STUDIA MATHEMATICA 197 (2) (2010)

Linear maps Lie derivable at zero on $\mathcal{J}$-subspace lattice algebras

by

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Abstract. A linear map L on an algebra is said to be Lie derivable at zero if $L([A,B]) = [L(A),B] + [A,L(B)]$ whenever $[A,B] = 0$. It is shown that, for a $\mathcal{J}$-subspace lattice $\mathcal{L}$ on a Banach space X satisfying $\dim K \neq 2$ whenever $K \in \mathcal{J}(\mathcal{L})$, every linear map on $\mathcal{F}(\mathcal{L})$ (the subalgebra of all finite rank operators in the JSL algebra $\operatorname{Alg}\mathcal{L}$) Lie derivable at zero is of the standard form $A \mapsto \delta(A) + \phi(A)$, where δ is a generalized derivation and ϕ is a center-valued linear map. A characterization of linear maps Lie derivable at zero on $\operatorname{Alg}\mathcal{L}$ is also obtained, which are not of the above standard form in general.

1. Introduction. Let $\mathcal{A}$ be an algebra (or a ring). Then $\mathcal{A}$ is a Lie algebra (resp. Lie ring) under the Lie product $[A,B] = AB - BA$. Recall that a linear (resp. an additive) map δ from $\mathcal{A}$ into itself is called a *linear* (resp. an *additive*) *derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A,B \in \mathcal{A}$. Linear (or additive) derivations are important both in theory and applications, and studied intensively. More generally, a linear (resp. an additive) map L from $\mathcal{A}$ into itself is called a *linear* (resp. an *additive*) *Lie derivation* if $L([A,B]) = [L(A),B] + [A,L(B)]$ for all $A,B \in \mathcal{A}$.

Note that all derivations are Lie derivations, but the converse is not true. The problem of how to characterize the linear (or additive) Lie derivations has received many mathematicians' attention for many years. Brešar [1] proved that every additive Lie derivation on a prime ring $\mathcal{R}$ with characteristic not 2 can be decomposed as $\tau + \zeta$, where τ is a derivation from $\mathcal{R}$ into its central closure and ζ is an additive map of $\mathcal{R}$ into the extended centroid $\mathcal{C}$ sending commutators to zero. Johnson [4] proved that every continuous linear Lie derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule M is *standard*, that is, can be decomposed as $\tau + h$, where $\tau : \mathcal{A} \to M$ is a derivation and h is a linear map from $\mathcal{A}$ into the center of M vanishing at each commutator. Mathieu and Villena [14] showed that every linear Lie derivation on a C^*-algebra is standard. In [11] Qi and Hou proved that

2010 *Mathematics Subject Classification*: 47L35, 17B40.
Key words and phrases: $\mathcal{J}$-subspace lattice algebra, linear maps Lie derivable at zero.

DOI: 10.4064/sm197-2-3

the same is true for additive Lie derivations of nest algebras over Banach spaces. But linear Lie derivations on operator algebras are not necessarily standard: in [9], Lu proved that a linear map L on a *complex* $\mathcal{J}$-subspace lattice algebra (briefly, JSL algebra) $\operatorname{Alg}\mathcal{L}$ is a Lie derivation if and only if, for each $K \in \mathcal{J}(\mathcal{L})$, there exist an operator $T_K \in \mathcal{B}(K)$ and a linear functional $h_K : \operatorname{Alg}\mathcal{L} \to \mathbb{C}$ vanishing at every commutator such that $L(A)x = (T_K A - AT_K)x + h_K(A)x$ for all $x \in K$ and $A \in \operatorname{Alg}\mathcal{L}$.

Let $\mathcal{A}$ be an algebra (or a ring). Recall that a linear (resp. an additive) map $\gamma : \mathcal{A} \to \mathcal{A}$ is *derivable at* Z if $\gamma(A)B + A\gamma(B) = \gamma(Z)$ for any $A, B \in \mathcal{A}$ with $AB = Z$; an element $Z \in \mathcal{A}$ is a *full-derivable point* of $\mathcal{A}$ if every linear (resp. additive) map derivable at Z is a derivation. The problem of characterizing maps, especially on operator algebras, derivable at a certain point and the problem of finding full-derivable points have been studied by several authors (for example, see [12, 15, 16]). Motivated by this, we say that a linear (resp. an additive) map $L : \mathcal{A} \to \mathcal{A}$ is *Lie derivable at a point* Z if $L([A,B]) = [L(A), B] + [A, L(B)]$ for any $A, B \in \mathcal{A}$ with $[A,B] = Z$. Clearly, this definition is vacuous for some Z, for instance, for $Z = I$, as the unit I cannot be a commutator $[A,B]$. It is also obvious that the condition of being Lie derivable at some point is much weaker than being a Lie derivation.

Let $\mathcal{U}$ be a triangular algebra and $L : \mathcal{U} \to \mathcal{U}$ an additive map Lie derivable at zero. Hou and Qi [13] proved that, under some mild conditions, L has the form $L(X) = ZX + \tau(X) + \nu(X)$ for all $X \in \mathcal{U}$, where $Z \in \mathcal{Z}(\mathcal{U})$ (the center of $\mathcal{U}$), $\tau : \mathcal{U} \to \mathcal{U}$ is a derivation and ν is a map from $\mathcal{U}$ into $\mathcal{Z}(\mathcal{U})$. The purpose of this paper is to discuss the question of characterizing linear maps on JSL algebras Lie derivable at zero.

JSL algebras are an important class of subspace lattice algebras. Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X over the real or complex field $\mathbb{F}$ with $\dim X \geq 3$ and $\operatorname{Alg}\mathcal{L}$ be the associated $\mathcal{J}$-subspace lattice algebra. Generally, a linear map on $\operatorname{Alg}\mathcal{L}$ Lie derivable at zero has a wild behavior (see Proposition 2.5, Example 2.6 and Example 3.7). However, we show that, under some mild conditions on the lattice, every linear map L derivable at zero on $\mathcal{F}(\mathcal{L})$ has the form $L(A) = \delta(A) + \phi(A)$ for all $A \in \mathcal{F}(\mathcal{L})$, where δ is a *generalized derivation* and $\phi : \mathcal{F}(\mathcal{L}) \to \mathcal{Z}(\mathcal{F}(\mathcal{L}))$ is a linear map (not necessarily sending commutators to zero) (Theorem 3.1 and Corollary 3.2); every linear map L derivable at zero on $\operatorname{Alg}\mathcal{L}$ has the property that, for each $K \in \mathcal{J}(\mathcal{L})$, there exist an operator $T_K \in \mathcal{B}(K)$, a scalar λ_K and a linear functional $h_K : \operatorname{Alg}\mathcal{L} \to \mathbb{F}$ such that $L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$ for all $x \in K$ and $A \in \operatorname{Alg}\mathcal{L}$ (Theorem 3.4). It is clear that if $\lambda_K = 0$ and if h_K vanishes at each commutator, i.e., $h_K([A,B]) = 0$ for all A, B, then L is a Lie derivation. In particular, we get a characterization of linear Lie derivations on both real and complex JSL algebras (Corollaries 3.5–3.6), which generalizes the corresponding results in [9].

To prove our main results, we first, in Section 2, discuss the question for $\mathcal{F}(X)$, the algebra of all finite rank operators. By using a result of [1] (see Lemma 2.1), we show that every linear map Lie derivable at zero on $\mathcal{F}(X)$ is of the standard form, that is, $A \mapsto TA - AT + \lambda A + f(A)I$, where $T \in \mathcal{B}(X)$, λ is a scalar and f is a linear functional on $\mathcal{F}(X)$ with $f = 0$ if $\dim X = \infty$ (Corollaries 2.2–2.4), provided X is a (real or complex) Banach space with $\dim X > 2$. For $\dim X = 2$, we show that a linear map L on $\mathcal{F}(X) = M_2(\mathbb{F})$ is Lie derivable at zero if and only if $L(I) = cI$ for some scalar c, which is not necessarily the standard form stated above (Proposition 2.5). The results of Section 2 are then used to prove our main results, Theorems 3.1 and 3.4.

Now let us recall some notions and fix some notations used in this paper. Let X be a Banach space over the real or complex field $\mathbb{F}$. A family $\mathcal{L}$ of subspaces of X is called a *subspace lattice* on X if it contains $\{0\}$ and X, and is closed under taking closed linear span $\vee$ and intersection $\wedge$ in the sense that $\bigvee_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ and $\bigwedge_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ for every family $\{L_\gamma : \gamma \in \Gamma\}$ of elements in $\mathcal{L}$. For a subspace lattice $\mathcal{L}$ on X, the associated *subspace lattice algebra* $\operatorname{Alg}\mathcal{L}$ is the set of operators on X leaving every subspace in $\mathcal{L}$ invariant. Given a subspace lattice $\mathcal{L}$ on X, put

$$\mathcal{J}(\mathcal{L}) = \{K \in \mathcal{L} : K \neq \{0\} \text{ and } K_- \neq X\},$$

where $K_- = \bigvee\{L \in \mathcal{L} : K \nsubseteq L\}$. Call $\mathcal{L}$ a *$\mathcal{J}$-subspace lattice* (simply, JSL) on X if it satisfies the following conditions:

(1) $\bigvee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$;
(2) $\bigwedge\{K_- : K \in \mathcal{J}(\mathcal{L})\} = \{0\}$;
(3) $K \vee K_- = X$, $\forall K \in \mathcal{J}(\mathcal{L})$;
(4) $K \wedge K_- = \{0\}$, $\forall K \in \mathcal{J}(\mathcal{L})$.

If $\mathcal{L}$ is a JSL, the associated subspace lattice algebra $\operatorname{Alg}\mathcal{L}$ is called a *$\mathcal{J}$-subspace lattice algebra*, or briefly, *JSL algebra*. It should be mentioned that both atomic Boolean subspace lattices and pentagon subspace lattices are $\mathcal{J}$-subspace lattices [8].

For $L \in \mathcal{L}$, denote $L_-^\perp = (L_-)^\perp$, where $L^\perp$ denotes the annihilator of L. Denote by $\langle \mathcal{J}(\mathcal{L})\rangle$ and $\langle \mathcal{J}(\mathcal{L})_-^\perp\rangle$ the (not necessarily closed) linear spans of $\bigcup\{K : K \in \mathcal{J}(\mathcal{L})\}$ and of $\bigcup\{K_-^\perp : K \in \mathcal{J}(\mathcal{L})\}$, respectively. For $x \in X$ and $f \in X^*$, $x \otimes f$ stands for the operator on X with rank not greater than one defined by $(x\otimes f)y = f(y)x$. Sometimes we use $\langle x, f\rangle$ to represent the value $f(x)$ of f at x. For $K \in \mathcal{J}(\mathcal{L})$, let $\mathcal{F}_\mathcal{L}(K)$ denote the subspace spanned by all rank one operators $x \otimes f$ with $x \in K$ and $f \in K_-^\perp$. Let $\mathcal{F}(\mathcal{L})$ denote the algebra of all finite rank operators in $\operatorname{Alg}\mathcal{L}$. For an algebra $\mathcal{A}$, we use $\mathcal{Z}(\mathcal{A})$ to denote the center of $\mathcal{A}$.

The following facts are well-known.

Lemma 1.1 ([6]). *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Then $x \otimes f \in \operatorname{Alg}\mathcal{L}$ if and only if there exists a subspace $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^{\perp}$.*

Thus $\mathcal{F}_{\mathcal{L}}(K) \subset \operatorname{Alg}\mathcal{L}$ for every $K \in \mathcal{J}(\mathcal{L})$. Also, we have

Lemma 1.2 ([8]). *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. The following statements hold true.*

(1) *For any $K, L \in \mathcal{J}(\mathcal{L})$, $K \neq L$ implies that $K \subseteq L_-$.*
(2) *For any $K, L \in \mathcal{J}(\mathcal{L})$, $K \neq L$ implies that $K \cap L = \{0\}$.*
(3) *Let $K \in \mathcal{J}(\mathcal{L})$. Then, for any nonzero vector $x \in K$, there exists $f \in K_-^{\perp}$ such that $f(x) = 1$; dually, for any nonzero functional $f \in K_-^{\perp}$, there exists $x \in K$ such that $f(x) = 1$.*

We refer the readers to [3, 5, 7, 8] for more properties of JSL algebras.

2. Preliminary results: Maps Lie derivable at zero on $\mathcal{F}(X)$. In this section, we first give a characterization of linear maps Lie derivable at zero on $\mathcal{F}(X)$, the algebra of all finite rank operators on a Banach space X, which is needed in the next section. To do this, we need a lemma concerning additive maps on prime rings Lie derivable at zero, which is a consequence of [1, Theorem 4].

Let $\mathcal{R}$ be a ring and $\mathcal{Z}(\mathcal{R})$ its center. Recall that an element $A \in \mathcal{R}$ is *algebraic over $\mathcal{Z}(\mathcal{R})$* if there exists a nonzero polynomial $p \in \mathcal{P}(\mathcal{Z}(\mathcal{R}))$ such that $p(A) = 0$, that is, there exist $Z_0, Z_1, \ldots, Z_n \in \mathcal{Z}(\mathcal{R})$ such that $Z_n \neq 0$ and $p(A) = Z_0 + Z_1A + \cdots + Z_nA^n = 0$. In this case $n = \deg(p)$ is called the *degree* of p, and $\min\{\deg(p) : p(A) = 0\}$ is called the *degree of algebraicity of A over $\mathcal{Z}(\mathcal{R})$*, denoted by $\deg(A)$. If A is not algebraic over $\mathcal{Z}(\mathcal{R})$, then we shall write $\deg(A) = \infty$. The *degree of algebraicity* of $\mathcal{R}$ is

$$\deg(\mathcal{R}) = \sup\{\deg(A) : A \in \mathcal{R}\}.$$

Lemma 2.1. *Let $\mathcal{R}$ be a prime ring of characteristic neither 2 nor 3 and with $\deg(\mathcal{R}) \geq 3$. Suppose that $L : \mathcal{R} \to \mathcal{R}$ is an additive map Lie derivable at zero. Then there exists an element α in $\mathcal{C}$, the extended centroid of $\mathcal{R}$, an additive derivation τ of $\mathcal{R}$ into its center closure, and an additive map $\nu : \mathcal{R} \to \mathcal{C}$ such that $L(A) = \tau(A) + \alpha A + \nu(A)$ for all $A \in \mathcal{R}$.*

By the above result, we see that an additive map Lie derivable at zero is not a Lie derivation in general since α need not be 0 and ν may not vanish at every commutator.

Let X be a Banach space over the real or complex field $\mathbb{F}$ with $\dim X > 2$ and let $\mathcal{B}(X)$ stand for the algebra of all bounded linear operators on X. It is clear that $\mathcal{B}(X)$ is prime with characteristic 0 and $\deg(\mathcal{B}(X)) \geq 3$. Thus by Lemma 2.1, every additive (resp. linear) map on $\mathcal{B}(X)$ Lie derivable at zero has the form $A \mapsto \tau(A) + \lambda A + h(A)I$, where τ is an additive (resp. a linear) derivation, λ is a scalar and h is an additive (resp. a linear) functional. Also, note that linear derivations on $\mathcal{B}(X)$ are inner. So, in particular, for the finite-dimensional case, we have

COROLLARY 2.2. *Let $\mathbb{F}$ be the real or complex field and $n > 2$. Suppose that $L : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a linear map. Then L is Lie derivable at zero if and only if there exists a scalar $\lambda \in \mathbb{F}$, a matrix $T \in M_n(\mathbb{F})$ and a linear functional f on $M_n(\mathbb{F})$ such that $L(A) = TA - AT + \lambda A + f(A)I$ for all $A \in M_n(\mathbb{F})$.*

Corollary 2.2 is needed in the next section to prove our main result. The following result is also true and needed in the next section.

COROLLARY 2.3. *Let X be an infinite-dimensional Banach space over the real or complex field $\mathbb{F}$ and let $\mathcal{F}(X)$ be the subalgebra of all finite rank operators in $\mathcal{B}(X)$. Suppose that $L : \mathcal{F}(X) \to \mathcal{F}(X)$ is a linear map. Then L is Lie derivable at zero if and only if there exists a scalar $\lambda \in \mathbb{F}$ and an operator $T \in \mathcal{B}(X)$ such that $L(A) = TA - AT + \lambda A$ for all $A \in \mathcal{F}(X)$.*

Proof. The "if" part is obvious. To check the "only if" part, assume that L is Lie derivable at zero. Since the extended centroid of the prime algebra $\mathcal{F}(X)$ is $\mathbb{F}I$ and the derivations on $\mathcal{F}(X)$ are inner, by Lemma 2.1 there exist $\lambda \in \mathbb{F}$, $T \in \mathcal{B}(X)$ and a linear functional f on $\mathcal{F}(X)$ such that

$$L(A) = TA - AT + \lambda A + f(A)I \quad \text{for all } A \in \mathcal{F}(X). \tag{2.1}$$

Because $L(\mathcal{F}(X)) \subseteq \mathcal{F}(X)$, it follows from (2.1) that $f(A) = 0$ for all $A \in \mathcal{F}(X)$, completing the proof. ∎

Note that every factor von Neumann algebra is prime and all linear derivations of von Neumann algebras are inner. By Lemma 2.1, the following corollary is also immediate.

COROLLARY 2.4. *Let $\mathcal{M}$ be a factor von Neumann algebra with* $\deg(\mathcal{M}) > 2$ *and $L : \mathcal{M} \to \mathcal{M}$ a linear map. Then L is Lie derivable at zero if and only if there exists an element $T \in \mathcal{M}$, a scalar λ and a linear functional h on $\mathcal{M}$ such that $L(A) = TA - AT + \lambda A + h(A)I$ for all $A \in \mathcal{M}$.*

The story for 2×2 matrices is quite different.

PROPOSITION 2.5. *Let $L : M_2(\mathbb{F}) \to M_2(\mathbb{F})$ be a linear map. Then L is Lie derivable at zero if and only if $L(I) = \lambda I$ for some $\lambda \in \mathbb{F}$.*

Proof. We need only check the "if" part. For any $A \in M_2(\mathbb{F})$, if $A = \alpha I \in \mathbb{F}I$, then $[A, B] = 0$ for all $B \in M_2(\mathbb{F})$. Since L is linear, we have

$$[L(A), B] + [A, L(B)] = [\alpha L(I), B] + [\alpha I, L(B)] = [\alpha\lambda I, B] + [\alpha I, L(B)] = 0.$$

Now assume that $A \notin \mathbb{F}I$. First note that, if $[A, B] = AB - BA = 0$, then $B = \mu(B)A + \nu(B)I$ for some $\mu(B), \nu(B) \in \mathbb{F}$. In fact, one can easily check that, for $A = (a_{ij}) \notin \mathbb{F}I$ and $B = (b_{ij})$, $AB = BA$ implies that

$$B = \begin{cases} b_{21}a_{21}^{-1}A + (b_{22} - b_{21}a_{21}^{-1}a_{22})I & \text{if } a_{21} \neq 0, \\ b_{12}a_{12}^{-1}A + (b_{22} - b_{12}a_{12}^{-1}a_{22})I & \text{if } a_{12} \neq 0, \\ \dfrac{b_{11} - b_{22}}{a_{11} - a_{22}}A + \left(b_{11} - \dfrac{b_{11} - b_{22}}{a_{11} - a_{22}}a_{11}\right)I & \text{if } a_{12} = a_{21} = 0. \end{cases}$$

It follows that

$$\begin{aligned}[L(A), B] + [A, L(B)] &= [L(A), \mu(B)A + \nu(B)I] + [A, \mu(B)L(A) + \nu(B)\lambda I] \\ &= \mu(B)[L(A), A] + \mu(B)[A, L(A)] = 0.\end{aligned}$$

Thus L is Lie derivable at zero. ∎

Therefore, unlike linear Lie derivations (every linear Lie derivation on $M_2(\mathbb{F})$ has a standard form), linear maps on $M_2(\mathbb{F})$ Lie derivable at zero behave wildly and are not always of the form stated in Corollary 2.2. To illustrate this, we give a simple example.

Example 2.6. Let $\mathbb{F}$ be the real or complex field. For any $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) \in M_2(\mathbb{F})$, we define a map $L : M_2(\mathbb{F}) \to M_2(\mathbb{F})$ by $L(A) = \left(\begin{smallmatrix} 0 & a_{12} \\ a_{21} & 0 \end{smallmatrix}\right)$. We will check that L is a linear map Lie derivable at zero, but there do not exist $\lambda \in \mathbb{F}$, $T \in M_2(\mathbb{F})$ and a linear functional f on $M_2(\mathbb{F})$ such that $L(A) = TA - AT + \lambda A + f(A)I$ for all $A \in M_2(\mathbb{F})$.

Since $L(I) = 0$, by Proposition 2.5, L is Lie derivable at zero. Suppose, on the contrary, that there exist λ, T and f as above. Let $A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. By the definition of L, we have $0 = L(A) = TA - AT + \lambda A + f(A)I$. By a simple calculation, we get $\lambda = 0$ and $T = \left(\begin{smallmatrix} t_{11} & 0 \\ 0 & t_{22} \end{smallmatrix}\right)$ for some $t_{11}, t_{22} \in \mathbb{F}$.

Now for any $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$, we have

$$\begin{aligned}\begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} &= \begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix} \\ &\quad + \begin{pmatrix} f(A) & 0 \\ 0 & f(A) \end{pmatrix} \\ &= \begin{pmatrix} f(A) & (t_{11} - t_{22})a_{12} \\ (t_{22} - t_{11})a_{21} & f(A) \end{pmatrix}.\end{aligned}$$

This yields $t_{11} - t_{22} = -1$ and $t_{11} - t_{22} = 1$, a contradiction.

3. Linear maps Lie derivable at zero on JSL algebras. In this section, we discuss linear maps Lie derivable at zero on $\mathcal{J}$-subspace lattice algebras. Note that a JSL algebra may not be prime.

Our first result in this section gives a characterization of linear maps Lie derivable at zero on $\mathcal{F}(\mathcal{L})$.

THEOREM 3.1. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X over the real or complex field $\mathbb{F}$ with $\dim X \geq 3$. Suppose that $\dim K \neq 2$ for every $K \in \mathcal{J}(\mathcal{L})$ and $L : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$ is a linear map. Then L is Lie derivable at zero if and only if there exist linear operators $T, S : \langle \mathcal{J}(\mathcal{L})\rangle \to \langle \mathcal{J}(\mathcal{L})\rangle$ with $T|_K \in \mathcal{B}(K)$, $S|_K \in \mathcal{B}(K)$ and $T|_K + S|_K \in \mathbb{F}I_K$ for each $K \in \mathcal{J}(\mathcal{L})$, and a linear map $\phi : \mathcal{F}(\mathcal{L}) \to \mathcal{Z}(\mathcal{F}(\mathcal{L}))$ such that $L(A)x = (TA + AS + \phi(A))x$ for all $A \in \mathcal{F}(\mathcal{L})$ and $x \in \langle \mathcal{J}(\mathcal{L})\rangle$.*

Recall that a linear map δ is called a *generalized derivation* if there exists a linear derivation τ such that $\delta(AB) = \delta(A)B + A\tau(B)$ for all A, B.

COROLLARY 3.2. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X over the real or complex field $\mathbb{F}$ with $\dim X \geq 3$. Suppose that $\dim K \neq 2$ for every $K \in \mathcal{J}(\mathcal{L})$ and $L : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$ is a linear map. If L is Lie derivable at zero, then there exists a generalized derivation $\delta : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$ and a linear map $\phi : \mathcal{F}(\mathcal{L}) \to \mathcal{Z}(\mathcal{F}(\mathcal{L}))$ such that $L(A) = \delta(A) + \phi(A)$ for all $A \in \mathcal{F}(\mathcal{L})$.*

Proof. For any $A \in \mathcal{F}(\mathcal{L})$, let $\delta(A)|_{\langle \mathcal{J}(\mathcal{L})\rangle} = (TA + AS)|_{\langle \mathcal{J}(\mathcal{L})\rangle}$, where T and S are as in Theorem 3.1. We will show that $\delta : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$ is a linear generalized derivation.

Define a map $\tau : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$ by $\tau(A)|_{\langle \mathcal{J}(\mathcal{L})\rangle} = (AS - SB)|_{\langle \mathcal{J}(\mathcal{L})\rangle}$ for all $A \in \mathcal{F}(\mathcal{L})$. For any $A, B \in \mathcal{F}(\mathcal{L})$, by the definition of τ, on the one hand, we have

$$\tau(AB)|_{\langle \mathcal{J}(\mathcal{L})\rangle} = ABS - SA|_{\langle \mathcal{J}(\mathcal{L})\rangle}B|_{\langle \mathcal{J}(\mathcal{L})\rangle}.$$

On the other hand,

$$\begin{aligned}(\tau(A)B &+ A\tau(B))|_{\langle \mathcal{J}(\mathcal{L})\rangle} \\ &= ASB|_{\langle \mathcal{J}(\mathcal{L})\rangle} - SA|_{\langle \mathcal{J}(\mathcal{L})\rangle}B|_{\langle \mathcal{J}(\mathcal{L})\rangle} + ABS - ASB|_{\langle \mathcal{J}(\mathcal{L})\rangle} \\ &= ABS - SA|_{\langle \mathcal{J}(\mathcal{L})\rangle}B|_{\langle \mathcal{J}(\mathcal{L})\rangle}.\end{aligned}$$

Comparing the above two equations, we get $\tau(AB) = \tau(A)B + A\tau(B)$ since $\langle \mathcal{J}(\mathcal{L})\rangle$ is dense in X. Hence τ is a derivation. Now for any $A, B \in \mathcal{F}(\mathcal{L})$, by the definition of δ, on the one hand, we have

$$\delta(AB)|_{\langle \mathcal{J}(\mathcal{L})\rangle} = TA|_{\langle \mathcal{J}(\mathcal{L})\rangle}B|_{\langle \mathcal{J}(\mathcal{L})\rangle} + ABS.$$

On the other hand,

$$\begin{aligned}(\delta(A)B &+ A\tau(B))|_{\langle \mathcal{J}(\mathcal{L})\rangle} \\ &= TA|_{\langle \mathcal{J}(\mathcal{L})\rangle}B|_{\langle \mathcal{J}(\mathcal{L})\rangle} + ASB|_{\langle \mathcal{J}(\mathcal{L})\rangle} + ABS - ASB|_{\langle \mathcal{J}(\mathcal{L})\rangle} \\ &= TA|_{\langle \mathcal{J}(\mathcal{L})\rangle}B|_{\langle \mathcal{J}(\mathcal{L})\rangle} + ABS.\end{aligned}$$

Comparing the above two equations, we get $\delta(AB) = \delta(A)B + A\tau(B)$ for all $A, B \in \mathcal{F}(\mathcal{L})$. Hence δ is a generalized derivation and τ is the related derivation.

By Theorem 3.1, we have $L(A)x = (\delta(A) + \phi(A))x$ for all $A \in \mathcal{F}(\mathcal{L})$ and $x \in \langle \mathcal{J}(\mathcal{L}) \rangle$. This implies that $L(A) = \delta(A) + \phi(A)$ for all $A \in \mathcal{F}(\mathcal{L})$. The proof is complete. ∎

To prove Theorem 3.1, we need the following lemma.

LEMMA 3.3 ([10, Lemma 3.6]). *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and let $A_1, \ldots, A_n \in \mathcal{F}_{\mathcal{L}}(K)$ for some $K \in \mathcal{J}(\mathcal{L})$ with $\dim K \geq 2$. Then there is an idempotent $P \in \mathcal{F}_{\mathcal{L}}(K)$ such that $A_i = PA_iP$, $i = 1, \ldots, n$.*

Proof of Theorem 3.1. The "if" part is obvious. We will check the "only if" part by proving two claims.

CLAIM 1. *There exist linear operators $T, S : \langle \mathcal{J}(\mathcal{L}) \rangle \to \langle \mathcal{J}(\mathcal{L}) \rangle$ with $T|_K \in \mathcal{B}(K)$, $S|_K \in \mathcal{B}(K)$ and $T|_K + S|_K \in \mathbb{F}I_K$ for each $K \in \mathcal{J}(\mathcal{L})$ and with the following property: for any $K \in \mathcal{J}(\mathcal{L})$, there is a linear functional $f_K : \mathcal{F}_{\mathcal{L}}(K) \to \mathbb{F}$ such that*

$$L(A)|_{\langle \mathcal{J}(\mathcal{L}) \rangle} = (TA + AS + f_K(A)P_K)|_{\langle \mathcal{J}(\mathcal{L}) \rangle}$$

for all $A \in \mathcal{F}_{\mathcal{L}}(K)$, where $P_K \in \mathcal{F}_{\mathcal{L}}(K)$ is an idempotent satisfying $P_K A = AP_K = A$ if $\dim K < \infty$, and $f_K = 0$ if $\dim K = \infty$.

In fact, if $\dim K = 1$, then $\dim K_-^{\perp} = 1$. So $\dim \mathcal{F}_{\mathcal{L}}(K) = 1$. It is easy to check that, for any $A \in \mathcal{F}_{\mathcal{L}}(K)$, we have $L(A) = \lambda A$ for some $\lambda \in \mathbb{F}$. So Claim 1 holds true in this case.

Now assume that $\dim K > 2$. For any $A \in \mathcal{F}_{\mathcal{L}}(K)$, by Lemma 3.3, there is an idempotent operator $P = \sum_{i=1}^n y_i \otimes g_i \in \mathcal{F}_{\mathcal{L}}(K)$ such that $A = PAP$, where $y_i \in K$ and $g_i \in K_-^{\perp}$. We may require that both $\{y_1, \ldots, y_n\}$ and $\{g_1, \ldots, g_n\}$ are linearly independent sets. Since $P^2 = P$, it is easily checked that

$$g_i(y_j) = \delta_{ij} \quad \text{for } 1 \leq i, j \leq n.$$

Define

$$\mathcal{D}_P = \Big\{ C \in \mathcal{F}_{\mathcal{L}}(K) : C = \sum_{i,j=1}^n \lambda_{ij} y_i \otimes g_j,\ \lambda_{ij} \in \mathbb{F} \Big\}.$$

It is clear that $\mathcal{D}_P$ is a subalgebra of $\mathcal{F}_{\mathcal{L}}(K)$ and $\mathcal{D}_P$ is isomorphic to $M_n(\mathbb{F})$ via $C \mapsto [\lambda_{ij}]_{n \times n}$. Since $\dim K > 2$, we can choose P so that $\dim \operatorname{ran}(P) > 2$. For any $A, B \in \mathcal{F}_{\mathcal{L}}(K)$, by Lemma 3.3 again, there exists some $P \in \mathcal{F}_{\mathcal{L}}(K)$ such that $A = PAP$ and $B = PBP$. Hence $\mathcal{F}_{\mathcal{L}}(K)$ is a local matrix algebra. Since $K \wedge K_- = \{0\}$ and $K \vee K_-^{\perp} = X$, we may regard $K_-^{\perp}$ as the dual space K^* of K. So $\mathcal{F}_{\mathcal{L}}(K)$ is isomorphic to $\mathcal{F}(K) \subset \mathcal{B}(K)$, the algebra of all bounded finite rank operators from K into K. Thus, if $\dim K < \infty$, then $\mathcal{F}_{\mathcal{L}}(K)$ is isomorphic to $M_n(\mathbb{F})$. Hence by Corollary 2.2, there exist $T_K, S_K, P_K \in \mathcal{F}_{\mathcal{L}}(K)$ with $T_K + S_K \in \mathbb{F}I_K$ and P_K being an idempotent onto K, and a linear functional $f_K : \mathcal{F}_{\mathcal{L}}(K) \to \mathbb{F}$, such that

$$L(A) = T_K A + AS_K + f_K(A)P_K \quad \text{for all } A \in \mathcal{F}_{\mathcal{L}}(K). \tag{3.1}$$

If $\dim K = \infty$, then $\mathcal{F}_{\mathcal{L}}(K)$ is isomorphic to $\mathcal{F}(K) \subset \mathcal{B}(K)$, and by Corollary 2.3, there exist $\lambda_K \in \mathbb{F}$ and $T_K \in \mathcal{B}(K)$ such that $L(A) = T_K A - AT_K + \lambda_K A$ for all $A \in \mathcal{F}_{\mathcal{L}}(K)$. Let $S_K = \lambda_K I_K - T_K$, where I_K is the identity on K. We get

$$L(A) = T_K A + AS_K \quad \text{for all } A \in \mathcal{F}_{\mathcal{L}}(K). \tag{3.2}$$

Now, by (3.1)–(3.2), we can define linear maps $T, S : \langle \mathcal{J}(\mathcal{L}) \rangle \to \langle \mathcal{J}(\mathcal{L}) \rangle$ by $T|_K = T_K$ and $S|_K = S_K$ for every $K \in \mathcal{J}(\mathcal{L})$. Since $\mathcal{L}$ is a JSL, T and S are well defined, and we have

$$L(A)|_{\langle \mathcal{J}(\mathcal{L}) \rangle} = (TA + AS + f_K(A)P_K)|_{\langle \mathcal{J}(\mathcal{L}) \rangle} \quad \text{for all } A \in \mathcal{F}_{\mathcal{L}}(K), \tag{3.3}$$

where $f_K = 0$ whenever $\dim K = \infty$. Also note that, by the definition of T and S, $T|_K \in \mathcal{B}(K)$ and $S|_K \in \mathcal{B}(K)$ for every $K \in \mathcal{J}(\mathcal{L})$. Hence Claim 1 holds true.

Claim 2. *L has the form stated in the theorem.*

For any $A \in \mathcal{F}(\mathcal{L})$, there uniquely exist finite distinct $K_1, \ldots, K_n$ in $\mathcal{J}(\mathcal{L})$ such that $A = A_1 + \cdots + A_n$ with $A_i \in \mathcal{F}_{\mathcal{L}}(K)$, $i = 1, \ldots, n$. Take linear operators $T, S : \langle \mathcal{J}(\mathcal{L}) \rangle \to \langle \mathcal{J}(\mathcal{L}) \rangle$ as in Claim 1. Then, for each i, we have $T|_{K_i} \in \mathcal{B}(K_i)$, $S|_{K_i} \in \mathcal{B}(K_i)$ and $T|_{K_i} + S|_{K_i} \in \mathbb{F} I_{K_i}$. Also, for each i there is a linear functional $f_{K_i} : \mathcal{F}_{\mathcal{L}}(K_i) \to \mathbb{F}$ such that

$$L(A_i)|_{\langle \mathcal{J}(\mathcal{L}) \rangle} = (TA_i + A_i S + f_{K_i}(A_i) P_{K_i})|_{\langle \mathcal{J}(\mathcal{L}) \rangle}$$

for all $A_i \in \mathcal{F}_{\mathcal{L}}(K_i)$, where $P_{K_i} \in \mathcal{F}_{\mathcal{L}}(K_i)$ is an idempotent satisfying $P_{K_i} B_i = B_i P_{K_i} = B_i$ for all $B_i \in \mathcal{F}_{\mathcal{L}}(K_i)$ if $\dim K_i < \infty$, and $f_{K_i} = 0$ if $\dim K_i = \infty$. Let $\phi(A) = \sum_{i=1}^n f_{K_i}(A_i) P_{K_i}$. Since $A_i P_{K_j} = P_{K_j} A_i = 0$ if $i \neq j$, and $A_i P_{K_j} = P_{K_j} A_i = A_i$ if $i = j$, and since $A \in \mathcal{F}(\mathcal{L})$ is arbitrary, we see that ϕ is well-defined and $\phi(A) \in \mathcal{Z}(\mathcal{F}(\mathcal{L}))$. It follows that

$$L(A) = \sum_{i=1}^n L(A_i) = \sum_{i=1}^n (TA_i + A_i S + f_{K_i}(A_i) P_{K_i}) = TA + AS + \phi(A)$$

on $\langle \mathcal{J}(\mathcal{L}) \rangle$, that is, $L(A)x = (TA + AS + \phi(A))x$ for all $A \in \mathcal{F}(\mathcal{L})$ and $x \in \langle \mathcal{J}(\mathcal{L}) \rangle$. Now, it is obvious that ϕ is linear as L is linear.

The proof is complete. ∎

Next, we discuss linear maps Lie derivable at zero on $\mathcal{J}$-subspace lattice algebras. The following is our second main result in this section.

Theorem 3.4. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X over the real or complex field $\mathbb{F}$ with $\dim X \geq 3$. Suppose that $\dim K \neq 2$ for every $K \in \mathcal{J}(\mathcal{L})$ and $L : \operatorname{Alg} \mathcal{L} \to \operatorname{Alg} \mathcal{L}$ is a linear map. Then the following are equivalent.*

(1) *L is Lie derivable at zero.*
(2) *For each $K \in \mathcal{J}(\mathcal{L})$, there exists an operator $T_K \in \mathcal{B}(K)$, a scalar λ_K and a linear functional $h_K : \mathrm{Alg}\,\mathcal{L} \to \mathbb{F}$ such that $L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$ for all $x \in K$ and $A \in \mathrm{Alg}\,\mathcal{L}$.*

Proof. The "if" part is obvious. We check the "only if" part by proving several claims.

Claim 1. *For any $K \in \mathcal{J}(\mathcal{L})$, there exist $T_K, S_K \in \mathcal{B}(K)$ with $T_K + S_K \in \mathbb{F}I_K$ and a linear functional $f_K : \mathcal{F}_{\mathcal{L}}(K) \to \mathbb{F}$ such that*

$$L(A)|_K = (T_K A + AS_K + f_K(A)P_K)|_K \tag{3.4}$$

for all $A \in \mathcal{F}_{\mathcal{L}}(K)$, where $P_K \in \mathcal{F}_{\mathcal{L}}(K)$ is an idempotent satisfying $P_K A = AP_K = A$ if $\dim K < \infty$, and $f_K = 0$ if $\dim K = \infty$.

The proof is the same as that of Claim 1 in Theorem 3.1.

Claim 2. *For each $K \in \mathcal{J}(\mathcal{L})$, there exist $T_K \in \mathcal{B}(K)$ and a scalar λ_K such that, for every invertible $A \in \mathrm{Alg}\,\mathcal{L}$,*

$$L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$$

for some scalar $h_K(A)$ and all $x \in K$.

Assume that $A \in \mathrm{Alg}\,\mathcal{L}$ is invertible. For any $K \in \mathcal{J}(\mathcal{L})$ and any nonzero $x \in K$, we have $0 \neq Ax \in K$. By Lemma 1.2, there exists $f \in K_-^\perp$ such that $\langle Ax, f\rangle = 1$. Since

$$[A - Ax \otimes A^*f, x \otimes f] = Ax \otimes f - Ax \otimes f + x \otimes A^*f - x \otimes A^*f = 0,$$

we have

$$[L(A - Ax \otimes A^*f), x \otimes f] + [A - Ax \otimes A^*f, L(x \otimes f)] = 0.$$

Thus, by (3.4), we get

$$\begin{aligned}
&[L(A) - T_K(Ax \otimes A^*f) - (Ax \otimes A^*f)S_K, x \otimes f] \\
&\qquad + [A - Ax \otimes A^*f, T_K(x \otimes f) + (x \otimes f)S_K + f_K(x \otimes f)P_K] = 0.
\end{aligned} \tag{3.5}$$

Note that $f_K(x \otimes f)(AP_K - P_K A)x = 0$. It follows from (3.5) that

$$\begin{aligned}
&((L(A) - T_K A + AT_K - (\langle AS_K x, f\rangle + \langle AT_K x, f\rangle)A)x \otimes f)x \\
&\qquad = (x \otimes f)(L(A) + S_K A - AS_K - (\langle T_K Ax, f\rangle + \langle S_K Ax, f\rangle)A)x.
\end{aligned}$$

Since $T_K + S_K = \lambda_K I_K$ for each $K \in \langle \mathcal{J}(\mathcal{L})\rangle$ and $\langle Ax, f\rangle = 1$, the above equation becomes

$$((L(A) - T_K A + AT_K - \lambda_K A)x \otimes f)x = (x \otimes f)(L(A) + S_K A - AS_K - \lambda_K A)x.$$

Hence we see that $(L(A) - T_K A + AT_K - \lambda_K A)x$ and x are linearly dependent for every $x \in K$. It follows that there exist scalars λ_K and $h_K(A)$ such that

$$L(A) - T_K A + AT_K - \lambda_K A = h_K(A)I_K$$

on K. That is,

$$L(A) = T_K A - AT_K + \lambda_K A + h_K(A) I_K \tag{3.6}$$

on K for all invertible A.

CLAIM 3. *For each $K \in \mathcal{J}(\mathcal{L})$, there exist $T_K \in \mathcal{B}(K)$, a scalar λ_K and a linear functional $h_K : \operatorname{Alg}\mathcal{L} \to \mathbb{F}$ such that $L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$ for all $x \in K$ and all $A \in \operatorname{Alg}\mathcal{L}$.*

For any $A \in \operatorname{Alg}\mathcal{L}$, take a scalar c such that $|c| > \|A\|$. Then $cI - A$ is invertible with inverse still in $\operatorname{Alg}\mathcal{L}$. By (3.6), we have

$$\begin{aligned} &L(cI - A)|_K \\ &\quad = T_K(cI - A)|_K - (cI - A)T_K + \lambda_K(cI - A)|_K + h_K(cI - A)I|_K \\ &\quad = -T_K A|_K + AT_K + c\lambda_K I|_K - \lambda_K A|_K + h_K(cI - A)I|_K. \end{aligned}$$

On the other hand,

$$L(cI - A)|_K = L(cI)|_K - L(A)|_K = c\lambda_K I|_K + h_K(cI)I|_K - L(A)|_K.$$

Combining the above two equations, we get

$$\begin{aligned} L(A)|_K &= T_K A|_K - AT_K + \lambda_K A|_K + (h_K(cI) - h_K(cI - A))I|_K \\ &= T_K A|_K - AT_K + \lambda_K A|_K + h_K(A)I|_K, \end{aligned}$$

where $h_K(A) = h_K(cI) - h_K(cI - A)$. Thus there exists a functional h_K on $\operatorname{Alg}\mathcal{L}$ such that

$$L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$$

for all $A \in \operatorname{Alg}\mathcal{L}$ and all $x \in K$. Since L is linear, we see that h_K is linear. The proof of Theorem 3.4 is complete. ∎

In particular, if L is a linear map on a pentagon subspace lattice algebra $\operatorname{Alg}\mathcal{L}$ over a Banach space, then L is Lie derivable at zero if and only if L has the form stated in Theorem 3.4, since $\dim K = \infty$ for every $K \in \mathcal{J}(\mathcal{L})$.

Recall that every Lie derivation on $M_n(\mathbb{F})$ ($n \geq 1$) is a sum of a derivation and of I multiplied by a linear functional vanishing at all commutators (for example, see [2, Corollary 6.9]). From this fact, together with Theorem 3.1, Theorem 3.4 and their proofs (including the case $\dim K = 2$ for some $K \in \mathcal{J}(\mathcal{L})$), one can immediately get a characterization of linear Lie derivations on real or complex JSL algebras, which generalizes [9, Theorems 3.1, 4.1], where only the complex case was dealt with.

COROLLARY 3.5. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a real or complex Banach space X and $L : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$ be a linear map. Then L is a Lie derivation if and only if there exists a derivation $\tau : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$ and a linear map $\phi : \mathcal{F}(\mathcal{L}) \to \mathcal{Z}(\mathcal{F}(\mathcal{L}))$ vanishing at each commutator such that $L(A) = \tau(A) + \phi(A)$ for all $A \in \mathcal{F}(\mathcal{L})$.*

Corollary 3.6. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a real or complex Banach space X. Suppose that $L : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a linear map. Then L is a Lie derivation if and only if for each $K \in \mathcal{J}(\mathcal{L})$, there exists $T_K \in \mathcal{B}(K)$, a scalar λ_K and a linear functional $h_K : \operatorname{Alg}\mathcal{L} \to \mathbb{F}$ vanishing at every commutator such that $L(A)x = (T_K A - AT_K + \lambda_K A + h_K(A)I)x$ for all $x \in K$ and $A \in \operatorname{Alg}\mathcal{L}$.*

Finally, we remark that, in Theorems 3.1 and 3.4, the condition that $\dim K \neq 2$ for every $K \in \mathcal{J}(\mathcal{L})$ is necessary, as the following example shows.

Example 3.7. Let X be a Banach space over the real or complex field $\mathbb{F}$ with $\dim X = 4$. Assume that $\mathcal{L} = \{(0), K_1, K_2, X\}$ with $\dim K_1 = \dim K_2 = 2$ and $K_1 \cap K_2 = \{0\}$. Then $\mathcal{L}$ is an atomic Boolean lattice and $\operatorname{Alg}\mathcal{L}$ is the associated atomic Boolean lattice algebra. It is clear that $\mathcal{J}(\mathcal{L}) = \{K_1, K_2\}$ and $X = K_1 \oplus K_2$. Moreover, every $A \in \operatorname{Alg}\mathcal{L}$ is of the form $A = A_1 \oplus A_2$, where $A_i \in \mathcal{B}(K_i)$, $i = 1, 2$. Take any linear maps $L_i : \mathcal{B}(K_i) \to \mathcal{B}(K_i)$ satisfying $L_1(I_i) = c_i I_i$ for some $c_i \in \mathbb{F}$, $i = 1, 2$. By Proposition 2.5, both L_1 and L_2 are Lie derivable at zero. Now let $L(A) = L_1(A_1) \oplus L_2(A_2)$, where $A = A_1 \oplus A_2 \in \operatorname{Alg}\mathcal{L}$. It is clear that $L : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a linear map Lie derivable at zero. However, it is easily seen from Example 2.6 that L is not always of the form stated in Theorem 3.4.

Acknowledgments. The authors wish to thank the referees for valuable comments that greatly improved the presentation of this paper.

This work is partially supported by National Natural Science Foundation of China (10771157) and Research Fund of Shanxi for Returning Scholars (2007–38).

References

[1] M. Brešar, *Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings*, Trans. Amer. Math. Soc. 335 (1993), 525–546.

[2] M. Brešar, M. A. Chebotar and W. S. Martindale III, *Functional Identities*, Birkhäuser, Basel, 2006.

[3] A. Katavolos, M. S. Lambrou and W. E. Longstaff, *Pentagon subspace lattices on Banach spaces*, J. Operator Theory 46 (2001), 355–380.

[4] B. E. Johnson, *Symmetric amenability and the nonexistence of Lie and Jordan derivations*, Math. Proc. Cambridge Philos. Soc. 120 (1996), 455–473.

[5] W. E. Longstaff, *Operators of rank one in reflexive algebras*, Canad. J. Math. 28 (1976), 9–23.

[6] —, *Strongly reflexive lattices*, J. London Math. Soc. 11 (1975), 491–498.

[7] W. E. Longstaff, J. B. Nation and O. Panaia, *Abstract reflexivity subspace lattices and completely distributive collapsibility*, Bull. Austral. Math. Soc. 58 (1998), 245–260.

[8] W. E. Longstaff and O. Panaia, *$\mathcal{J}$-subspace lattices and subspace M-bases*, Studia Math. 139 (2000), 197–212.
[9] F. Y. Lu, *Lie derivations of $\mathcal{J}$-subspace lattice algebras*, Proc. Amer. Math. Soc. 135 (2007), 2581–2590.
[10] F. Y. Lu and P. T. Li, *Algebraic isomorphisms and Jordan derivations of $\mathcal{J}$-subspace lattice algebras*, Studia Math. 158 (2003), 287–301.
[11] X. F. Qi and J. C. Hou, *Additive Lie (ξ-Lie) derivations and generalized Lie (ξ-Lie) derivations on nest algebras*, Linear Algebra Appl. 431 (2009), 843–854.
[12] —, —, *Characterizations of derivations of Banach space nest algebras: all-derivable points*, Linear Algebra Appl. (2010), doi:10.1016/j.laa.2010.01.020.
[13] —, —, *Additive maps ξ-Lie derivable at zero point on triangular algebras*, preprint.
[14] M. Mathieu and A. R. Villena, *The structure of Lie derivations on C^*-algebras*, J. Funct. Anal. 202 (2003), 504–525.
[15] J. Zhu and C. P. Xiong, *Derivable mappings at unit operator on nest algebras*, Linear Algebra Appl. 422 (2007), 721–735.
[16] —, —, *All-derivable points in continuous nest algebras*, J. Math. Anal. Appl. 340 (2008), 845–853.

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Received April 20, 2009
Revised version September 24, 2009 (6604)