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Quotients of Banach spaces and surjectively universal spaces

by

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Abstract. We characterize those classes $\mathcal{C}$ of separable Banach spaces for which there exists a separable Banach space Y not containing ℓ_1 and such that every space in the class $\mathcal{C}$ is a quotient of Y.

1. Introduction. There are two classical universality results in Banach space theory. The first one, known to Stefan Banach [5], asserts that the space $C(2^{\mathbb{N}})$, where $2^{\mathbb{N}}$ stands for the Cantor set, is isometrically universal for all separable Banach spaces; that is, every separable Banach space is isometric to a subspace of $C(2^{\mathbb{N}})$. The second result, also known to Banach, is "dual" to the previous one and asserts that every separable Banach space is isometric to a quotient of ℓ_1.

By now, it is well understood that there are natural classes of separable Banach spaces for which one can get nothing better than what is quoted above (see [1, 8, 16, 31]). For instance, if a separable Banach space Y is universal for the separable reflexive Banach spaces, then Y must contain an isomorphic copy of $C(2^{\mathbb{N}})$, and so it is universal for all separable Banach spaces. However, there are non-trivial classes of separable Banach spaces which do admit "smaller" universal spaces (see [2, 11, 12, 13, 15, 23, 24, 27]).

Recently, in [11], a characterization was obtained of those classes of separable Banach spaces admitting a universal space which is not universal for all separable Banach spaces. One of the goals of the present paper is to obtain the corresponding characterization for the "dual" problem concerning quotients instead of embeddings. To proceed with our discussion it is useful to introduce the following definition.

DEFINITION 1. We say that a Banach space Y is a *surjectively universal space* for a class $\mathcal{C}$ of Banach spaces if every space in the class $\mathcal{C}$ is a quotient (1) of Y.

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(1) If X and Y are Banach spaces, then we say that X is a *quotient* of Y if there exists a bounded, linear and onto operator $Q : Y \to X$.

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We can now state the main problem addressed in this paper.

(P) Let $\mathcal{C}$ be a class of separable Banach space. When can we find a separable Banach space Y which is surjectively universal for the class $\mathcal{C}$ and does not contain a copy of ℓ_1?

We notice that if a separable Banach space Y does not contain a copy of ℓ_1, then ℓ_1 is not a quotient of Y (see [21, Proposition 2.f.7]) and therefore Y is not surjectively universal for all separable Banach spaces.

To state our results we recall the following (more or less standard) notation and terminology. We denote by SB the standard Borel space of separable Banach spaces defined by B. Bossard [7], by NC_{ℓ_1} the subset of SB consisting of all $X \in \mathrm{SB}$ not containing an isomorphic copy of ℓ_1, and finally by $\phi_{\mathrm{NC}_{\ell_1}}$ Bourgain's ℓ_1 index [8] (these concepts are properly defined in §2). We show the following.

Theorem 2. *Let $\mathcal{C} \subseteq \mathrm{SB}$. Then the following are equivalent.*

(i) *There exists a separable Banach space Y which is surjectively universal for the class $\mathcal{C}$ and does not contain a copy of ℓ_1.*
(ii) $\sup\{\phi_{\mathrm{NC}_{\ell_1}}(X) : X \in \mathcal{C}\} < \omega_1$.
(iii) *There exists an analytic subset $\mathcal{A}$ of NC_{ℓ_1} with $\mathcal{C} \subseteq \mathcal{A}$.*

We notice that stronger versions of Theorem 2 are valid provided that all spaces in the class $\mathcal{C}$ have some additional property (see §5).

A basic ingredient of the proof of Theorem 2 (probably of independent interest) is the construction, for every separable Banach space X, of a Banach space E_X with special properties. Specifically we show the following.

Theorem 3. *Let X be a separable Banach space. Then there exists a separable Banach space E_X with the following properties:*

(i) (Existence of a Schauder basis) *The space E_X has a normalized monotone Schauder basis (e_n^X).*
(ii) (Existence of a quotient map) *There exists a norm-one linear and onto operator $Q_X : E_X \to X$.*
(iii) (Subspace structure) *If Y is an infinite-dimensional subspace of E_X and the operator $Q_X : Y \to X$ is strictly singular, then Y contains a copy of c_0.*
(iv) (Representability of X) *For every normalized basic sequence (w_k) in X there exists a subsequence $(e_{n_k}^X)$ of (e_n^X) such that $(e_{n_k}^X)$ is equivalent to (w_k).*
(v) (Uniformity) *The set $\mathcal{E} \subseteq \mathrm{SB} \times \mathrm{SB}$ defined by*

$$(X, Y) \in \mathcal{E} \Leftrightarrow Y \textit{ is isometric to } E_X$$

is analytic.

(vi) (Preservation of separability of the dual) *E_X^* is separable if and only if X^* is separable.*

We notice that there are a large number of related results in the literature; see, for instance, [9, 14, 15, 19, 23, 24, 32]. The novelty in Theorem 3 is that, besides functional-analytic tools, its proof is enriched with descriptive set theory and the combinatorial machinery developed in [3] and [4].

The paper is organized as follows. In §2 we gather some background material. In §3 we define the space E_X and prove Theorem 3. The proof of Theorem 2 (actually of a more detailed version of it) is given in §4. Finally, in §5 we present some related results and we discuss open problems.

2. Background material. Our general notation and terminology is standard as can be found, for instance, in [21] and [20]. By $\mathbb{N} = \{0, 1, 2, \dots\}$ we denote the natural numbers.

We will frequently need to compute the descriptive set-theoretic complexity of various sets and relations. To this end, we will use the "Kuratowski–Tarski algorithm". We assume that the reader is familiar with this classical method. For more details we refer to [20, p. 353].

2.1. Trees. Let Λ be a non-empty set. We denote by $\Lambda^{<\mathbb{N}}$ the set of all finite sequences in Λ, and by $\Lambda^{\mathbb{N}}$ the set of all infinite sequences in Λ (the empty sequence is denoted by $\emptyset$ and is included in $\Lambda^{<\mathbb{N}}$). We view $\Lambda^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. Two nodes $s, t \in \Lambda^{<\mathbb{N}}$ are said to be *comparable* if either $s \sqsubseteq t$ or $t \sqsubseteq s$. Otherwise, s and t are said to be *incomparable*. A subset of $\Lambda^{<\mathbb{N}}$ consisting of pairwise comparable nodes is said to be a *chain*, while a subset of $\Lambda^{<\mathbb{N}}$ consisting of pairwise incomparable nodes is said to be an *antichain*.

A *tree* T on Λ is a subset of $\Lambda^{<\mathbb{N}}$ which is closed under initial segments. We denote by $\mathrm{Tr}(\Lambda)$ the set of all trees on Λ. Hence

$$T \in \mathrm{Tr}(\Lambda) \;\Leftrightarrow\; \forall s, t \in \Lambda^{<\mathbb{N}} \ (s \sqsubseteq t \text{ and } t \in T \Rightarrow s \in T).$$

The *body* of a tree T on Λ is the set $\{\sigma \in \Lambda^{\mathbb{N}} : \sigma|n \in T\ \forall n \in \mathbb{N}\}$, denoted by $[T]$. A tree T is said to be *well-founded* if $[T] = \emptyset$. By $\mathrm{WF}(\Lambda)$ we denote the set of all well-founded trees on Λ. For every $T \in \mathrm{WF}(\Lambda)$ we let $T' = \{s \in T : \exists t \in T \text{ with } s \sqsubset t\} \in \mathrm{WF}(\Lambda)$. By transfinite recursion, we define the iterated derivatives T^{ξ} $(\xi < \kappa^+)$ of T, where κ stands for the cardinality of Λ. The *order* $o(T)$ of T is defined to be the least ordinal ξ such that $T^{\xi} = \emptyset$.

Let S and T be trees on two non-empty sets Λ_1 and Λ_2 respectively. A map $\psi : S \to T$ is said to be *monotone* if for every $s_0, s_1 \in S$ with $s_0 \sqsubset s_1$ we have $\psi(s_0) \sqsubset \psi(s_1)$. We notice that if there exists a monotone map $\psi : S \to T$ and T is well-founded, then S is well-founded and $o(S) \leq o(T)$.

2.2. Dyadic subtrees and related combinatorics. Let $2^{<\mathbb{N}}$ be the Cantor tree, i.e. the set of all finite sequences of 0's and 1's. For every $s,t \in 2^{<\mathbb{N}}$ we let $s \wedge t$ be the $\sqsubset$-maximal node w of $2^{<\mathbb{N}}$ with $w \sqsubseteq s$ and $w \sqsubseteq t$. If $s,t \in 2^{<\mathbb{N}}$ are incomparable with respect to $\sqsubseteq$, then we write $s \prec t$ provided that $(s \wedge t)^\frown 0 \sqsubseteq s$ and $(s\wedge t)^\frown 1 \sqsubseteq t$. We say that a subset D of $2^{<\mathbb{N}}$ is a *dyadic subtree* of $2^{<\mathbb{N}}$ if D can be written in the form $\{d_t : t \in 2^{<\mathbb{N}}\}$ so that for every $t_0, t_1 \in 2^{<\mathbb{N}}$ we have $t_0 \sqsubset t_1$ (respectively $t_0 \prec t_1$) if and only if $d_{t_0} \sqsubset d_{t_1}$ (respectively $d_{t_0} \prec d_{t_1}$). It is easy to see that such a representation of D as $\{d_t : t \in 2^{<\mathbb{N}}\}$ is unique. When we write $D = \{d_t : t \in 2^{<\mathbb{N}}\}$, where D is a dyadic subtree, we will assume that this is the canonical representation of D described above.

For every dyadic subtree D of $2^{<\mathbb{N}}$ we denote by $[D]_{\mathrm{chains}}$ the set of all infinite chains of D. Notice that $[D]_{\mathrm{chains}}$ is a G_δ, hence Polish, subspace of $2^{2^{<\mathbb{N}}}$. We will need the following partition theorem due to J. Stern (see [30]).

Theorem 4. *Let D be a dyadic subtree of $2^{<\mathbb{N}}$ and $\mathcal{X}$ be an analytic subset of $[D]_{\mathrm{chains}}$. Then there exists a dyadic subtree S of $2^{<\mathbb{N}}$ with $S \subseteq D$ and such that either $[S]_{\mathrm{chains}} \subseteq \mathcal{X}$ or $[S]_{\mathrm{chains}} \cap \mathcal{X} = \emptyset$.*

2.3. Separable Banach spaces with non-separable dual. We will need a structural result concerning separable Banach spaces with non-separable dual. To state this result and to facilitate future references to it, it is convenient to introduce the following definition.

Definition 5. Let X be a Banach space and $(x_t)_{t\in 2^{<\mathbb{N}}}$ be a sequence in X indexed by the Cantor tree. We say that $(x_t)_{t\in 2^{<\mathbb{N}}}$ is *topologically equivalent to the basis of James tree* if the following conditions are satisfied:

(1) The sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ is semi-normalized.
(2) For every infinite antichain A of $2^{<\mathbb{N}}$ the sequence $(x_t)_{t\in A}$ is weakly null.
(3) For every $\sigma \in 2^{\mathbb{N}}$ the sequence $(x_{\sigma|n})$ is weak* convergent to an element $x^{**}_\sigma \in X^{**} \setminus X$. Moreover, if $\sigma, \tau \in 2^{\mathbb{N}}$ with $\sigma \neq \tau$, then $x^{**}_\sigma \neq x^{**}_\tau$.

The archetypical example of such a sequence is the standard Schauder basis of the space JT (see [17]). There are also classical Banach spaces having a natural Schauder basis which is topologically equivalent to the basis of James tree; the space $C(2^{\mathbb{N}})$ is an example. We isolate, for future use, the following fact.

Fact 6. *Let X be a Banach space and $(x_t)_{t\in 2^{<\mathbb{N}}}$ be a sequence in X which is topologically equivalent to the basis of James tree. Then for every dyadic subtree $D = \{d_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ the sequence $(x_{d_t})_{t\in 2^{<\mathbb{N}}}$ is also topologically equivalent to the basis of James tree.*

We notice that if a Banach space X contains a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ which is topologically equivalent to the basis of James tree, then X^* is not separable. The following theorem establishes the converse for separable Banach spaces not containing a copy of ℓ_1 (see [3, Theorem 40] or [4, Theorem 17]).

THEOREM 7. *Let X be a separable Banach space not containing a copy of ℓ_1 and with non-separable dual. Then X contains a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ which is topologically equivalent to the basis of James tree.*

2.4. Co-analytic ranks. Let (X,Σ) be a standard Borel space; that is, X is a set, Σ is a σ-algebra on X and the measurable space (X,Σ) is Borel isomorphic to the reals. A subset A of X is said to be *analytic* if there exists a Borel map $f:\mathbb{N}^{\mathbb{N}}\to X$ with $f(\mathbb{N}^{\mathbb{N}})=A$. A subset of X is said to be *co-analytic* if its complement is analytic. Now let B be a co-analytic subset of X. A map $\phi: B\to\omega_1$ is said to be a *co-analytic rank on B* if there exist relations $\le_\Sigma$ and $\le_\Pi$ in $X\times X$ which are analytic and co-analytic respectively and such that for every $y\in B$ we have

$$x\in B \text{ and } \phi(x)\le\phi(y) \;\Leftrightarrow\; x\le_\Sigma y \;\Leftrightarrow\; x\le_\Pi y.$$

For our purposes, the most important property of co-analytic ranks is that they satisfy *boundedness*. This means that if $\phi: B\to\omega_1$ is a co-analytic rank on a co-analytic set B and $A\subseteq B$ is analytic, then $\sup\{\phi(x): x\in A\}<\omega_1$. For a proof as well as for a thorough presentation of rank theory we refer to [20, §34].

2.5. The standard Borel space of separable Banach spaces. Let $F(C(2^{\mathbb{N}}))$ be the set of all closed subsets of $C(2^{\mathbb{N}})$ and Σ be the Effros–Borel structure on $F(C(2^{\mathbb{N}}))$; that is, Σ is the σ-algebra generated by the sets

$$\{F\in F(C(2^{\mathbb{N}})) : F\cap U\neq\emptyset\}$$

where U ranges over all open subsets of $C(2^{\mathbb{N}})$. Consider the set

$$\mathrm{SB}=\{X\in F(C(2^{\mathbb{N}})) : X \text{ is a linear subspace}\}.$$

It is easy to see that SB equipped with the relative Effros–Borel structure is a standard Borel space (see [7] for more details). The space SB is referred to in the literature as the *standard Borel space of separable Banach spaces.* We will need the following consequence of the Kuratowski–Ryll-Nardzewski selection theorem (see [20, Theorem 12.13]).

PROPOSITION 8. *There exists a sequence $S_n:\mathrm{SB}\to C(2^{\mathbb{N}})$ $(n\in\mathbb{N})$ of Borel maps such that for every $X\in\mathrm{SB}$ with $X\neq\{0\}$ we have $S_n(X)\in S_X$ and the sequence $(S_n(X))$ is norm dense in S_X, where S_X stands for the unit sphere of X.*

2.6. The class NC_Z and Bourgain's indices. Let Z be a Banach space with a Schauder basis (2). We fix a normalized Schauder basis (z_n) of Z. If Z is one of the classical sequence spaces c_0 and ℓ_p ($1 \le p < \infty$), then we let (z_n) be the standard unit vector basis. We consider the set

$$\mathrm{NC}_Z = \{X \in \mathrm{SB} : X \text{ does not contain an isomorphic copy of } Z\}.$$

Let $\delta \ge 1$ and let Y be an arbitrary separable Banach space. Following Bourgain [8], we introduce a tree $\mathbf{T}(Y, Z, (z_n), \delta)$ on Y defined by the rule

$$(y_n)_{n=0}^k \in \mathbf{T}(Y, Z, (z_n), \delta) \;\Leftrightarrow\; (y_n)_{n=0}^k \text{ is } \delta\text{-equivalent to } (z_n)_{n=0}^k.$$

In particular, if Z is the space ℓ_1, then for every $\delta \ge 1$ and every finite sequence $(y_n)_{n=0}^k$ in Y we have $(y_n)_{n=0}^k \in \mathbf{T}(Y, \ell_1, (z_n), \delta)$ if and only if for every $a_0, \ldots, a_k \in \mathbb{R}$,

$$\frac{1}{\delta} \sum_{n=0}^{k} |a_n| \le \Big\| \sum_{n=0}^{k} a_n y_n \Big\| \le \delta \sum_{n=0}^{k} |a_n|.$$

We notice that $Y \in \mathrm{NC}_Z$ if and only if for every $\delta \ge 1$ the tree $\mathbf{T}(Y, Z, (z_n), \delta)$ is well-founded. We set $\phi_{\mathrm{NC}_Z}(Y) = \omega_1$ if $Y \notin \mathrm{NC}_Z$, while if $Y \in \mathrm{NC}_Z$ we define

$$\phi_{\mathrm{NC}_Z}(Y) = \sup\{o(\mathbf{T}(Y, Z, (z_n), \delta)) : \delta \ge 1\}. \tag{1}$$

In [8], Bourgain proved that for every Banach space Z with a Schauder basis and every $Y \in \mathrm{SB}$ we have $Y \in \mathrm{NC}_Z$ if and only if $\phi_{\mathrm{NC}_Z}(Y) < \omega_1$. We need the following refinement of this result (see [7, Theorem 4.4]).

Theorem 9. *Let Z be a Banach space with a Schauder basis. Then the set NC_Z is co-analytic and the map $\phi_{\mathrm{NC}_Z} : \mathrm{NC}_Z \to \omega_1$ is a co-analytic rank on NC_Z.*

We will also need the following quantitative strengthening of the classical fact that ℓ_1 has the lifting property.

Lemma 10. *Let X and Y be separable Banach spaces and assume that X is a quotient of Y. Then $\phi_{\mathrm{NC}_{\ell_1}}(X) \le \phi_{\mathrm{NC}_{\ell_1}}(Y)$.*

Proof. Clearly we may assume that Y does not contain a copy of ℓ_1. We fix a quotient map $Q : Y \to X$. There exists a constant $C \ge 1$ such that

(a) $\|Q\| \le C$,
(b) for every $x \in X$ there exists $y \in Y$ with $Q(y) = x$ and $\|y\| \le C\|x\|$.

For every $x \in X$ we select $y_x \in Y$ such that $Q(y_x) = x$ and $\|y_x\| \le C\|x\|$. We define a map $\psi : X^{<\mathbb{N}} \to Y^{<\mathbb{N}}$ as follows. We set $\psi(\emptyset) = \emptyset$. If $s =$

(2) Throughout the paper when we say that a Banach space X has a Schauder basis or the bounded approximation property, then we implicitly assume that X is infinite-dimensional.

$(x_n)_{n=0}^k \in X^{<\mathbb{N}} \setminus \{\emptyset\}$, then we set

$$\psi(s) = (y_{x_n})_{n=0}^k.$$

We notice that the map ψ is monotone. Denote by (z_n) the standard unit vector basis of ℓ_1.

Claim 11. *For every $\delta \geq 1$ if $s \in \mathbf{T}(X, \ell_1, (z_n), \delta)$, then $\psi(s) \in \mathbf{T}(Y, \ell_1, (z_n), C\delta)$.*

Granting Claim 11, we can complete the proof of the lemma. Indeed, by Claim 11, for every $\delta \geq 1$ the map ψ is monotone from $\mathbf{T}(X, \ell_1, (z_n), \delta)$ into $\mathbf{T}(Y, \ell_1, (z_n), C\delta)$. Hence

$$o(\mathbf{T}(X, \ell_1, (z_n), \delta)) \leq o(\mathbf{T}(Y, \ell_1, (z_n), C\delta)).$$

This clearly implies that $\phi_{\mathrm{NC}_{\ell_1}}(X) \leq \phi_{\mathrm{NC}_{\ell_1}}(Y)$.

It remains to prove Claim 11. Let $s = (x_n)_{n=0}^k \in \mathbf{T}(X, \ell_1, (z_n), \delta)$ and $a_0, \ldots, a_k \in \mathbb{R}$. Then

$$Q(a_0 y_{x_0} + \cdots + a_k y_{x_k}) = a_0 x_0 + \cdots + a_k x_k.$$

Hence, by (a), we get

$$\frac{1}{\delta} \sum_{n=0}^k |a_n| \leq \Big\| \sum_{n=0}^k a_n x_n \Big\| = \Big\| Q\Big(\sum_{n=0}^k a_n y_{x_n} \Big) \Big\| \leq C \Big\| \sum_{n=0}^k a_n y_{x_n} \Big\|. \tag{2}$$

Observe that $\|x_n\| \leq \delta$ for every $n \in \{0, \ldots, k\}$. Therefore,

$$\Big\| \sum_{n=0}^k a_n y_{x_n} \Big\| \leq \sum_{n=0}^k |a_n| \cdot \|y_{x_n}\| = \sum_{n=0}^k |a_n| \cdot \|Q(x_n)\| \leq C\delta \sum_{n=0}^k |a_n|. \tag{3}$$

Since the coefficients $a_0, \ldots, a_k \in \mathbb{R}$ were arbitrary, inequalities (2) and (3) imply that $\psi(s) = (y_{x_n})_{n=0}^k \in \mathbf{T}(Y, \ell_1, (z_n), C\delta)$. This completes the proof of Claim 11 and of the lemma. ∎

2.7. Separable spaces with the B.A.P. and Lusky's theorem. By the results in [18] and [26], a separable Banach space X has the bounded approximation property (for short B.A.P.) if and only if X is isomorphic to a complemented subspace of a Banach space Y with a Schauder basis. W. Lusky found an effective way to produce the space Y. To state his result we need, first, to recall the definition of the space C_0 due to W. B. Johnson. Let (F_n) be a sequence of finite-dimensional spaces dense in the Banach–Mazur distance in the class of all finite-dimensional spaces. We set

$$C_0 = \Big(\sum_{n \in \mathbb{N}} \oplus F_n \Big)_{c_0} \tag{4}$$

and we notice that C_0 is hereditarily c_0 (i.e. every infinite-dimensional subspace of C_0 contains a copy of c_0). We can now state Lusky's theorem (see [22]).

THEOREM 12. *Let X be a separable Banach space with the bounded approximation property. Then $X \oplus C_0$ has a Schauder basis.*

Theorem 12 will be used in the following parameterized form.

LEMMA 13. *Let Z be a minimal* (3) *Banach space not containing a copy of c_0. Let $\mathcal{A}$ be an analytic subset of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$. Then there exists a (possibly empty) subset $\mathcal{D}$ of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$ with the following properties.*

(i) *The set $\mathcal{D}$ is analytic.*
(ii) *Every $Y \in \mathcal{D}$ has a Schauder basis.*
(iii) *For every $X \in \mathcal{A}$ with the bounded approximation property there exists $Y \in \mathcal{D}$ such that X is isomorphic to a complemented subspace of Y.*

Proof. The result is essentially known, and so we will be rather sketchy. First we consider the set $\mathcal{B} \subseteq \mathrm{SB}$ defined by

$$X \in \mathcal{B} \Leftrightarrow X \text{ has the bounded approximation property.}$$

Using the characterization of B.A.P. mentioned above, it is easy to check that the set $\mathcal{B}$ is analytic. Next, consider the set $\mathcal{C} \subseteq \mathrm{SB} \times \mathrm{SB}$ defined by

$$(X, Y) \in \mathcal{C} \Leftrightarrow Y \text{ is isomorphic to } X \oplus C_0.$$

It is also easy to see that $\mathcal{C}$ is analytic (see [2] for more details). Define $\mathcal{D} \subseteq \mathrm{SB}$ by the rule

$$Y \in \mathcal{D} \Leftrightarrow \exists X\ [X \in \mathcal{A} \cap \mathcal{B} \text{ and } (X, Y) \in \mathcal{C}]$$

and notice that $\mathcal{D}$ is analytic. By Theorem 12, the set $\mathcal{D}$ is as desired. ∎

2.8. Amalgamated spaces. A recurrent theme in the proof of various universality results found in the literature (a theme that goes back to the classical results of A. Pełczyński [25]) is the use at a certain point of a “gluing” procedure. A number of different “gluing” procedures have been proposed by several authors. We will need the following result (see [2, Theorem 71]).

THEOREM 14. *Let $1 < p < \infty$ and $\mathcal{C}$ be an analytic subset of* SB *such that every $Y \in \mathcal{C}$ has a Schauder basis. Then there exists a Banach space V with a Schauder basis that contains a complemented copy of every space in the class $\mathcal{C}$.*

Moreover, if W is an infinite-dimensional subspace of V, then either

(i) *W contains a copy of ℓ_p, or*
(ii) *there exist $Y_0, \dots, Y_n$ in the class $\mathcal{C}$ such that W is isomorphic to a subspace of $Y_0 \oplus \cdots \oplus Y_n$.*

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(3) We recall that an infinite-dimensional Banach space Z is said to be *minimal* if every infinite-dimensional subspace of Z contains an isomorphic copy of Z; e.g. the classical sequence spaces c_0 and ℓ_p $(1 \leq p < \infty)$ are minimal spaces.

The space V obtained above is called in [2] the *p-amalgamation space* of the class $\mathcal{C}$. The reader can find in [2, §8] an extensive study of its properties.

3. Quotients of Banach spaces

3.1. Definitions. We start with the following.

DEFINITION 15. Let X be a separable Banach space and (x_n) be a sequence (with possible repetitions) which is norm dense in the unit sphere S_X. We denote by E_X the completion of $c_{00}(\mathbb{N})$ under the norm

$$\|z\|_{E_X} = \sup\Big\{\Big\|\sum_{n=0}^{m} z(n)x_n\Big\|_X : m \in \mathbb{N}\Big\}. \tag{5}$$

Let (e_n^X) denote the standard Hamel basis of $c_{00}(\mathbb{N})$ regarded as a sequence in E_X. If $X = \{0\}$, then by convention we set $E_X = c_0$.

The construction of the space E_X is somehow "classical" and its motivation can be traced back to the proof of the fact that every separable Banach space is a quotient of ℓ_1 (see [21, p. 108]). A similar construction was presented by G. Schechtman in [29] for different, though related, purposes.

We isolate two elementary properties of the space E_X. First, we observe that the sequence (e_n^X) defines a normalized monotone Schauder basis of E_X. It is also easy to see that the map $E_X \ni e_n^X \mapsto x_n \in X$ extends to a norm-one linear operator. This operator will be denoted as follows.

DEFINITION 16. We denote by $Q_X : E_X \to X$ the (unique) bounded linear operator satisfying $Q_X(e_n^X) = x_n$ for every $n \in \mathbb{N}$.

Let us make two comments about the above definitions. Let (y_n) be a basic sequence in a Banach space Y and assume that the map

$$\overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} \ni y_n \mapsto x_n \in X$$

extends to a bounded linear operator. Then it is easy to see that there exists a constant $C \geq 1$ such that the sequence (e_n^X) is C-dominated (4) by (y_n). In other words, among all basic sequences (y_n) such that the map $\overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} \ni y_n \mapsto x_n \in X$ extends to a bounded linear operator, the sequence (e_n^X) is *minimal* with respect to domination.

Notice also that the space E_X depends on the choice of the sequence (x_n). For our purposes, however, the dependence is not important, as can be seen from the following simple observation. Let (d_n) be another norm dense

(4) We recall that if (v_n) and (y_n) are two basic sequences in two Banach spaces V and Y respectively, then (v_n) is said to be *C-dominated* by (y_n) if for every $k \in \mathbb{N}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have $\|\sum_{n=0}^{k} a_n v_n\|_V \leq C\|\sum_{n=0}^{k} a_n y_n\|_Y$.

sequence in S_X and let E'_X be the completion of $c_{00}(\mathbb{N})$ under the norm

$$\|z\|_{E'_X} = \sup\Big\{\Big\|\sum_{n=0}^{m} z(n)d_n\Big\|_X : m\in\mathbb{N}\Big\}.$$

Then it is easy to check that E_X embeds isomorphically into E'_X and vice versa. Actually, it is possible to modify the construction to obtain a different space sharing most of the properties of E_X and not depending on the choice of the dense sequence. We could not find, however, any application of this construction, and since it is involved and conceptually less natural to grasp we prefer not to bother the reader with it.

The rest of the section is organized as follows. In §3.2 we present some preliminary tools needed for the proof of Theorem 3. The proof itself is given in §3.3, while in §3.4 we present some of its consequences. Finally, in §3.5 we make some comments.

3.2. Preliminary tools. We start by introducing a few pieces of notation that will be used only in this section. Let F and G be two non-empty finite subsets of $\mathbb{N}$. We write $F<G$ if $\max F<\min G$. Let (e_n) be a basic sequence in a Banach space E and let $v\in\overline{\mathrm{span}}\{e_n : n\in\mathbb{N}\}$. There exists a (unique) sequence (a_n) of reals such that $v=\sum_{n\in\mathbb{N}} a_n e_n$. The *support* of the vector v, denoted by $\mathrm{supp}(v)$, is defined to be the set $\{n\in\mathbb{N} : a_n\neq 0\}$. The *range* of v, denoted by $\mathrm{range}(v)$, is the minimal interval of $\mathbb{N}$ that contains $\mathrm{supp}(v)$.

In what follows, X will be a separable Banach space and (x_n) will be the sequence in X which is used to define the space E_X. The following propositions will be basic tools for the analysis of E_X.

Proposition 17. *Let (v_k) be a semi-normalized block sequence of (e^X_n) and assume that $\|Q_X(v_k)\|_X\leq 2^{-k}$ for every $k\in\mathbb{N}$. Then (v_k) is equivalent to the standard unit vector basis of c_0.*

Proof. We select a constant $C>0$ such that $\|v_k\|_{E_X}\leq C$ for every $k\in\mathbb{N}$. Let $F=\{k_0<\cdots<k_j\}$ be a finite subset of $\mathbb{N}$. We will show that

$$\Big\|\sum_{i=0}^{j} v_{k_i}\Big\|_{E_X}\leq 2+C.$$

This will finish the proof. To this end we argue as follows. First we set

(a) $G_i=\mathrm{supp}(v_{k_i})$ and $m_i=\min G_i$ for every $i\in\{0,\dots,j\}$.

Let (a_n) be the unique sequence of reals such that

(b) $a_n=0$ if $n\notin G_0\cup\cdots\cup G_j$,
(c) $v_{k_i}=\sum_{n\in G_i} a_n e^X_n$ for every $i\in\{0,\dots,j\}$.

Notice that for every $l \in \{0, \dots, j\}$ and $m \in \mathbb{N}$ with $m \in \mathrm{range}(v_{k_l})$ we have

$$\Big\| \sum_{n=m_l}^{m} a_n x_n \Big\|_X \le \|v_{k_l}\|_{E_X} \le C. \tag{6}$$

We select $p \in \mathbb{N}$ such that

$$\Big\| \sum_{i=0}^{j} v_{k_i} \Big\|_{E_X} = \Big\| \sum_{n=0}^{p} a_n x_n \Big\|_X$$

and we distinguish the following cases.

CASE 1: $p \in \mathrm{range}(v_{k_0})$. Using (6), we see that

$$\Big\| \sum_{i=0}^{j} v_{k_i} \Big\|_{E_X} = \Big\| \sum_{n=m_0}^{p} a_n x_n \Big\|_X \le C.$$

CASE 2: $p \in \mathrm{range}(v_{k_l})$ *for some* $l \in \{1, \dots, j\}$. Using our hypotheses on the sequence (v_k) and inequality (6), we get

$$\begin{aligned} \Big\| \sum_{i=0}^{j} v_{k_i} \Big\|_{E_X} &= \Big\| \sum_{i=0}^{l-1} \sum_{n \in G_i} a_n x_n + \sum_{n=m_l}^{p} a_n x_n \Big\|_X \\ &\le \sum_{i=0}^{l-1} \Big\| \sum_{n \in G_i} a_n x_n \Big\|_X + \Big\| \sum_{n=m_l}^{p} a_n x_n \Big\|_X \\ &= \sum_{i=0}^{l-1} \|Q_X(v_{k_i})\|_X + \Big\| \sum_{n=m_l}^{p} a_n x_n \Big\|_X \le 2 + C. \end{aligned}$$

CASE 3: $p \notin \mathrm{range}(v_{k_i})$ *for every* $i \in \{0, \dots, j\}$. In this case there exists $l \in \{0, \dots, j\}$ such that $\mathrm{range}(v_{k_i}) < \{p\}$ if $i \in \{0, \dots, l\}$, while $\{p\} < \mathrm{range}(v_{k_i})$ otherwise. Using this observation we see that

$$\Big\| \sum_{i=0}^{j} v_{k_i} \Big\|_{E_X} = \Big\| \sum_{i=0}^{l} \sum_{n \in G_i} a_n x_n \Big\|_X \le \sum_{i=0}^{l} \|Q_X(v_{k_i})\|_X \le 2.$$

The above cases are exhaustive, and so the proof is complete. ∎

PROPOSITION 18. *Let (v_k) be a bounded block sequence of (e_n^X). If $(Q_X(v_k))$ is weakly null, then (v_k) is also weakly null.*

For the proof of Proposition 18 we will need the following "unconditional" version of Mazur's theorem.

LEMMA 19. *Let (v_k) be a weakly null sequence in a Banach space V. Then for every $\varepsilon > 0$ there exist $k_0 < \dots < k_j$ in $\mathbb{N}$ and $\lambda_0, \dots, \lambda_j$ in $\mathbb{R}_+$ with $\sum_{i=0}^{j} \lambda_i = 1$ and such that*

$$\max\{\lambda_i : 0 \le i \le j\} \le \varepsilon$$

and

$$\max\Big\{\Big\|\sum_{i\in F}\lambda_i v_{k_i}\Big\| : F\subseteq\{0,\dots,j\}\Big\}\le\varepsilon.$$

Proof. Clearly we may assume that $V=\overline{\mathrm{span}}\{v_k : k\in\mathbb{N}\}$, and so we may also assume that V is a subspace of $C(2^{\mathbb{N}})$. Therefore, each v_k is a continuous function on $2^{\mathbb{N}}$ and the norm of V is the usual $\|\cdot\|_\infty$ norm. By Lebesgue's dominated convergence theorem, a sequence (f_k) in $C(2^{\mathbb{N}})$ is weakly null if and only if (f_k) is bounded and pointwise convergent to 0. Hence, setting $y_k=|v_k|$ for every $k\in\mathbb{N}$, we see that the sequence (y_k) is weakly null. Therefore, using Mazur's theorem, we find $k_0<\dots<k_j$ in $\mathbb{N}$ and $\lambda_0,\dots,\lambda_j$ in $\mathbb{R}_+$ with $\sum_{i=0}^j\lambda_i=1$ and such that

$$\max\{\lambda_i : 0\le i\le j\}\le\varepsilon$$

and $\|\sum_{i=0}^j\lambda_i y_{k_i}\|_\infty\le\varepsilon$. Noticing that

$$\max\Big\{\Big\|\sum_{i\in F}\lambda_i v_{k_i}\Big\|_\infty : F\subseteq\{0,\dots,j\}\Big\}\le\Big\|\sum_{i=0}^j\lambda_i y_{k_i}\Big\|_\infty\le\varepsilon$$

completes the proof. ∎

We proceed to the proof of Proposition 18.

Proof of Proposition 18. We argue by contradiction. So, assume that $(Q_X(v_k))$ is weakly null while (v_k) is not. We select $C\ge 1$ such that $\|v_k\|_{E_X}\le C$ for every $k\in\mathbb{N}$. By passing to a subsequence of (v_k) if necessary, we find $e^*\in E_X^*$ and $\delta>0$ such that $e^*(v_k)\ge\delta$ for every $k\in\mathbb{N}$. This implies that

(a) $\|z\|_{E_X}\ge\delta$ for every z in $\mathrm{conv}\{v_k : k\in\mathbb{N}\}$.

We apply Lemma 19 to the weakly null sequence $(Q_X(v_k))$ and $\varepsilon=\delta\cdot(4C)^{-1}$ to find $k_0<\dots<k_j$ in $\mathbb{N}$ and $\lambda_0,\dots,\lambda_j$ in $\mathbb{R}_+$ with $\sum_{i=0}^j\lambda_i=1$ and such that

$$\max\{\lambda_i : 0\le i\le j\}\le\delta/4C \tag{7}$$

and

$$\max\Big\{\Big\|\sum_{i\in F}\lambda_i Q_X(v_{k_i})\Big\|_X : F\subseteq\{0,\dots,j\}\Big\}\le\delta/4C. \tag{8}$$

Since $\|v_k\|_{E_X}\le C$ for every $k\in\mathbb{N}$, inequality (7) implies that

(b) $\|\lambda_i v_{k_i}\|_{E_X}\le\delta/4$ for every $i\in\{0,\dots,j\}$.

We define

$$w=\sum_{i=0}^j\lambda_i v_{k_i}\in\mathrm{conv}\{v_k : k\in\mathbb{N}\}.$$

We will show that $\|w\|_{E_X}\le\delta/2$, which contradicts property (a) above.

To this end we argue as in the proof of Proposition 17. First we set

(c) $G_i = \mathrm{supp}(v_{k_i})$ and $m_i = \min G_i$ for every $i \in \{0, \ldots, j\}$

and we let (a_n) be the unique sequence of reals such that

(d) $a_n = 0$ if $n \notin G_0 \cup \cdots \cup G_j$,
(e) $\lambda_i v_{k_i} = \sum_{n \in G_i} a_n e_n^X$ for every $i \in \{0, \ldots, j\}$.

Using (b), we see that if $l \in \{0, \ldots, j\}$ and $m \in \mathbb{N}$ with $m \in \mathrm{range}(v_{k_l})$, then

$$\Big\| \sum_{n=m_l}^{m} a_n x_n \Big\|_X \le \|\lambda_l v_{k_l}\|_{E_X} \le \delta/4. \tag{9}$$

We select $p \in \mathbb{N}$ such that

$$\|w\|_{E_X} = \Big\| \sum_{n=0}^{p} a_n x_n \Big\|_X$$

and, as in the proof of Proposition 17, we consider the following three cases.

Case 1: $p \in \mathrm{range}(v_{k_0})$. Using (9), we see that

$$\|w\|_{E_X} = \Big\| \sum_{n=m_0}^{p} a_n x_n \Big\|_X \le \delta/4.$$

Case 2: $p \in \mathrm{range}(v_{k_l})$ *for some* $l \in \{1, \ldots, j\}$. In this case the desired estimate will be obtained by combining (8) and (9). Specifically, let $F = \{0, \ldots, l-1\}$ and notice that

$$\begin{aligned}
\|w\|_{E_X} &= \Big\| \sum_{i=0}^{l-1} \sum_{n \in G_i} a_n x_n + \sum_{n=m_l}^{p} a_n x_n \Big\|_X \\
&\le \Big\| \sum_{i \in F} \sum_{n \in G_i} a_n x_n \Big\|_X + \Big\| \sum_{n=m_l}^{p} a_n x_n \Big\|_X \\
&= \Big\| \sum_{i \in F} \lambda_i Q_X(v_{k_i}) \Big\|_X + \Big\| \sum_{n=m_l}^{p} a_n x_n \Big\|_X \\
&\overset{(8)}{\le} \frac{\delta}{4C} + \Big\| \sum_{n=m_l}^{p} a_n x_n \Big\|_X \\
&\overset{(9)}{\le} \frac{\delta}{4C} + \frac{\delta}{4} \le \frac{\delta}{2}.
\end{aligned}$$

Case 3: $p \notin \mathrm{range}(v_{k_i})$ *for some* $l \in \{0, \ldots, j\}$. In this case we will only use (8). Indeed, there exists $l \in \{0, \ldots, j\}$ such that $\mathrm{range}(v_{k_i}) < \{p\}$ if $i \in \{0, \ldots, l\}$, while $\{p\} < \mathrm{range}(v_{k_i})$ otherwise. Setting $H = \{0, \ldots, l\}$, we see that

$$\|w\|_{E_X} = \Big\| \sum_{i=0}^{l} \sum_{n\in G_i} a_n x_n \Big\|_X = \Big\| \sum_{i\in H} \lambda_i Q_X(v_{k_i}) \Big\|_X \overset{(8)}{\leq} \frac{\delta}{4C} \leq \frac{\delta}{4}.$$

The above cases are exhaustive, and so $\|w\|_{E_X} \leq \delta/2$, completing the proof. ∎

3.3. Proof of Theorem 3. Let X be a separable Banach space, and (x_n) the sequence used to define E_X.

(i) It is straightforward.

(ii) We have already noticed that $\|Q_X\| = 1$. To see that Q_X is onto, observe that the image of the closed unit ball of E_X under Q_X contains $\{x_n : n \in \mathbb{N}\}$ and therefore it is dense in the closed unit ball of X.

(iii) Let Y be an infinite-dimensional subspace of E_X and assume that the operator $Q_X : Y \to X$ is strictly singular. Using a standard sliding hump argument we find a block subspace V of E_X and a subspace Y' of Y with V isomorphic to Y' and such that $Q_X : V \to X$ is strictly singular. Hence, we may select a normalized block sequence (v_k) of (e_n^X) with $v_k \in V$ and $\|Q_X(v_k)\|_X \leq 2^{-k}$ for every $k \in \mathbb{N}$. By Proposition 17, the sequence (v_k) is equivalent to the standard unit vector basis of c_0, and the result follows.

(iv) This part was essentially observed in [29]. We reproduce the argument for completeness. So, let (w_k) be a normalized basic sequence in X. The sequence (x_n) is dense in the unit sphere of X. Therefore it is possible to select an infinite subset $N = \{n_0 < n_1 < \cdots\}$ of $\mathbb{N}$ such that the subsequence (x_{n_k}) is basic and equivalent to (w_k) (see [21]). Let $K \geq 1$ be the basis constant of (x_{n_k}). Let $j \in \mathbb{N}$ and $a_0, \ldots, a_j \in \mathbb{R}$, and notice that

$$\begin{aligned}\Big\| \sum_{k=0}^{j} a_k x_{n_k} \Big\|_X &\leq \Big\| \sum_{k=0}^{j} a_k e_{n_k}^X \Big\|_{E_X} \\ &= \max_{0\leq i\leq j} \Big\| \sum_{k=0}^{i} a_k x_{n_k} \Big\|_X \leq K \Big\| \sum_{k=0}^{j} a_k x_{n_k} \Big\|_X.\end{aligned}$$

Therefore, (x_{n_k}) is K-equivalent to $(e_{n_k}^X)$, and the result follows.

(v) First we consider the relation $\mathcal{S} \subseteq C(2^{\mathbb{N}})^{\mathbb{N}} \times \mathrm{SB}$ defined by

$$\big((y_n), Y\big) \in \mathcal{S} \;\Leftrightarrow\; (\forall n\ y_n \in Y) \text{ and } \overline{\mathrm{span}}\{y_n : n \in \mathbb{N}\} = Y.$$

The relation $\mathcal{S}$ is analytic (see [7, Lemma 2.6]). We apply Proposition 8 to get a sequence $S_n : \mathrm{SB} \to C(2^{\mathbb{N}})$ $(n \in \mathbb{N})$ of Borel maps such that for every $X \in \mathrm{SB}$ with $X \neq \{0\}$ the sequence $(S_n(X))$ is norm dense in S_X. Now notice that

$$(X,Y)\in\mathcal{E} \Leftrightarrow \exists (y_n)\in C(2^{\mathbb{N}})^{\mathbb{N}} \text{ with } \big((y_n),Y\big)\in\mathcal{S} \text{ and either}$$
$$\Big(X=\{0\} \text{ and } \forall k\in\mathbb{N}\ \forall a_0,\dots,a_k\in\mathbb{Q} \text{ we have}$$
$$\Big\|\sum_{n=0}^{k} a_n y_n\Big\|_\infty = \max_{0\le n\le k}|a_n|\Big), \text{ or}$$
$$\Big(X\neq\{0\} \text{ and } \forall k\in\mathbb{N}\ \forall a_0,\dots,a_k\in\mathbb{Q} \text{ we have}$$
$$\Big\|\sum_{n=0}^{k} a_n y_n\Big\|_\infty = \max_{0\le m\le k}\Big\|\sum_{n=0}^{m} a_n S_n(X)\Big\|_\infty\Big).$$

The above formula implies that the set $\mathcal{E}$ is analytic.

(vi) By (ii), X is a quotient of E_X. Therefore, if E_X^* is separable, then X^* is also separable. For the converse, we argue by contradiction. So, assume that there exists a Banach space X with separable dual such that E_X^* is non-separable. Our strategy is to show that there exists a sequence $(w_t)_{t\in 2^{<\mathbb{N}}}$ in E_X which is topologically equivalent to the basis of James tree (see Definition 5) and the sequence $(Q_X(w_t))_{t\in 2^{<\mathbb{N}}}$ has the same property. As already indicated in §2.3, this implies that X^* is non-separable and yields a contradiction.

We argue as follows. First, X does not contain a copy of ℓ_1. Therefore, by (iii), E_X does not contain a copy of ℓ_1 either. Hence, Theorem 7 applied to E_X yields a sequence $(e_t)_{t\in 2^{<\mathbb{N}}}$ in E_X which is topologically equivalent to the basis of James tree. We need to replace $(e_t)_{t\in 2^{<\mathbb{N}}}$ with another sequence having an additional property. Specifically, let us say that a sequence $(v_t)_{t\in 2^{<\mathbb{N}}}$ in E_X is a *tree-block* if $(v_{\sigma|n})$ is a block sequence of (e_n^X) for every $\sigma\in 2^{\mathbb{N}}$. Notice that this notion is hereditary with respect to dyadic subtrees: if $(v_t)_{t\in 2^{<\mathbb{N}}}$ is a tree-block and $D=\{d_t : t\in 2^{<\mathbb{N}}\}$ is a dyadic subtree of $2^{<\mathbb{N}}$, then $(v_{d_t})_{t\in 2^{<\mathbb{N}}}$ is also a tree-block.

CLAIM 20. *There exists a sequence $(v_t)_{t\in 2^{<\mathbb{N}}}$ in E_X which is topologically equivalent to the basis of James tree and a tree-block.*

Proof of Claim 20. We select $C\ge 1$ such that $C^{-1}\le \|e_t\|_{E_X}\le C$ for every $t\in 2^{<\mathbb{N}}$. Let $s\in 2^{<\mathbb{N}}$. There exists an infinite antichain A of $2^{<\mathbb{N}}$ such that $s\sqsubset t$ for every $t\in A$. The sequence $(e_t)_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree, and so $(e_t)_{t\in A}$ is weakly null. Using this observation, we can recursively construct a dyadic subtree $R=\{r_t : t\in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ and a tree-block sequence $(v_t)_{t\in 2^{<\mathbb{N}}}$ in E_X with $\|e_{r_t}-v_t\|_{E_X}\le (2C)^{-|t|+1}$ for every $t\in 2^{<\mathbb{N}}$. Clearly, $(v_t)_{t\in 2^{<\mathbb{N}}}$ is as desired. ■

CLAIM 21. *There exist a dyadic subtree S_0 of $2^{<\mathbb{N}}$ and a constant $\Theta\ge 1$ such that $\Theta^{-1}\le\|Q_X(v_t)\|_X\le\Theta$ for every $t\in S_0$.*

Proof of Claim 21. Let $K \geq 1$ be such that $\|v_t\|_{E_X} \leq K$ for every $t \in 2^{<\mathbb{N}}$. We will show that there exist $s_0 \in 2^{<\mathbb{N}}$ and $\theta > 0$ such that for every $t \in 2^{<\mathbb{N}}$ with $s_0 \sqsubseteq t$ we have $\|Q_X(v_t)\|_X \geq \theta$. Then we set $S_0 = \{s_0^\frown t : t \in 2^{<\mathbb{N}}\}$ and $\Theta = \max\{\theta^{-1}, K\}$ and notice that S_0 and Θ satisfy the requirements of the claim.

We argue by contradiction. Assume that for every $s \in 2^{<\mathbb{N}}$ and $\theta > 0$ there exists $t \in 2^{<\mathbb{N}}$ with $s \sqsubseteq t$ and such that $\|Q_X(v_t)\|_X \leq \theta$. Hence, we may select a sequence (t_k) in $2^{<\mathbb{N}}$ such that for every $k \in \mathbb{N}$ we have

(a) $t_k \sqsubset t_{k+1}$,
(b) $\|Q_X(v_{t_k})\|_X \leq 2^{-k}$.

By (a), the set $\{t_k : k \in \mathbb{N}\}$ is a chain, while, by Claim 20, the sequence $(v_t)_{t\in 2^{<\mathbb{N}}}$ is semi-normalized and a tree-block. Therefore, (v_{t_k}) is a semi-normalized block sequence of (e_n^X). By Proposition 17 and (b), (v_{t_k}) is equivalent to the standard unit vector basis of c_0, and so it is weakly null. By Claim 20, however, $(v_t)_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree. Since $\{t_k : k \in \mathbb{N}\}$ is a chain, (v_{t_k}) must be a non-trivial weak* Cauchy sequence. This yields a contradiction. ∎

Claim 22. *There exists a dyadic subtree S_1 of $2^{<\mathbb{N}}$ with $S_1 \subseteq S_0$ and such that for every infinite chain $\{t_0 \sqsubset t_1 \sqsubset \cdots\}$ of S_1 the sequence $(Q_X(v_{t_n}))$ is basic.*

Proof of Claim 22. By Claims 20 and 21, for every $s \in S_0$ there exists an infinite antichain A of S_0 with $s \sqsubset t$ for every $t \in A$ and such that the sequence $(Q_X(v_t))_{t\in A}$ is semi-normalized and weakly null. Now use the classical procedure of Mazur for selecting basic sequences (see [21]). ∎

Claim 23. *There exists a dyadic subtree S_2 of $2^{<\mathbb{N}}$ with $S_2 \subseteq S_1$ and such that for every infinite chain $\{t_0 \sqsubset t_1 \sqsubset \cdots\}$ of S_2 the sequence $(Q_X(v_{t_n}))$ is weak* Cauchy.*

Proof of Claim 23. Let

$$\mathcal{X} = \big\{c \in [S_1]_{\text{chains}} : \text{the sequence } (Q_X(v_t))_{t\in c} \text{ is weak* Cauchy}\big\}.$$

The set $\mathcal{X}$ is co-analytic (see [30] for more details). Therefore, by Theorem 4, there exists a dyadic subtree S_2 of $2^{<\mathbb{N}}$ with $S_2 \subseteq S_1$ and such that $[S_2]_{\text{chains}}$ is monochromatic. It is enough to show that $[S_2]_{\text{chains}} \cap \mathcal{X} \neq \emptyset$. Recall that X does not contain a copy of ℓ_1. Therefore, by Rosenthal's dichotomy [28], we may find an infinite chain c of S_2 such that $(Q_X(v_t))_{t\in c}$ is weak* Cauchy, and the result follows. ∎

Let S_2 be the dyadic subtree of $2^{<\mathbb{N}}$ obtained in Claim 23 and let $\{s_t : t \in 2^{<\mathbb{N}}\}$ be the canonical representation of S_2. We can now define the sequence $(w_t)_{t\in 2^{<\mathbb{N}}}$ mentioned at the beginning of the proof of (vi): we set

$$w_t = v_{s_t}$$

for every $t \in 2^{<\mathbb{N}}$. By Claim 20 and Fact 6, the sequence $(w_t)_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree and a tree-block. The final claim is the following.

CLAIM 24. *The sequence $(Q_X(w_t))_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree.*

Proof of Claim 24. By Claim 21, $(Q_X(w_t))_{t\in 2^{<\mathbb{N}}}$ is semi-normalized. Notice also that $(Q_X(w_t))_{t\in A}$ is weakly null for every infinite antichain A of $2^{<\mathbb{N}}$.

Let $\sigma \in 2^{\mathbb{N}}$. By Claim 23, $(Q_X(w_{\sigma|n}))$ is weak* convergent to an $x^{**}_\sigma \in X^{**}$. First notice that $x^{**}_\sigma \neq 0$. Indeed, $(w_{\sigma|n})$ is a semi-normalized block sequence of (e^X_n) which is weak* convergent to a $w^{**}_\sigma \in E^{**}_X \setminus E_X$. If $x^{**}_\sigma = 0$, then Proposition 18 would imply that $(w_{\sigma|n})$ is weakly null. Hence $x^{**}_\sigma \neq 0$. Next we observe that $x^{**}_\sigma \in X^{**} \setminus X$. Indeed, by Claim 22, the sequence $(Q_X(w_{\sigma|n}))$ is basic. Therefore, if $(Q_X(w_{\sigma|n}))$ were weakly convergent to an $x \in X$, then necessarily $x = 0$, which, however, is ruled out by the previous reasoning, and so $x^{**}_\sigma \in X^{**} \setminus X$.

Finally, suppose, towards a contradiction, that there exist $\sigma, \tau \in 2^{\mathbb{N}}$ with $\sigma \neq \tau$ and such that $x^{**}_\sigma = x^{**}_\tau$. Then one can select two sequences (s_n) and (t_n) in $2^{<\mathbb{N}}$ such that:

(a) $s_n \sqsubset s_{n+1} \sqsubset \sigma$ for every $n \in \mathbb{N}$.
(b) $t_n \sqsubset t_{n+1} \sqsubset \tau$ for every $n \in \mathbb{N}$.
(c) If we set $z_n = w_{s_n} - w_{t_n}$ for every $n \in \mathbb{N}$, then (z_n) is a semi-normalized block sequence of (e^X_n).

Our assumption that $x^{**}_\sigma = x^{**}_\tau$ reduces to the fact that the sequence $(Q_X(z_n))$ is weakly null. By (c), we may apply Proposition 18 to infer that (z_n) is also weakly null. Therefore, the sequences $(w_{\sigma|n})$ and $(w_{\tau|n})$ are weak* convergent to the same element of E^{**}_X. This contradicts the fact that $(w_t)_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree. The proof is complete. ∎

As already indicated, Claim 24 yields a contradiction. This completes the proof of part (vi) of Theorem 3, and so the entire proof is complete.

3.4. Consequences. We now isolate three corollaries of Theorem 3. The second one will be of particular importance in the next section.

COROLLARY 25. *Let Z be a minimal Banach space not containing a copy of c_0. If X is a separable Banach space not containing a copy of Z, then E_X does not contain a copy of Z either.*

Proof. Follows immediately from Theorem 3(iii). ∎

Corollary 26. *Let Z be a minimal Banach space not containing a copy of c_0 and let $\mathcal{A}$ be an analytic subset of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$. Then there exists a subset $\mathcal{B}$ of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$ with the following properties:*

(i) *The set $\mathcal{B}$ is analytic.*
(ii) *Every $Y \in \mathcal{B}$ has a Schauder basis.*
(iii) *For every $X \in \mathcal{A}$ there exists $Y \in \mathcal{B}$ such that X is a quotient of Y.*

Proof. Let $\mathcal{E}$ be the set defined in Theorem 3(v). We define $\mathcal{B} \subseteq \mathrm{SB}$ by the rule

$$Y \in \mathcal{B} \Leftrightarrow \exists X\ [X \in \mathcal{A} \text{ and } (X,Y) \in \mathcal{E}].$$

The set $\mathcal{B}$ is clearly analytic. Invoking parts (i) and (ii) of Theorem 3 and Corollary 25, we see that $\mathcal{B}$ is as desired. ∎

Corollary 27. *There exists a map $f : \omega_1 \to \omega_1$ such that for every countable ordinal ξ and every separable Banach space X with $\phi_{\mathrm{NC}_{\ell_1}}(X) \le \xi$ the space X is a quotient of a Banach space Y with a Schauder basis satisfying $\phi_{\mathrm{NC}_{\ell_1}}(Y) \le f(\xi)$.*

Proof. Fix a countable ordinal ξ and set

$$\mathcal{A}_\xi = \{X \in \mathrm{SB} : \phi_{\mathrm{NC}_{\ell_1}}(X) \le \xi\}.$$

By Theorem 9, $\phi_{\mathrm{NC}_{\ell_1}} : \mathrm{NC}_{\ell_1} \to \omega_1$ is a co-analytic rank on NC_{ℓ_1}. Hence the set $\mathcal{A}_\xi$ is analytic (in fact Borel, see [20]). We apply Corollary 26 to the space $Z = \ell_1$ and the analytic set $\mathcal{A}_\xi$ to get an analytic subset $\mathcal{B}$ of NC_{ℓ_1} such that for every $X \in \mathcal{A}_\xi$ there exists $Y \in \mathcal{B}$ with a Schauder basis and having X as quotient. By boundedness, there exists a countable ordinal ζ such that

$$\sup\{\phi_{\mathrm{NC}_{\ell_1}}(Y) : Y \in \mathcal{B}\} = \zeta.$$

We define $f(\xi) = \zeta$. Clearly the map $f : \omega_1 \to \omega_1$ is as desired. ∎

3.5. Comments. By a well-known result due to W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński [9], if X is a Banach space with separable dual, then X is a quotient of a Banach space V_X with a *shrinking* Schauder basis. By Theorem 3, the space E_X has a Schauder basis, separable dual and admits X as quotient. We point out, however, that the natural Schauder basis (e_n^X) of E_X is *not* shrinking. On the other hand, the subspace structure of E_X is very well understood. The space V_X mentioned above is defined using the interpolation techniques developed in [9] and it is not clear which are the isomorphic types of its subspaces.

We would also like to comment on the proof of the separability of the dual of E_X. Our strategy was to construct a sequence $(w_t)_{t\in 2^{<\mathbb{N}}}$ in E_X which is topologically equivalent to the basis of James tree and is such that its image under the operator Q_X has the same property; in other words, Q_X fixes a copy of this basic object. This kind of reasoning can be applied to a more general framework. Specifically, let Y and Z be separable Banach spaces and $T : Y \to Z$ be a bounded linear operator. There are a number of problems in functional analysis which boil down to understanding when the dual operator T^* has non-separable range. Using the combinatorial tools developed in [3] and an analysis similar to the one in the present paper, it can be shown that if Y does not contain a copy of ℓ_1, then T^* has non-separable range if and only if T fixes a copy of a sequence which is topologically equivalent to the basis of James tree.

4. Proof of the main result. In this section we prove Theorem 2 stated in the introduction. The proof will be based on the following, more detailed, result.

THEOREM 28. *Let Z be a minimal Banach space not containing a copy of c_0 and $\mathcal{A}$ be an analytic subset of* $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$. *Then there exists a Banach space $V \in \mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$ with a Schauder basis which is surjectively universal for the class $\mathcal{A}$. Moreover, if $X \in \mathcal{A}$ has the bounded approximation property, then X is isomorphic to a complemented subspace of V.*

Let us point out that the assumption on the complexity of $\mathcal{A}$ in Theorem 28 is optimal. Notice also that if E is any Banach space with a Schauder basis, then the set of all $X \in \mathcal{A}$ which are isomorphic to a complemented subspace of E is contained in the set of all $X \in \mathcal{A}$ having the bounded approximation property. Therefore, the "moreover" part of the above result is optimal too.

Proof of Theorem 28. Since Z is minimal, there exists $1 < p < \infty$ such that Z does not contain a copy of ℓ_p. We fix such a p. We apply Lemma 13 to the space Z and the analytic set $\mathcal{A}$ to get a subset $\mathcal{D}$ of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$ such that:

(a) The set $\mathcal{D}$ is analytic.
(b) Every $Y \in \mathcal{D}$ has a Schauder basis.
(c) For every $X \in \mathcal{A}$ with the bounded approximation property there exists $Y \in \mathcal{D}$ such that X is isomorphic to a complemented subspace of Y.

Next we apply Corollary 26 to the space Z and the analytic set $\mathcal{A}$ to get a subset $\mathcal{B}$ of $\mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$ with the following properties:

(d) The set $\mathcal{B}$ is analytic.
(e) Every $Y \in \mathcal{B}$ has a Schauder basis.
(f) For every $X \in \mathcal{A}$ there exists $Y \in \mathcal{B}$ such that X is a quotient of Y.

We set $\mathcal{C} = \mathcal{B} \cup \mathcal{D}$ and we notice that $\mathcal{C} \subseteq \mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$. By (a) and (d), the set $\mathcal{C}$ is analytic, while, by (b) and (e), every $Y \in \mathcal{C}$ has a Schauder basis.

The desired space V is the p-amalgamation space of the class $\mathcal{C}$ obtained in Theorem 14. It remains to check that V has the desired properties. Notice, first, that V has a Schauder basis.

CLAIM 29. *The space V is surjectively universal for the class $\mathcal{A}$.*

Proof of Claim 29. Let $X \in \mathcal{A}$. By (f), there exists $Y \in \mathcal{B}$ such that X is a quotient of Y. We fix a quotient map $Q : Y \to X$. Next we observe that V contains a complemented copy of Y. Therefore, we can find a subspace E of V, a projection $P : V \to E$ and an isomorphism $T : E \to Y$. Let $Q' : V \to X$ be defined by $Q' = Q \circ T \circ P$ and notice that Q' is onto. Hence, X is a quotient of V, and the result follows. ∎

CLAIM 30. *We have $V \in \mathrm{NC}_Z \cap \mathrm{NC}_{\ell_1}$.*

Proof of Claim 30. We will show that V does not contain a copy of Z (the proof that V does not contain a copy of ℓ_1 is identical). We argue by contradiction. Assume that there exists a subspace W of V which is isomorphic to Z. By the choice of p, we see that W does not contain a copy of ℓ_p. Therefore, by Theorem 14, there exist $Y_0, \dots, Y_n$ in the class $\mathcal{C}$ such that W is isomorphic to a subspace of $Y_0 \oplus \dots \oplus Y_n$. There exist an infinite-dimensional subspace W' of W and $i_0 \in \{0, \dots, n\}$ such that W' is isomorphic to a subspace of Y_{i_0}. Since Z is minimal, Y_{i_0} must contain a copy of Z. This contradicts $\mathcal{C} \subseteq \mathrm{NC}_Z$, and so the claim is proved. ∎

Finally, we notice that if $X \in \mathcal{A}$ has the bounded approximation property, then, by (c) above and Theorem 14, X is isomorphic to a complemented subspace of V. This shows that the space V has the desired properties. The proof of Theorem 28 is complete. ∎

We proceed to the proof of Theorem 2.

Proof of Theorem 2. Let $\mathcal{C} \subseteq \mathrm{SB}$.

(i)⇒(ii). Assume that there exists a separable Banach space Y not containing a copy of ℓ_1 which is surjectively universal for the class $\mathcal{C}$. The space Y does not contain a copy of ℓ_1, and so $\phi_{\mathrm{NC}_{\ell_1}}(Y) < \omega_1$. Moreover, every space in $\mathcal{C}$ is a quotient of Y. Therefore, by Lemma 10,

$$\sup\{\phi_{\mathrm{NC}_{\ell_1}}(X) : X \in \mathcal{C}\} \leq \phi_{\mathrm{NC}_{\ell_1}}(Y) < \omega_1.$$

(ii)⇒(iii). Let ξ be a countable ordinal such that $\sup\{\phi_{\mathrm{NC}_{\ell_1}}(X) : X \in \mathcal{C}\} = \xi$. By Theorem 9, the map $\phi_{\mathrm{NC}_{\ell_1}} : \mathrm{NC}_{\ell_1} \to \omega_1$ is a co-analytic rank on NC_{ℓ_1}. It follows that the set

$$\mathcal{A} = \{V \in \mathrm{SB} : \phi_{\mathrm{NC}_{\ell_1}}(V) \leq \xi\}$$

is a Borel subset of NC_{ℓ_1} (see [20]) and clearly $\mathcal{C} \subseteq \mathcal{A}$.

(iii)⇒(i). Assume that there exists an analytic subset $\mathcal{A}$ of NC_{ℓ_1} with $\mathcal{C} \subseteq \mathcal{A}$. We apply Theorem 28 for $Z = \ell_1$ and the class $\mathcal{A}$ to get a Banach space V with a Schauder basis which does not contain a copy of ℓ_1 and is surjectively universal for $\mathcal{A}$. A fortiori, V is surjectively universal for $\mathcal{C}$, and the result follows. ∎

5. A related result and open problems. Let us recall the following notion (see [2, Definition 90]).

DEFINITION 31. A class $\mathcal{C} \subseteq \mathrm{SB}$ is said to be *strongly bounded* if for every analytic subset $\mathcal{A}$ of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ which is universal for the class $\mathcal{A}$.

This is a quite strong structural property. It turns out, however, that many natural classes of separable Banach spaces are strongly bounded.

Part of the research in this paper grew out from our attempt to find natural instances of the "dual" phenomenon, abstracted in the following definition.

DEFINITION 32. A class $\mathcal{C} \subseteq \mathrm{SB}$ is said to be *surjectively strongly bounded* if for every analytic subset $\mathcal{A}$ of $\mathcal{C}$ there exists $Y \in \mathcal{C}$ which is surjectively universal for the class $\mathcal{A}$.

Thus, Theorem 28 has the following consequence.

COROLLARY 33. *Let Z be a minimal Banach space not containing a copy of c_0. Then the class $\mathrm{NC}_{\ell_1} \cap \mathrm{NC}_Z$ is surjectively strongly bounded.*

The following proposition provides two more natural examples.

PROPOSITION 34. *The class* REFL *of separable reflexive Banach spaces and the class* SD *of Banach spaces with separable dual are surjectively strongly bounded.*

Proposition 34 follows by combining a number of results already existing in the literature, and so instead of giving a formal proof we only give the guidelines. To see that REFL is surjectively strongly bounded, let $\mathcal{A}$ be an analytic subset of REFL and consider the *dual class* $\mathcal{A}^*$ defined by

$$Y \in \mathcal{A}^* \;\Leftrightarrow\; \exists X \in \mathcal{A} \text{ with } Y \text{ isomorphic to } X^*.$$

The set $\mathcal{A}^*$ is analytic (see [10]) and $\mathcal{A}^* \subseteq \mathrm{REFL}$. Since REFL is strongly bounded (see [12]), there exists a separable reflexive Banach space Z which is universal for the class $\mathcal{A}^*$. Therefore, every X in $\mathcal{A}$ is a quotient of Z^*. The referee suggested that, alternatively, one can use the universality results obtained in [24].

The argument for the class SD is somewhat different and uses the parameterized version of the Davis–Figiel–Johnson–Pełczyński construction due to Bossard, as well as an idea already employed in the proof of Theorem 28. Let $\mathcal{A}$ be an analytic subset of SD. By the results in [9] and [6], there exists an analytic subset $\mathcal{B}$ of Banach spaces with a shrinking Schauder basis such that for every $X \in \mathcal{A}$ there exists $Y \in \mathcal{B}$ having X as a quotient. It is then possible to apply the machinery developed in [2] to obtain a Banach space E with a shrinking Schauder basis that contains a complemented copy of every space in $\mathcal{B}$. By the choice of $\mathcal{B}$, the space E is surjectively universal for the class $\mathcal{A}$.

Although, by Theorem 2, the class NC_{ℓ_1} is surjectively strongly bounded, we should point out that it is not known whether it is strongly bounded. We close this section by mentioning the following related problems.

Problem 1. *Is it true that every separable Banach space X not containing a copy of ℓ_1 embeds into a space Y with a Schauder basis and not containing a copy of ℓ_1?*

Problem 2. *Does there exist a map $g : \omega_1 \to \omega_1$ such that for every countable ordinal ξ and every separable Banach space X with $\phi_{\mathrm{NC}_{\ell_1}}(X) \leq \xi$ the space X embeds into a Banach space Y with a Schauder basis satisfying $\phi_{\mathrm{NC}_{\ell_1}}(Y) \leq g(\xi)$?*

Problem 3. *Is the class NC_{ℓ_1} strongly bounded?*

We notice that an affirmative answer to Problem 2 can be used to provide an affirmative answer to Problem 3 (to see this combine Theorem 9 and Theorem 14 stated in §2).

It seems reasonable to conjecture that the above problems have an affirmative answer. Our optimism is based on the following facts. Firstly, the answer to Problem 3 is known to be "yes" within the category of Banach spaces with a Schauder basis (see [2]). Secondly, it is known that for every minimal Banach space Z not containing a copy ℓ_1 the class NC_Z is strongly bounded (see [11]).

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