:*

(1) *T is a joint p-isometry.*
(2) *T is a joint q-isometry with $q = \min\{k, p\}$.*

Proof. This follows from Corollary 3.7. ∎

Example 4.5. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a completely monotone sequence of order k such that $\beta_0 = 1$. Assume further that $\{\beta_{n+1}/\beta_n\}_{n\in\mathbb{N}}$ is bounded and let m be a positive integer. Consider the positive definite kernel given by

$$\kappa_\beta(z, w) = \sum_{n=0}^{\infty} \frac{\binom{n+m-1}{n}}{\beta_n} \langle z, w\rangle^n$$

defined for z, w in the open unit ball in $\mathbb{C}^m$. Consider the reproducing kernel Hilbert space $\mathscr{H}(\kappa_\beta)$ associated with κ_β and let $M_{z,\beta}$ denote the m-tuple of bounded linear multiplication operators $M_{z_1}, \ldots, M_{z_m}$ defined on $\mathscr{H}(\kappa_\beta)$. We check that $M_{z,\beta}$ is a complete hypercontraction of order k. Indeed, by an application of the multinomial theorem, we have

$$\kappa_\beta(z,w)=\sum_{n=0}^{\infty}\frac{\binom{n+m-1}{n}}{\beta_n}\sum_{|\alpha|=n}\frac{n!}{\alpha!}z^\alpha\overline{w}^\alpha=\sum_{\alpha\geq 0}\bigg(\frac{\binom{|\alpha|+m-1}{|\alpha|}}{\beta_{|\alpha|}}\,\frac{|\alpha|!}{\alpha!}\bigg)z^\alpha\overline{w}^\alpha.$$

It follows that

$$\|z^\alpha\|^2=\frac{\beta_{|\alpha|}}{\binom{|\alpha|+m-1}{|\alpha|}}\,\frac{\alpha!}{|\alpha|!},$$

and hence $\sum_{i=1}^m\|z^{\alpha+\epsilon_i}\|^2/\|z^\alpha\|^2=\beta_{|\alpha|+1}/\beta_{|\alpha|}$ for any $\alpha\in\mathbb{N}^m$. By [14, Lemma 3.1], $M_{z,\beta}$ is a complete hypercontraction of order k if and only if the one-variable weighted shift with weight sequence $\{\beta_{n+1}/\beta_n\}_{n\in\mathbb{N}}$ is a complete hypercontraction of order k. The latter holds true since the sequence $\{\beta_n\}_{n\in\mathbb{N}}$ is completely monotone of order k. In case $\beta_n=\binom{n+m-1}{n}$, the multiplication m-tuple $M_{z,\beta}$ is nothing but the Drury–Arveson m-shift.

LEMMA 4.6. *The approximate-point spectrum and the Taylor spectrum of a complete hypercontraction of finite order is contained in the closed unit ball. In particular, the spectral radius $r(T)=\sup\{\|z\|_2 : z\in\sigma(T)\}$ of T is at most* 1.

Proof. Let $\lambda\in\sigma_{\mathrm{ap}}(T)$. It is observed in the discussion prior to [19, Lemma 3.2] that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of unit vectors such that $\|T^\alpha x_n\|\to|\lambda^\alpha|$ as $n\to\infty$ for any $\alpha\in\mathbb{N}^m$. Now since $\lim_{n\to\infty}\langle B_q(Q_T)x_n,x_n\rangle\geq 0$ for odd positive integers q, we conclude that $\|\lambda\|_2\leq 1$, that is, $\sigma_{\mathrm{ap}}(T)\subseteq\overline{\mathbb{B}}$. Since the convex envelopes of $\sigma_{\mathrm{ap}}(T)$ and $\sigma(T)$ coincide [17], the Taylor spectrum of T is also contained in the closed unit ball $\overline{\mathbb{B}}$. ∎

Here is a rigidity theorem.

PROPOSITION 4.7. *Let T be a complete hypercontraction of order k and let Q_T denote the spherical generating* 1*-tuple associated with T. Then the following statements are equivalent:*

(1) *T is a joint contraction.*
(2) *T is joint subnormal.*
(3) *The spectral radius of T is at least $\sqrt{\|Q_T(I)\|}$.*

Proof. (1) implies (2): One may apply Corollary 3.20 to the sequence $\{\langle Q_T^n(I)h,h\rangle\}_{n\in\mathbb{N}}$, $h\in\mathcal{H}$, to conclude that T is a joint complete hypercontraction (refer also to [28, Lemma 2]). It follows from [6, Theorem 5.2] that T is joint subnormal.

(2) implies (3): We check that for any joint subnormal m-tuple T,

$$\sqrt{\|Q_T(I)\|}\leq r(T).$$

It is observed in the proof of [14, Proposition 4.9] that $\sqrt{\|Q_T(I)\|} \leq r(T)$ provided T satisfies the inequality

$$\langle Q_T^k(I)h, h\rangle \leq \langle Q_T^{k-1}(I)h, h\rangle^{1/2} \langle Q_T^{k+1}(I)h, h\rangle^{1/2}$$

for all $h \in \mathcal{H}$ and for all integers $k \geq 1$. It is easy to see using the Cauchy–Schwarz inequality that every joint subnormal m-tuple T satisfies the last inequality.

(3) implies (1): Suppose that $r(T) \geq \sqrt{\|Q_T(I)\|}$. By Lemma 4.6, $r(T)$ is at most 1. It follows that T is a joint contraction. ∎

Remark 4.8. By [27, Theorem 1] and [16, Theorem 1], the geometric spectral radius $r(T)$ of T is given by

$$r(T) = \lim_{k\to\infty} \left\|Q_T^k(I)\right\|^{1/(2k)}. \tag{4.11}$$

With the help of this formula, it can be seen that $r(T) \leq \sqrt{\|Q_T(I)\|}$ for any commuting m-tuple T of bounded linear operators on $\mathcal{H}$.

Since norm and spectral radius agree for hyponormal operators, we immediately obtain the following interesting fact:

Corollary 4.9. *Any hyponormal complete hypercontraction operator of finite order is necessarily a subnormal contraction. In particular, any completely monotone sequence* $\{\beta_n\}_{n\in\mathbb{N}}$ *of finite order, for which* $\{\beta_{n+1}/\beta_n\}_{n\in\mathbb{N}}$ *is increasing, is necessarily completely monotone.*

It is interesting to know whether or not a multi-variable analog of the last result holds.

Proposition 4.10. *Let k be an odd integer. Then any complete hypercontraction of order k is a complete hypercontraction of order $k-1$. In particular, any joint k-isometry is a complete hypercontraction of order $k-1$.*

Proof. This is a consequence of Corollary 3.18. ∎

Theorem 4.11. *Let k be a positive integer. Let T be a commuting m-tuple of bounded linear operators $T_1, \ldots, T_m$ on $\mathcal{H}$, and let Q_T denote the spherical generating 1-tuple associated with T. Then the following statements are equivalent:*

(1) *T is a complete hypercontraction of order k.*
(2) *There exist a polynomial p_k of degree $k-1$ with coefficients in $B(\mathcal{H})$ and a semispectral measure E on $[0,1]$ such that for all non-negative*

integers n,

$$\begin{aligned} Q_T^n(I) &= p_k(n) + (-1)^k \binom{n}{k} E(\{1\}) \\ &\quad + \int\limits_{[0,1)} \frac{1}{(1-x)^k} \left(x^n - \sum_{j=0}^{k-1} \frac{(x-1)^j}{j!} (n)_j \right) dE \\ &= p_k(n) + \int\limits_{[0,1]} \sum_{j=0}^{n-k} \left((-1)^k \frac{(x-1)^j}{(j+k)!} (n)_{j+k} \right) dE, \end{aligned}$$

where the integral in the last expression is absent if $n < k$.

If this holds then the integral representation in (2) *is unique in the sense that the coefficients of* p_k *and the semispectral measure* $E(\cdot)$ *are completely determined by* T.

Proof. This is immediate from Theorem 3.6 and the polarization technique employed in [20, Theorem 4.2]. For the sake of completeness, we briefly outline the proof.

Since $\{\psi_f(n) := \langle Q_T^n(I)f, f\rangle\}_{n\in\mathbb{N}}$ is completely monotone of order k for every $f \in \mathcal{H}$, by Theorem 3.6 there exist a polynomial $p_{k,f}$ of degree $k-1$ and a positive Radon measure μ_f on $[0,1]$ such that for every $n \in \mathbb{N}$,

$$\begin{aligned} \psi_f(n) &= p_{k,f}(n) + (-1)^k \binom{n}{k} \mu_f(\{1\}) \\ &\quad + \int\limits_{[0,1)} \frac{1}{(1-x)^k} \left(x^n - \sum_{j=0}^{k-1} \frac{(x-1)^j}{j!} (n)_j \right) d\mu_f(x) \\ &= p_{k,f}(n) + \int\limits_{[0,1]} \sum_{j=0}^{n-k} \left((-1)^k \frac{(x-1)^j}{(j+k)!} (n)_{j+k} \right) d\mu_f(x), \end{aligned}$$

where the integral in the last expression is absent if $n < k$. Moreover, an examination of the proof of Theorem 3.6 shows that

$$p_{k,f}(n) = \sum_{j=0}^{k-1} \frac{\Delta^j \psi_f(0)}{j!} (n)_j \qquad (n \in \mathbb{N}).$$

By the polarization formula, we obtain

$$\begin{aligned} \psi_{f,g}(n) := \langle Q_T^n(I)f, g\rangle &= \frac{1}{4} \big(\psi_{f+g}(n) - \psi_{f-g}(n) + i\psi_{f+ig}(n) - i\psi_{f-ig}(n) \big) \\ &= \sum_{j=0}^{k-1} \frac{\Delta^j \psi_{f,g}(0)}{j!} (n)_j + \int\limits_{[0,1]} \sum_{j=0}^{n-k} \left((-1)^k \frac{(x-1)^j}{(j+k)!} (n)_{j+k} \right) d\mu_{f,g}(x), \end{aligned}$$

where $\mu_{f,g} = \frac{1}{4}(\mu_{f+g} - \mu_{f-g} + i\mu_{f+ig} - i\mu_{f-ig})$ is a regular complex Borel measure. As in the proof of [20, Theorem 4.2], for every Borel measurable subset σ of $[0,1]$, the mapping $\phi_\sigma : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ given by

$$\phi_\sigma(f,g) = \mu_{f,g}(\sigma) \quad (f,g \in \mathcal{H})$$

is sesquilinear and satisfies $\phi_\sigma(f,f) = \mu_f(\sigma)$ $(f \in \mathcal{H})$. Since

$$\begin{aligned}\phi_\sigma(f,f) = \mu_f(\sigma) \le \int\limits_{[0,1]} d\mu_f &= (-1)^k\bigg(\psi_f(k) - \sum_{j=0}^{k-1} \frac{\Delta^j \psi_f(0)}{j!}(k)_j\bigg) \\ &= (-1)^k\bigg(\langle Q_T^k(I)f,f\rangle - \sum_{j=0}^{k-1}(-1)^j \frac{\langle B_j(Q_T)f,f\rangle}{j!}(k)_j\bigg),\end{aligned}$$

ϕ_σ is easily seen to be jointly continuous. Now the existence of the semispectral measure E may be deduced from the Fischer–Riesz Theorem (for more details, the reader is referred to [20]). To see the uniqueness part, note first that $p_k(n) = \sum_{j=0}^{k-1}(-1)^j \frac{B_j(Q_T)}{j!}(n)_j$. Suppose

$$I_l := \int\limits_{[0,1]} \sum_{j=0}^{l-k}\bigg((-1)^k \frac{(x-1)^j}{(j+k)!}(l)_{j+k}\bigg)\, dF(x) = 0 \quad (l \ge k),$$

where $F(\cdot)$ is a complex $B(\mathcal{H})$-valued measure. In view of the Riesz Representation Theorem and the Weierstrass Approximation Theorem, it now suffices to check that $J_n := \int_{[0,1]} x^n\, dF(x) = 0$ for every integer $n \ge 0$. As in the proof of Theorem 3.6, one can prove this by induction on n. ∎

REMARK 4.12. Suppose $m = 1$. Let $\mathcal{N}$ denote the intersection of the kernels of the coefficients of p_k except the constant coefficient. Let $\mathcal{M}$ denote the intersection of $\mathcal{N}$ with the kernel of $E(\{1\})$. If $\frac{1}{(1-x)^k} \in L^1(dE)$ then one may imitate the argument of [12, proof of Corollary 4.6] to see that $\mathcal{M}$ is a hyperinvariant subspace of T. We remark that the statement of [12, Corollary 4.6] is erroneous. In fact, one needs to add the condition $\frac{1}{1-x} \in L^1(dE)$ used crucially in its proof.

We refer to $E(\cdot)$ as the *Lévy measure associated with* T.

The following result is a consequence of the integral representation for complete hypercontractions of finite order.

COROLLARY 4.13. *Let T be a complete hypercontraction of finite order with associated Lévy measure $E(\cdot)$. If $\frac{1}{(1-x)^k} \in L^1(dE)$ then $\|Q_T^n(I)\|$ has polynomial growth, that is, there exists a polynomial $p : \mathbb{R} \to (0,\infty)$ such that $\|Q_T^n(I)\| \le p(n)$ for every non-negative integer n.*

REMARK 4.14. If $\|Q_T^n(I)\|$ has polynomial growth then by the spectral radius formula (4.11), the spectral radius of T is at most 1.

Corollary 4.15. *For any complete hypercontraction m-tuple T of order 2, $\|Q_T^n(I)\|$ has polynomial growth.*

Proof. Note that

$$\int_{[0,1)} \frac{\left(x^n - 1 - n(x-1) - \binom{n}{2}(x-1)^2\right)}{(1-x)^2}\, dE$$
$$= \int_{[0,1)} \frac{\left(-(1+x+\cdots+x^{n-1}) + n - \binom{n}{2}(x-1)\right)}{(1-x)}\, dE$$
$$= \int_{[0,1)} \left(\sum_{j=0}^{n-1}(1+x+\cdots+x^{j-1}) + \binom{n}{2}\right) dE.$$

Since $E(\cdot)$ is a finite measure, $\|Q_T^n(I)\|$ has polynomial growth. ∎

Question 4.16. *Does the conclusion of the last corollary hold true for order $k \geq 3$?*

A subclass of complete hypercontraction operators of finite order, namely, the class $\mathcal{ECH}_2$ of expansive complete hypercontractions of order 2, is more tractable. One basic example of this class is the multiplication operator M_z acting on the reproducing kernel Hilbert space $\mathscr{H}(\kappa)$, where

$$\kappa(z,w) = \sum_{n=0}^{\infty} \frac{n+1}{n^2+n+1} z^n \overline{w}^n$$

for z, w in the open unit disc in $\mathbb{C}$.

We now specialize to the class $\mathcal{ECH}_2$ and make a few observations.

Proposition 4.17. *Every invertible bounded linear operator in $\mathcal{ECH}_2$ is unitary. In particular, the spectrum $\sigma(T)$ of any T in $\mathcal{ECH}_2$ admits the following spectral dichotomy:*

$$\sigma(T) = \overline{\mathbb{D}} \quad \text{or} \quad \sigma(T) \subseteq \partial\mathbb{D},$$

where $\mathbb{D}$ denotes the open unit disc in the complex plane $\mathbb{C}$, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$.

Proof. If T is an invertible expansion then T^{-1} is a contraction. Note that $B_3(Q_{T^{-1}}) \leq 0$, and hence by [28, Lemma 2], $B_2(Q_{T^{-1}}) \leq 0$. However, by hypothesis, we have $B_2(Q_{T^{-1}}) \geq 0$. Thus T^{-1} is an invertible 2-isometry, and hence it is unitary [2].

The remaining assertion may be deduced from the fact that the approximate-point spectrum of an expansion is disjoint from the open unit disc. ∎

Remark 4.18. It turns out that any operator in $\mathcal{ECH}_2$ is *operator close to isometry* in the sense of [12].

Corollary 4.19. *Any operator in $\mathcal{ECH}_2$ with finite-dimensional cokernel is essentially normal.*

Proof. Note that an expansive operator with finite-dimensional cokernel is Fredholm. Now apply the preceding proposition to the image of $T \in \mathcal{ECH}_2$ under the Calkin map π. ∎

In connection with the preceding corollary, the authors do not know whether the self-commutator of an operator in $\mathcal{ECH}_2$ with finite-dimensional cokernel is necessarily a trace-class operator.

4.2. A dilation theorem. We have already seen that not all complete hypercontractions of finite order admit normal extensions (see Proposition 4.7). On the other hand, since complete hypercontractions of finite order miss out only finitely many Agler's positivity conditions, it is reasonable to expect that such tuples would resemble subnormals in some sense. In particular, it is interesting to look for some association of normal tuples with complete hypercontractions of finite order. In this subsection, we apply Stinespring's dilation theorem to show that the spherical generating 1-tuples associated with such tuples dilate to so-called multiplicative tuples (refer to [13]).

Let $\mathcal{A}$ denote a unital C^*-algebra. By an *operator system* we mean a self-adjoint subspace of $\mathcal{A}$ containing the unit. Let $M_n(\mathcal{A})$ denote the C^*-algebra of all $n \times n$ matrices with entries from $\mathcal{A}$. A mapping ϕ from $\mathcal{A}$ into another C^*-algebra $\mathcal{B}$ is said to be *positive* if it maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$. Let $\mathcal{S} \subseteq \mathcal{A}$ denote an operator system. If $\phi : \mathcal{S} \to \mathcal{B}$ is a linear map, then we define $\phi_n : M_n(\mathcal{S}) \to M_n(\mathcal{B})$ by $\phi_n([a_{i,j}]) := [\phi(a_{i,j})]$, where $[a_{i,j}] \in M_n(\mathcal{S})$. We say that ϕ is *completely positive* if ϕ_n is positive for all $n \geq 1$.

A special case of [13, Theorem 4.6] may be derived from the following result.

Theorem 4.20. *Let k be a positive integer. Let T be a complete hypercontraction m-tuple of finite order acting on $\mathcal{H}$ and let Q_T be the spherical generating 1-tuple associated with T. Suppose further that the measure F given by*

$$F(\sigma) := \int_\sigma \frac{dE(x)}{(1-x)^k} \qquad (\sigma \text{ a Borel subset of } [0,1))$$

is a $B(\mathcal{H})$-valued Borel measure, where E is the semispectral measure appearing in the integral representation of Q_T (see (2) of Theorem 4.11). Then there exist a normal operator N on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ such that

$$p(Q_T)(I) = V^*(p(N))V \qquad \text{for any } p \in \mathcal{S},$$

where $\mathcal{S}$ is the self-adjoint subspace given by

$$\mathcal{S} := \{p \in C([0,1]) : p \text{ polynomial in } t \text{ such that } p^{(l)}(1) = 0 \ (0 \le l \le k)\}.$$

Moreover,

$$\mathcal{K} = \bigvee \{N^n h : n \in \mathbb{N},\, h \in \mathcal{H}\}.$$

Proof. Consider the mapping $\phi : \mathcal{S} \to B(\mathcal{H})$ given by

$$\phi(p) := \phi\Big(\sum_{j=0}^{n} \alpha_j t^j\Big) := \sum_{j=0}^{n} \alpha_j Q_T^j(I) \quad (n \in \mathbb{N},\, \alpha_j \in \mathbb{C}).$$

Since $p \in \mathcal{S}$, we have $\sum_{j=0}^{n} j^i \alpha_j = 0$ for $i = 0, \dots, k$. It is now easy to see from the integral representation of Theorem 4.11 that

$$\phi(p) = \int_{[0,1)} p(t)\, dF(t) \tag{4.12}$$

for every $p \in \mathcal{S}$. Thus ϕ is positive. We may extend ϕ linearly to the linear span of $\mathcal{S}$ and $\mathbb{C}$ by setting

$$\tilde{\phi}(p + \alpha) := \phi(p) + \alpha A \quad (p \in \mathcal{S}, \alpha \in \mathbb{C}),$$

where A is the total mass $F([0,1))$ of F. Then $\tilde{\phi}$ is also positive since

$$\tilde{\phi}(p + \alpha) = \int_{[0,1)} (p(t) + \alpha)\, dF(t)$$

for any $p \in \mathcal{S}$ and any $\alpha \in \mathbb{C}$. By the Stone–Weierstrass Theorem, we may extend $\tilde{\phi}$ to a positive map (still denoted by $\tilde{\phi}$) on the whole of $C([0,1])$. By a result of Stinespring, any positive map from $C([0,1])$ into a C^*-algebra is always completely positive [29, Theorem 3.11], so $\tilde{\phi}$ is completely positive. We can now apply Stinespring's Dilation Theorem to conclude that there exists a minimal Stinespring representation $(\pi, V, \mathcal{K})$ of $\tilde{\phi}$ with $\|V\| = \sqrt{\|A\|}$. For the positive operator $N := \pi(t)$ in $B(\mathcal{K})$,

$$p(Q_T)(I) = V^*(p(N))V \quad \text{for any } p \in \mathcal{S}.$$

The last part follows since the triple $(\pi, V, \mathcal{K})$ is minimal. ∎

Remark 4.21. We outline an alternative proof of Theorem 4.20. Since $F(\cdot)$ is a semispectral measure, one may apply the Naimark Dilation Theorem [29, Theorem 4.6] to see that there exist a Hilbert space $\mathcal{K}$, a $B(\mathcal{K})$-valued spectral Borel measure $E(\cdot)$ on $[0,1)$, and a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ such that $F(\sigma) = V^* E(\sigma) V$ for every Borel subset σ of $[0,1)$. By letting N be the normal operator $\int_{[0,1)} t\, dE(t)$, the desired conclusion may be deduced from the integral representation (4.12).

Remark 4.22. Let $B_q(Q_T)$ be as defined in (2.5). Then we have $B_q(Q_T) = V^*(I - N)^q V$ for every $q \ge k$.

5. Concluding remarks. The present paper deals with a class of functions closely related to the class of completely monotone functions, and a corresponding class of operators. However, the theories of completely monotone functions and completely alternating functions have interesting connections (see [11], [30], [32], [35]). The operator theory associated with the class of completely alternating functions, resulting in the study of completely hyperexpansive operators, was initiated in [7] and further explored in [10], [33], [23], [31], [9], [20], [8], [21], [22], [24], [25], [13], [14] etc. The interplay between completely monotone functions and completely alternating functions and its reflection in the study of hypercontractions and hyperexpansions has turned out to be a very fruitful association. Motivated by this association, we shall carry out the analysis of completely alternating functions of finite order and attach to it a class of completely hyperexpansive tuples of finite order in the forthcoming work. The notion of a Cauchy dual of an operator, introduced in [31] and further generalized in [13] to several variables, plays a decisive role in emphasizing the relation between hypercontractive and hyperexpansive tuples. In the sequel to this work, we formulate an appropriate notion of the Cauchy dual that would relate completely hypercontractive shifts of order k to completely hyperexpansive shifts of order k.

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