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Compactness properties of weighted summation operators on trees

by

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Abstract. We investigate compactness properties of weighted summation operators $V_{\alpha,\sigma}$ as mappings from $\ell_1(T)$ into $\ell_q(T)$ for some $q \in (1,\infty)$. Those operators are defined by

$$(V_{\alpha,\sigma}x)(t) := \alpha(t) \sum_{s \succeq t} \sigma(s)x(s), \quad t \in T,$$

where T is a tree with partial order $\preceq$. Here α and σ are given weights on T. We introduce a metric d on T such that compactness properties of (T,d) imply two-sided estimates for $e_n(V_{\alpha,\sigma})$, the (dyadic) entropy numbers of $V_{\alpha,\sigma}$. The results are applied to concrete trees, e.g. moderately increasing, biased or binary trees and to weights with $\alpha(t)\sigma(t)$ decreasing either polynomially or exponentially. We also give some probabilistic applications to Gaussian summation schemes on trees.

1. Introduction. This work essentially stems from the article [L] where the entropy of linear Volterra integral operators was studied in a difficult critical case. Handling this case required a new technique and it turned out that this technique could be cleanly elaborated and better explained if we replace the Volterra operator by an analogous summation operator on a binary tree. Notice that trees appear naturally in the study of functional spaces because the Haar base and other similar wavelet bases indeed have a structure close to that of a binary tree.

The class of summation operators on trees is quite simple and natural but it has not been investigated at all, and we believe that a deeper study of its properties, as presented here, is not only interesting in its own right but might also be helpful as a model for studying more conventional classes of operators.

Thus let T be a finite or infinite tree and let $\preceq$ be the partial order generated by its structure, i.e. $t \preceq s$ means that t is situated on the path

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leading from the root of the tree to s. If $k : T \times T \to \mathbb{R}$ is a kernel satisfying

$$\sup_{s\in T} \sum_{t\preceq s} |k(t,s)|^q < \infty \tag{1.1}$$

for some $q \in [1,\infty)$, then the Volterra-type summation operator V with

$$(Vx)(t) := \sum_{s\succeq t} k(t,s)x(s), \quad t \in T,$$

is bounded from $\ell_1(T)$ into $\ell_q(T)$. Compactness properties of V surely depend on the kernel k as well as on the structure of the underlying tree. It seems to be hopeless to describe such properties of V in this general context. A first step could be the investigation of this problem in the case of special kernels k (and for quite general trees). Thus we restrict ourselves to kernels k which may be written as $k(t,s) = \alpha(t)\sigma(s)$ for some given weights $\alpha, \sigma : T \to (0,\infty)$ where we assume that σ is non-increasing. Condition (1.1) then reads

$$\sup_{s\in T} \Big(\sum_{r\preceq s} \alpha(r)^q\Big)^{1/q} \sigma(s) < \infty \tag{1.2}$$

and $V = V_{\alpha,\sigma}$ acts as

$$(V_{\alpha,\sigma}x)(t) = \alpha(t) \sum_{s\succeq t} \sigma(s)x(s), \quad t \in T. \tag{1.3}$$

Note that adding signs to α and σ does not change compactness properties (or any other property), thus assuming positive weights we do not lose generality.

In the linear case $T = \mathbb{N}_0$, those weighted summation operators have been investigated in [CL]. The main observation in that paper was that such operators may be regarded as special weighted integration operators, and consequently their properties follow from those of integration operators as proved in [EEH1], [EEH2], [LL], [Ma], etc.

The situation is completely different for general trees. Here known results about Volterra integration operators are not applicable. Therefore summation operators in this general context have to be treated independently and new interesting phenomena appear because the structure of the underlying tree plays an important role.

The main objective of the present paper is to investigate compactness properties of the operators $V_{\alpha,\sigma}$ defined in (1.3). Our basic observation is as follows. Suppose we are given weights α and σ satisfying (1.2) with σ non-decreasing and let $q \in (1,\infty)$. If $t \preceq s$ are in T we define their distance by

$$d(t,s) := \max_{t\prec v\preceq s} \Big(\sum_{t\prec r\preceq v} \alpha(r)^q\Big)^{1/q} \sigma(v).$$

Then d may be extended to a metric d on T. Let $N(T,d,\varepsilon)$ be the covering numbers of (T,d), i.e.,

$$N(T,d,\varepsilon) := \inf\Big\{n \geq 1 : T = \bigcup_{j=1}^{n} B_\varepsilon(t_j)\Big\}$$

with (open) ε-balls $B_\varepsilon(t_j)$ for certain $t_j \in T$. We prove that the behavior of $N(T,d,\varepsilon)$ as $\varepsilon \to 0$ is closely connected with the degree of compactness of $V_{\alpha,\sigma}$. More precisely, let $e_n(V_{\alpha,\sigma})$ be the sequence of dyadic entropy numbers defined by

$$e_n(V_{\alpha,\sigma}) := \inf\Big\{\varepsilon > 0 : \{V_{\alpha,\sigma}x : \|x\|_1 \leq 1\} \subseteq \bigcup_{j=1}^{2^{n-1}} B_\varepsilon(y_j)\Big\}$$

where the $B_\varepsilon(y_j)$ are open ε-balls in $\ell_q(T)$ centered at certain y_j. We refer to [CS] for more information about entropy numbers. Our main objective is to prove that upper (resp. lower) bounds for $N(T,d,\varepsilon)$ yield upper (resp. lower) bounds for $e_n(V_{\alpha,\sigma})$.

For example, as shown in Theorem 4.4, given $a > 0$ and $b \geq 0$ it follows that

$$N(T,d,\varepsilon) \leq c\varepsilon^{-a}|\log\varepsilon|^b \quad \text{implies} \quad e_n(V_{\alpha,\sigma}) \leq c' n^{-1/a-1/p'}(\log n)^{b/a}$$

with $p := \min\{2,q\}$ and $1/p' = 1 - 1/p$. In Theorem 5.3 we prove a similar result assuming $N(T,d,\varepsilon) \geq c\varepsilon^{-a}|\log\varepsilon|^b$. In particular, if $1 < q \leq 2$, then

$$N(T,d,\varepsilon) \approx \varepsilon^{-a}|\log\varepsilon|^b \quad \text{implies} \quad e_n(V_{\alpha,\sigma}) \approx n^{-1/a-1/q'}(\log n)^{b/a}.$$

We also treat the case that $N(T,d,\varepsilon)$ increases exponentially. Apart from some critical case, sharp estimates are obtained as well.

Thus in order to get precise estimates for $e_n(V_{\alpha,\sigma})$ it suffices to describe the behavior of $N(T,d,\varepsilon)$ in dependence on properties of the weights α and σ and on the structure of the tree. This question is investigated in Sections 6 and 7. There we prove quite precise estimates for $N(T,d,\varepsilon)$ in the case of moderate trees (the number of nodes in the nth generation increases polynomially) or for binary trees provided we know something about the behavior of $\alpha(t)\sigma(t)$. In Section 8 we investigate a class of trees where the branches die out very quickly. Here the behavior of $N(T,d,\varepsilon)$ is completely different from the one observed for trees where each node has at least one offspring. This example demonstrates the influence of the tree structure on compactness properties of $V_{\alpha,\sigma}$.

In Section 9 we sketch a probabilistic interpretation of our results by providing the asymptotics of small deviation probabilities for some tree-indexed Gaussian random functions, and at the end in Section 10 we state some open problems related to the topic of the present paper.

Let us finally mention that throughout this paper we always denote by c or C (with or without subscript) universal constants which may vary even in one line. The constants may depend on q but neither on n nor on the behavior of the weights.

2. Trees. Let us recall some basic notation related to trees which will be used later on. Throughout, T denotes a finite or an infinite tree. We suppose that T has a unique root which we denote by $\mathbf{0}$, and each element $t \in T$ has a finite number $\xi(t)$ of offsprings. We do not exclude the case $\xi(t) = 0$, i.e., some elements may "die out". The tree structure leads in the natural way to a partial order $\preceq$ by letting $t \preceq s$ (or $s \succeq t$) provided there are $t = t_0, t_1, \ldots, t_m = s$ in T such that t_j is an offspring of t_{j-1} for $1 \le j \le m$. The strict inequalities have the same meaning with the additional assumption $t \neq s$. Elements $t, s \in T$ are said to be *comparable* provided that either $t \preceq s$ or $s \preceq t$; otherwise they are *incomparable*.

For $t, s \in T$ with $t \preceq s$ the *order interval* $[t, s]$ is defined by

$$[t, s] := \{r \in T : t \preceq r \preceq s\},$$

and similarly for $(t, s]$.

A subset $B \subseteq T$ is said to be a *branch* provided that all elements in B are comparable, and moreover, if $t \preceq r \preceq s$ with $t, s \in B$, then $r \in B$ as well. Of course, finite branches are of the form $[t, s]$ for suitable $t \preceq s$.

Given $s \in T$ its *order* $|s| \ge 0$ is defined by

$$|s| := \#\{t \in T : t \prec s\}.$$

Then

$$R(n) := \#\{t \in T : |t| = n\}, \quad n \ge 0,$$

is the number of elements in the nth generation of T.

3. Metrics and ε-nets on trees. Suppose we are given two weight functions $\alpha : T \to (0, \infty)$ and $\sigma : T \to (0, \infty)$ where we assume that σ is non-increasing, i.e., if $t \preceq s$, then $\sigma(t) \ge \sigma(s)$.

Given $q \in [1, \infty)$ and $t, v \in T$ with $t \preceq v$, we set

$$\|\alpha \mathbf{1}_{(t,v]}\|_q := \Big(\sum_{t \prec r \preceq v} \alpha(r)^q \Big)^{1/q}.$$

Using this, we define a mapping $d : T \times T \to [0, \infty)$ as follows: if $t \preceq s$, then

$$d(t, s) := \max_{t \prec v \preceq s} \{ \|\alpha \mathbf{1}_{(t,v]}\|_q \sigma(v) \}. \tag{3.1}$$

We let $d(t, s) := d(s, t)$ provided that $t \succeq s$, and

$$d(t, s) := d(t \wedge s, t) + d(t \wedge s, s)$$

whenever t and s are incomparable. Here $t \wedge s$ denotes the infimum of t and s which may be defined as the maximal element in $[\mathbf{0}, t] \cap [\mathbf{0}, s]$.

Remark. Since σ is assumed to be non-increasing it follows that for $t \preceq s$,

$$d(t,s) = \max_{t \prec v \preceq s} \{\|\alpha \mathbf{1}_{(t,v]}\|_q \cdot \|\sigma \mathbf{1}_{[v,s]}\|_\infty\}.$$

A similar expression (for weights and intervals on $\mathbb{R}$) played an important role in the investigation of weighted integration operators (cf. [EEH1], [EEH2], [LL] and [LLS]).

Proposition 3.1. *The mapping d constructed above is a metric on T with the following monotonicity property: whenever $t' \preceq t \preceq s \preceq s'$, then $d(t,s) \le d(t',s')$.*

Proof. The monotonicity is a direct consequence of the definition of d.

Clearly $d(t,s) \ge 0$ and since we assumed $\alpha(t) > 0$ for $t \in T$ we see that $d(t,s) = 0$ implies $t = s$. By the construction we also have $d(t,s) = d(s,t)$. Thus it remains to prove the triangle inequality

$$d(t,s) \le d(t,r) + d(r,s)$$

whenever $t, s, r \in T$. To do this, one has to treat six cases in dependence on the relation between t, s and r. Among them only one is non-trivial, namely when t, s and r are on a common branch and satisfy $t \preceq r \preceq s$ or, equivalently, $s \preceq r \preceq t$. Therefore we only consider that situation.

Assume $t \preceq r \preceq s$ and choose v in T with $t \prec v \preceq s$ where the maximum in (3.1) is attained. Then we have to distinguish between the cases $v \preceq r$ and $r \prec v$.

In the first case we have

$$d(t,s) = \|\alpha \mathbf{1}_{(t,v]}\|_q \sigma(v) \le \max_{t \prec v' \preceq r} \{\|\alpha \mathbf{1}_{(t,v']}\|_q \sigma(v')\} = d(t,r) \le d(t,r) + d(r,s).$$

Suppose now $r \prec v$. Here we argue as follows:

$$d(t,s) = \|\alpha \mathbf{1}_{(t,v]}\|_q \sigma(v) \le (\|\alpha \mathbf{1}_{(t,r]}\|_q + \|\alpha \mathbf{1}_{(r,v]}\|_q)\sigma(v),$$

and since σ is non-increasing, it follows that

$$d(t,s) \le \|\alpha \mathbf{1}_{(t,r]}\|_q \sigma(r) + \|\alpha \mathbf{1}_{(r,v]}\|_q \sigma(v) \le d(t,r) + d(r,s). \blacksquare$$

Our next objective is to investigate ε-nets for T with respect to the metric d possessing an additional useful property. Given $\varepsilon > 0$, a set $S \subseteq T$ is said to be an *order ε-net* provided that for each $t \in T$ there is an $s \in S$ satisfying $d(s,t) < \varepsilon$ as well as $s \preceq t$. The corresponding *order covering numbers* of T are then

$$\tilde{N}(T,d,\varepsilon) := \inf\{\#S : S \text{ is an order } \varepsilon\text{-net of } T\}. \tag{3.2}$$

Recall that the usual covering numbers $N(T,d,\varepsilon)$ are defined by

$$N(T,d,\varepsilon)=\inf\Big\{\#S : S\subset T,\, T=\bigcup_{s\in S}B_\varepsilon(s)\Big\}$$

where $B_\varepsilon(s)$ is the open ε-ball centered at $s\in T$, i.e.

$$B_\varepsilon(s):=\{r\in T : d(r,s)<\varepsilon\}.$$

Clearly $N(T,d,\varepsilon)\le \tilde N(T,d,\varepsilon)$, but as we shall see now, a slightly weaker reverse estimate is valid as well. More precisely we have the following.

PROPOSITION 3.2. *Let d be the metric defined in* (3.1). *Then for any $\varepsilon>0$,*

$$\tilde N(T,d,2\varepsilon)\le N(T,d,\varepsilon). \tag{3.3}$$

Proof. Take any ε-net $S\subset T$. For each $s\in S$ we choose $r_s\in B_\varepsilon(s)$ such that $r_s\wedge s$ is the minimal element in $\{r\wedge s : r\in B_\varepsilon(s)\}$. Then

$$r_s\wedge s\preceq r\wedge s\preceq s \quad \text{whenever } r\in B_\varepsilon(s).$$

Set

$$\tilde S:=\{r_s\wedge s : s\in S\}.$$

Clearly, $\#\tilde S\le \#S$, hence it suffices to prove that $\tilde S$ is an order 2ε-net of T. To this end take any $t\in T$. Then there is an $s\in S$ such that $t\in B_\varepsilon(s)$, and by the choice of r_s it follows that $r_s\wedge s\preceq t\wedge s\preceq t$. Thus it remains to estimate the distance between $r_s\wedge s$ and t. Note that the definition of d implies $d(r_s\wedge s,s)\le d(r_s,s)<\varepsilon$. Thus the triangle inequality leads to

$$d(r_s\wedge s,t)\le d(r_s\wedge s,s)+d(t,s)<2\varepsilon$$

because of $t\in B_\varepsilon(s)$. This completes the proof. ∎

4. Upper entropy estimates for weighted summation operators. Here and later on, the basic assumption about the weight functions α and σ is that they satisfy (1.2) for some fixed $q\in(1,\infty)$ and that σ is non-increasing.

In a first step we investigate weights σ attaining only dyadic values, i.e., in $\{2^{-m} : m\in\mathbb{Z}\}$. Without losing generality assume $\sigma(\mathbf{0})=1$, hence there are subsets $I_m\subseteq T$, $m\ge 0$, such that

$$\sigma(s)=\sum_{m=0}^{\infty}2^{-m}\mathbf{1}_{I_m}(s),\quad s\in T. \tag{4.1}$$

Since σ is supposed to be non-increasing, the sets I_m have the following properties:

(1) $T=\bigcup_{m=0}^{\infty}I_m$ and $I_l\cap I_m=\emptyset$ provided that $l\ne m$.

(2) Whenever $B \subseteq T$ is a branch, for each $m \geq 0$ either $B \cap I_m = \emptyset$ or it is an order interval in T. Furthermore, if $l < m$, $t \in B \cap I_l$, and $s \in B \cap I_m$, then $t \prec s$.

Define an operator W on $\ell_1(T)$ by

$$(Wx)(t) := \alpha(t) \sum_{\substack{s \succeq t \\ s \in I_m}} \sigma(s)x(s) = \alpha(t)\, 2^{-m} \sum_{\substack{s \succeq t \\ s \in I_m}} x(s), \quad t \in I_m. \tag{4.2}$$

The mapping W acts as a "partial" weighted summation operator depending on the partition $(I_m)_{m \geq 0}$. We claim that condition (1.2) implies that W is a bounded operator from $\ell_1(T)$ into $\ell_q(T)$. To see this define the unit vectors $\delta_t \in \ell_1(T)$, $t \in T$, by

$$\delta_t(r) := \begin{cases} 1, & r = t, \\ 0, & r \neq t. \end{cases}$$

Then

$$W(\delta_t) = \sigma(t) \sum_{\substack{r \preceq t \\ r \in I_m}} \alpha(r)\delta_r = 2^{-m} \sum_{\substack{r \preceq t \\ r \in I_m}} \alpha(r)\delta_r, \quad t \in I_m,$$

hence (1.2) implies $\sup_{t \in T} \|W(\delta_t)\|_q < \infty$ and $W : \ell_1(T) \to \ell_q(T)$ is well-defined and bounded.

Define the set $E_W \subseteq \ell_q(T)$ by

$$E_W := \{W(\delta_t) : t \in T\}$$

and let the metric d on T be as in (3.1) with weights α and σ satisfying (1.2) and (4.1), respectively. Then the following holds.

Proposition 4.1. *We have*

$$N(E_W, \|\cdot\|_q, \varepsilon) \leq \tilde{N}(T, d, \varepsilon) + 1.$$

Proof. Fix $\varepsilon > 0$ and choose an arbitrary order ε-net S in T (with respect to the metric d). Given $t \in T$, there is a unique $m \geq 0$ with $t \in I_m$. By definition we find an $s \in S$ satisfying $d(s,t) < \varepsilon$ as well as $s \preceq t$. Assume first that $s \in I_m$ as well. Then we get

$$\|W(\delta_t) - W(\delta_s)\|_q = \Big(\sum_{s \prec r \preceq t} \alpha(r)^q \Big)^{1/q} \cdot 2^{-m} = \|\alpha \mathbf{1}_{(s,t]}\|_q \sigma(t) \leq d(s,t) < \varepsilon.$$

Otherwise, if $s \in I_l$ for a certain $l < m$, we argue as follows:

$$\|W(\delta_t)\|_q = \Big(\sum_{\substack{r \preceq t \\ r \in I_m}} \alpha(r)^q \Big)^{1/q} \cdot 2^{-m} \leq \Big(\sum_{s \prec r \preceq t} \alpha(r)^q \Big)^{1/q} \cdot \sigma(t) \leq d(s,t) < \varepsilon.$$

Consequently, the set $\{W(\delta_s) : s \in S\} \cup \{0\}$ is an ε-net of E_W in $\ell_q(T)$. This being true for any order net S completes the proof. ∎

Proposition 4.2. *For $q \in (1,\infty)$ let $p := \min\{q,2\}$ and $1/p' := 1-1/p$. Furthermore let $a > 0$ and $0 \le b < \infty$. If*

$$\tilde{N}(T,d,\varepsilon) \le c\varepsilon^{-a}|\log\varepsilon|^b, \tag{4.3}$$

then

$$e_n(W : \ell_1(T) \to \ell_q(T)) \le c' n^{-1/a-1/p'}(\log n)^{b/a}. \tag{4.4}$$

If instead of (4.3) *we only have*

$$\log \tilde{N}(T,d,\varepsilon) \le c\varepsilon^{-a}, \tag{4.5}$$

then

$$e_n(W : \ell_1(T) \to \ell_q(T)) \le c' n^{-1/p'}(\log n)^{1/p'-1/a} \tag{4.6}$$

whenever $a < p'$, while for $a > p'$ we have

$$e_n(W : \ell_1(T) \to \ell_q(T)) \le c' n^{-1/a}. \tag{4.7}$$

Proof. If we assume (4.3), by Proposition 4.1 we also have

$$N(E_W, \|\cdot\|_q, \varepsilon) \le c\varepsilon^{-a}|\log\varepsilon|^b. \tag{4.8}$$

Observe that $e_n(W) = e_n(\mathrm{aco}(E_W))$, where $\mathrm{aco}(B)$ denotes the absolutely convex hull of a set $B \subseteq \ell_q(T)$. Thus we may use known estimates for the entropy of absolutely convex hulls as can be found in [CKP] or [St]. For example, assuming (4.8) we may apply Corollary 5 in [St]. Recall that $\ell_q(T)$ is of type p with $p = \min\{2,q\}$. Hence we get

$$e_n(W) = e_n(\mathrm{aco}(E_W)) \le c' n^{-1/a-1/p'}(\log n)^{b/a},$$

which completes the proof of (4.4).

Assuming (4.5), estimates (4.6) and (4.7) follow by similar arguments using Corollaries 4 and 3 in [St], respectively. ∎

Our next objective is to apply the previous results to weighted summation operators. To this end let α and σ be weight functions satisfying (1.2), where σ is non-increasing. Then we define the weighted summation operator $V_{\alpha,\sigma}$ as in (1.3). Under the assumptions on the weights the operator $V_{\alpha,\sigma}$ is well-defined and bounded from $\ell_1(T)$ into $\ell_q(T)$.

The main goal is to relate the degree of compactness of $V_{\alpha,\sigma}$ to the behavior of $\tilde{N}(T,d,\varepsilon)$ as $\varepsilon \to 0$. Here the metric d is defined as in (3.1) by α and σ. In a first step we suppose that σ is of the special form (4.1) with sets $I_m \subseteq T$ defined there.

Given $t \in T$, set

$$K_t := \{k \ge 0 : I_k \cap [\mathbf{0}, t] \ne \emptyset\}.$$

Consequently, if $k \in K_t$, then $I_k \cap [\mathbf{0}, t] = [\lambda_k(t), \theta_k(t)]$ for some $\lambda_k(t) \preceq$

$\theta_k(t) \preceq t$. Note that $\theta_m(t) = t$ for $t \in I_m$ and

$$[\mathbf{0}, t] = \bigcup_{k\in K_t} [\lambda_k(t), \theta_k(t)]. \tag{4.9}$$

Define now an operator $Z : \ell_1(T) \to \ell_1(T)$ by

$$Z(\delta_t) := \sum_{k\in K_t} 2^{k-m} \delta_{\theta_k(t)}, \quad t \in I_m. \tag{4.10}$$

PROPOSITION 4.3. *Assume* (1.2) *and* (4.1) *and define* $W : \ell_1(T) \to \ell_q(T)$ *and* $Z : \ell_1(T) \to \ell_1(T)$ *as in* (4.2) *and* (4.10), *respectively. Then* Z *is bounded with* $\|Z\| \le 2$, *and the operator* $V_{\alpha,\sigma}$ *given by* (1.3) *admits a factorization*

$$V_{\alpha,\sigma} = W \circ Z. \tag{4.11}$$

Proof. By construction, for each $t \in I_m$ we have

$$\|Z(\delta_t)\|_1 \le \sum_{k\in K_t} 2^{k-m} \le \sum_{k=0}^{m} 2^{k-m} \le 2,$$

hence $\|Z\| = \sup_{t\in T} \|Z(\delta_t)\|_1$ implies $\|Z\| \le 2$ as asserted.

To prove (4.11) first note that for $t \in T$ and $k \in K_t$ we get

$$W(\delta_{\theta_k(t)}) = \sigma(\theta_k(t)) \sum_{r\in[\lambda_k(t),\theta_k(t)]} \alpha(r)\delta_r = 2^{-k} \sum_{r\in[\lambda_k(t),\theta_k(t)]} \alpha(r)\delta_r,$$

hence, if $t \in I_m$, then by (4.9),

$$W(Z(\delta_t)) = \sum_{k\in K_t} 2^{k-m} \Big[2^{-k} \sum_{r\in[\lambda_k(t),\theta_k(t)]} \alpha(r)\delta_r\Big] = 2^{-m} \sum_{r\in[0,t]} \alpha(r)\delta_r.$$

On the other hand,

$$V_{\alpha,\sigma}(\delta_t) = \sigma(t) \sum_{r\preceq t} \alpha(r)\delta_r = 2^{-m} \sum_{r\in[0,t]} \alpha(r)\delta_r,$$

and it follows that $V_{\alpha,\sigma}(\delta_t) = W(Z(\delta_t))$. This being true for any $t \in T$ proves (4.11). ∎

THEOREM 4.4. *Let* α *and* σ *be weight functions satisfying* (1.2) *where* σ *is non-increasing. If*

$$\tilde{N}(T, d, \varepsilon) \le c\varepsilon^{-a} |\log \varepsilon|^b$$

for some $a > 0$ *and* $b \ge 0$, *then*

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \le c n^{-1/a-1/p'} (\log n)^{b/a}$$

with p *as in Proposition* 4.2. *If*

$$\log \tilde{N}(T, d, \varepsilon) \le c\varepsilon^{-a}, \tag{4.12}$$

then

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \le c' n^{-1/p'} (\log n)^{1/p'-1/a}$$

whenever $a < p'$, *while for* $a > p'$ *we have*

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \le c' n^{-1/a}.$$

Proof. Suppose as before $\sigma(\mathbf{0}) = 1$ and for $m \ge 0$ define

$$I_m := \{t \in T : 2^{-m-1} < \sigma(t) \le 2^{-m}\}.$$

If

$$\hat{\sigma}(t) := \sum_{m \ge 0} 2^{-m} \mathbf{1}_{I_m}(t), \quad t \in T,$$

then $\hat{\sigma}$ is a non-increasing weight function as in (4.1). By construction,

$$\sigma(t) \le \hat{\sigma}(t) \le 2\sigma(t), \quad t \in T. \tag{4.13}$$

Define the metrics d and $\hat{d}$ as in (3.1) by α and by σ or $\hat{\sigma}$, respectively. In view of (4.13) we get

$$d(t,s) \le \hat{d}(t,s) \le 2\, d(t,s), \quad t, s \in T,$$

hence $\tilde{N}(T, \hat{d}, \varepsilon/2) \le \tilde{N}(T, d, \varepsilon)$, which implies $\tilde{N}(T, \hat{d}, \varepsilon) \le c\varepsilon^{-a} |\log \varepsilon|^b$ as well. But now we are in the situation of Proposition 4.2 and obtain

$$e_n(W : \ell_1(T) \to \ell_q(T)) \le c n^{-1/a - 1/p'} (\log n)^{b/a}. \tag{4.14}$$

An application of Proposition 4.3 yields

$$e_n(V_{\alpha,\hat{\sigma}}) = e_n(W \circ Z) \le e_n(W) \|Z\| \le 2 e_n(W),$$

hence by (4.14) it follows that also

$$e_n(V_{\alpha,\hat{\sigma}}) \le c n^{-1/a - 1/p'} (\log n)^{b/a}.$$

To complete the proof note that (4.13) implies that the diagonal operator $\Delta : \ell_1(T) \to \ell_1(T)$ defined by

$$\Delta(\delta_t) := \frac{\sigma(t)}{\hat{\sigma}(t)} \delta_t$$

is bounded with $\|\Delta\| \le 1$. Of course, $V_{\alpha,\sigma} = V_{\alpha,\hat{\sigma}} \circ \Delta$, hence

$$e_n(V_{\alpha,\sigma}) \le e_n(V_{\alpha,\hat{\sigma}}) \|\Delta\| \le e_n(V_{\alpha,\hat{\sigma}}), \tag{4.15}$$

completing the proof of the first part.

The second part is proved by exactly the same arguments. Indeed, (4.12) implies $\log \tilde{N}(T, \hat{d}, \varepsilon) \le c\varepsilon^{-a}$. An application of Proposition 4.3 now yields

$$e_n(V_{\alpha,\hat{\sigma}}) = e_n(W \circ Z) \le e_n(W) \|Z\| \le 2 e_n(W),$$

and the estimates follow by the second part of Proposition 4.2 via (4.15). ∎

REMARK. The critical case $a = p'$ is excluded in the second part of Theorem 4.4. This is due to the fact that in that case only weaker estimates for $e_n(\mathrm{aco}(E_W))$, hence for $e_n(W)$ and also for $e_n(V_{\alpha,\sigma})$, are available. Indeed, using Corollary 1.4 in [CSt] it follows that (4.12) only gives

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \le c' n^{-1/a} \log n.$$

But the results in [L] suggest that the right order in that case is $n^{-1/a}$, i.e., the above estimate probably contains an unnecessary log-term ([1]).

5. Lower entropy estimates. We start with a quite general lower estimate for weighted summation operators on trees.

PROPOSITION 5.1. *Suppose there are m pairs of elements t_i, s_i in T with the following properties.*

(1) $t_i \prec s_i$ *and* $(t_i, s_i] \cap (t_j, s_j] = \emptyset$ *for* $1 \le i, j \le m$, $i \ne j$.
(2) *For some* $\varepsilon > 0$ *we have* $d(t_i, s_i) \ge \varepsilon$, $1 \le i \le m$.

Then

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \ge c\varepsilon \left(\frac{\log(1 + m/n)}{n} \right)^{1/q'}$$

with some $c = c(q)$ *whenever* $\log m \le n \le m$.

Proof. The strategy of the following construction consists in "inscribing" the well studied identity operator from ℓ_1^m into ℓ_q^m into our operator $V_{\alpha,\sigma}$.

The definition of the metric d implies the existence of $v_i \in T$ such that $t_i \prec v_i \preceq s_i$ and

$$\Big(\sum_{t_i \prec r \preceq v_i} \alpha(r)^q \Big)^{1/q} \sigma(v_i) \ge \varepsilon, \quad 1 \le i \le m.$$

By assumption the intervals $J_i := (t_i, v_i]$, $1 \le i \le m$, are disjoint subsets of T.

Next define elements $y_i \in \ell_1(T)$ by

$$y_i := \delta_{v_i} - \frac{\sigma(v_i)}{\sigma(t_i)} \delta_{t_i}, \quad 1 \le i \le m,$$

as well as an operator $I : \ell_1^m \to \ell_1(T)$ by setting

$$I(\delta_i) := y_i, \quad 1 \le i \le m.$$

Here δ_i is the ith unit vector in $\ell_1^m = \ell_1(\{1, \dots, m\})$. Then $\sigma(v_i) \le \sigma(t_i)$ implies $\|y_i\|_1 \le 2$, hence $\|I\| \le 2$ as well.

([1]) This conjecture has indeed been recently proved by the authors: see arXiv: 1009.2339.

The image $z_i \in \ell_q(T)$ of y_i under $V_{\alpha,\sigma}$ equals

$$z_i := V_{\alpha,\sigma}(y_i) = \sigma(v_i) \sum_{t_i \prec r \preceq v_i} \alpha(r)\delta_r, \quad 1 \le i \le m.$$

In particular, the support of z_i is contained in J_i.

Finally, let

$$\beta_i := \Big(\sum_{t_i \prec r \preceq v_i} \alpha(r)^q \Big)^{1/q'}, \qquad b_i := \beta_i^{-1} \sum_{t_i \prec r \preceq v_i} \alpha(r)^{q-1}\delta_r.$$

By the choice of β_i we obtain $\|b_i\|_{q'} = 1$. Moreover, since the order intervals J_i are disjoint, it follows that

$$\langle z_i, b_j \rangle = 0, \quad 1 \le i, j \le m,\ i \ne j,$$

while

$$\langle z_i, b_i \rangle = \sigma(v_i) \Big(\sum_{t_i \prec r \preceq v_i} \alpha(r)^q \Big)^{1/q} \ge \varepsilon, \quad 1 \le i \le m. \tag{5.1}$$

Define $P : \ell_q(T) \to \ell_q^m$ by

$$P(z) := (\langle z, b_i \rangle)_{i=1}^m, \quad z \in \ell_q(T).$$

Then $\|P\| \le 1$. Indeed, if $z \in \ell_q(T)$, then

$$\|P(z)\|_q^q = \sum_{i=1}^m |\langle z, b_i \rangle|^q = \sum_{i=1}^m |\langle z, b_i \mathbf{1}_{J_i} \rangle|^q \le \sum_{i=1}^m \|z \mathbf{1}_{J_i}\|_q^q \|b_i\|_{q'}^q \le \|z\|_q^q.$$

Summing up,

$$PV_{\alpha,\sigma} I(\delta_i) = \langle z_i, b_i \rangle \delta_i, \quad 1 \le i \le m,$$

and because of (5.1) we obtain, for the identity Id_m from ℓ_1^m into ℓ_q^m,

$$\mathrm{Id}_m = \Delta \circ (PV_{\alpha,\sigma} I)$$

with a diagonal operator Δ satisfying $\|\Delta : \ell_q^m \to \ell_q^m\| \le \varepsilon^{-1}$. Consequently,

$$e_n(\mathrm{Id}_m : \ell_1^m \to \ell_q^m) \le \varepsilon^{-1} e_n(PV_{\alpha,\sigma} I) \le 2\varepsilon^{-1} e_n(V_{\alpha,\sigma}).$$

To complete the proof note that a result of Schütt (cf. [Sch]) asserts that

$$e_n(\mathrm{Id}_m : \ell_1^m \to \ell_q^m) \ge c \Big(\frac{\log(1 + m/n)}{n} \Big)^{1/q'}$$

as long as $\log m \le n \le m$. ∎

In order to apply Proposition 5.1 we have to find sufficiently many order intervals $(t_i, s_i]$ with the properties stated above. The next result shows that we can find at least $N(T, d, 2\varepsilon) - 1$ such intervals.

Proposition 5.2. *Let $\varepsilon > 0$. Then there are at least $N(T, d, 2\varepsilon) - 1$ order intervals $(t_i, s_i]$ such that $d(t_i, s_i) \ge \varepsilon$ and $(t_i, s_i] \cap (t_j, s_j] = \emptyset$ for $i \ne j$.*

Proof. Let $S = \{s_1, \ldots, s_n\} \subset T$ be a maximal 2ε-distant set, i.e., $d(s_i, s_j) \geq 2\varepsilon$ whenever $i \neq j$. Since S is chosen maximal, for any $t \in T$ there is an $s_i \in S$ with $d(t, s_i) < 2\varepsilon$. Thus S is a 2ε-net, and consequently $n \geq N(T, d, 2\varepsilon)$. There is at most one s_i in S with $d(\mathbf{0}, s_i) < \varepsilon$. By changing the numbering we may assume $d(\mathbf{0}, s_j) \geq \varepsilon$ for $1 \leq j \leq n-1$. For those j we now define $t_j \in T$ possessing the following properties: $t_j \prec s_j$ and $d(t_j, s_j) \geq \varepsilon$, but whenever $t \in T$ satisfies $t_j \prec t \preceq s_j$, then $d(t_j, s_j) < \varepsilon$. Such t_j exist (and are uniquely determined) by $d(\mathbf{0}, s_j) \geq \varepsilon$ and by the monotonicity property of d.

We now claim that the order intervals $(t_1, s_1], \ldots, (t_{n-1}, s_{n-1}]$ have the desired properties. By construction $d(t_j, s_j) \geq \varepsilon$ and it remains to prove that the intervals are disjoint. Assume to the contrary that there is some t in $(t_i, s_i] \cap (t_j, s_j)$ for certain $i \neq j$. Then $d(t, s_i) < \varepsilon$ as well as $d(t, s_j) < \varepsilon$ by the choice of t_j. This implies $d(s_i, s_j) \leq d(t, s_i) + d(t, s_j) < 2\varepsilon$, which contradicts the choice of the set S. ∎

Let us state a first consequence of Propositions 5.1 and 5.2.

THEOREM 5.3. *Suppose that for some $a > 0$ and $b \geq 0$,*

$$N(T, d, \varepsilon) \geq c\varepsilon^{-a} |\log \varepsilon|^b. \tag{5.2}$$

Then

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \geq \tilde{c} n^{-1/a - 1/q'} (\log n)^{b/a} \tag{5.3}$$

with a constant $\tilde{c} = \tilde{c}(c, q)$. In particular, if $1 < q \leq 2$, then

$$N(T, d, \varepsilon) \approx \varepsilon^{-a} |\log \varepsilon|^b \tag{5.4}$$

implies

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \approx n^{-1/a - 1/q'} (\log n)^{b/a}.$$

Proof. In view of Proposition 5.2 the assumption implies that there are m disjoint order intervals $(t_i, s_i]$ with $d(t_i, s_i) \geq \varepsilon$. We may choose m of order $\varepsilon^{-a} |\log \varepsilon|^b$. Next we apply Proposition 5.1 with $n = m$ to obtain

$$e_n(V_{\alpha,\sigma}) \geq c\varepsilon \left(\frac{\log 2}{n} \right)^{1/q'} \geq \tilde{c} n^{-1/a - 1/q'} (\log n)^{b/a}. \ ∎$$

REMARK. (1) Note that by (3.3), both in (5.2) and in (5.4) the covering numbers $N(T, d, \varepsilon)$ may be replaced by the order numbers $\widetilde{N}(T, d, \varepsilon)$.

(2) It remains open whether or not in (5.3) the expression $n^{-1/a - 1/q'}$ may be replaced by $n^{-1/a - 1/2}$ whenever $2 < q < \infty$. For those q, there remains a gap between the upper estimate in Theorem 4.4 and the lower one in Theorem 5.3.

Our next objective is an application of Propositions 5.1 and 5.2 in the case of rapidly increasing covering numbers.

THEOREM 5.4. *Suppose that*

$$\log N(T,d,\varepsilon) \geq c\varepsilon^{-a}$$

for a certain $a>0$. Then

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \geq cn^{-1/q'}(\log n)^{1/q'-1/a} \tag{5.5}$$

provided that $a<q'$. On the other hand, if $q' \leq a$, then

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \geq cn^{-1/a}. \tag{5.6}$$

Proof. First observe that Proposition 5.2 implies the existence of m disjoint order intervals $(t_i,s_i]$ with $d(t_i,s_i) \geq \varepsilon$ where $\log m \approx \varepsilon^{-a}$.

So let us prove (5.5). We use Proposition 5.1 with $n \approx \sqrt{m}$ and note that the choice of m and n implies $\varepsilon \approx (\log m)^{-1/a} \approx (\log n)^{-1/a}$. Of course, $\log m \leq n \leq m$, thus Proposition 5.1 applies and leads to

$$\begin{aligned} e_n(V_{\alpha,\sigma}) &\geq c\varepsilon\left(\frac{\log(1+\sqrt{m})}{n}\right)^{1/q'} \geq c(\log m)^{-1/a+1/q'}\, n^{-1/q'} \\ &\geq c(\log n)^{-1/a+1/q'} n^{-1/q'}. \end{aligned}$$

Inequality (5.6) follows by similar arguments. The number m is chosen as before but this time we take n of order $\log m$. This implies $\varepsilon \approx n^{-1/a}$ and we get

$$e_n(V_{\alpha,\sigma}) \geq c\varepsilon\left(\frac{\log(1+\frac{m}{\log m})}{n}\right)^{1/q'} \geq c\varepsilon \geq cn^{-1/a}. \ \blacksquare$$

REMARK. Note that (5.5) as well as (5.6) are valid for all $a>0$. But for $a \leq q'$ the first estimate is better while for $a \geq q'$ the second one leads to a better lower bound.

6. Examples of upper entropy estimates. The aim of this section is to apply the previous results to weights and trees satisfying certain growth assumptions. We start by assuming that there is a strictly decreasing, continuous function φ on $(0,\infty)$ with

$$\int_0^\infty \varphi(x)\,dx < \infty$$

such that for some fixed $q<\infty$,

$$(\alpha(t)\sigma(t))^q \leq \varphi(|t|), \quad t \in T. \tag{6.1}$$

The next objective is to construct order ε-nets on $\mathbb{N}$ for a metric generated by φ. Later on, those nets on $\mathbb{N}$ lead in a natural way to nets on trees. Given φ as above define Φ on $[0,\infty]$ by

$$\Phi(y) := \int_y^\infty \varphi(x)\,dx, \quad 0 \le y < \infty, \tag{6.2}$$

and $\Phi(\infty) := 0$. The generated metric $\bar{d}$ on $[0,\infty]$ is then defined by

$$\bar{d}(y_1, y_2) := \Phi(y_1) - \Phi(y_2) = \int_{y_1}^{y_2} \varphi(x)\,dx \tag{6.3}$$

provided that $y_1 \le y_2$. Given $\varepsilon > 0$ we construct a 2ε-net for $(\mathbb{N}, \bar{d})$ as follows. First we take all points in $\mathbb{N}$ up to the level $\varphi^{-1}(\varepsilon)$, i.e., as a first part of the net we choose

$$M_\varepsilon := \{n \ge 1 : n \le \varphi^{-1}(\varepsilon)\} = \{n \ge 1 : \varphi(n) \ge \varepsilon\}$$

and note that $\# M_\varepsilon \le \varphi^{-1}(\varepsilon)$.

It remains to find a suitable 2ε-cover for $\{n \ge 1 : n \ge \varphi^{-1}(\varepsilon)\}$. Here we proceed as follows. For $k = 1, \ldots, N$ set

$$\tilde{u}_k := \Phi^{-1}(k\varepsilon) \tag{6.4}$$

where the number N is chosen as

$$\begin{aligned} N := \max\{k \ge 1 : \tilde{u}_k \ge \varphi^{-1}(\varepsilon)\} &= \max\{k \ge 1 : k\varepsilon \le \Phi(\varphi^{-1}(\varepsilon))\} \\ &= \max\{k \ge 1 : k \le \Phi(\varphi^{-1}(\varepsilon))/\varepsilon\}. \end{aligned}$$

Note that $\tilde{u}_1 > \cdots > \tilde{u}_N$, and moreover, since in that region $\varphi(x) < \varepsilon$, we necessarily have $\tilde{u}_{k-1} - \tilde{u}_k > 1$, $k = 1, \ldots, N$. Hence, setting (here $[u]$ denotes the integer part of $u \in \mathbb{R}$)

$$u_k := [\tilde{u}_k], \quad k = 1, \ldots, N-1,$$

it follows that $\tilde{u}_1 \ge u_1 > \tilde{u}_2 \ge \cdots \ge u_{N-1} > \tilde{u}_N$. It remains to define u_N. If $[\tilde{u}_N] \ge \varphi^{-1}(\varepsilon)$ we set $u_N := [\tilde{u}_N]$. Otherwise we take $u_N := [\varphi^{-1}(\varepsilon)] + 1$. By the construction it follows that $\bar{d}(u_k, m) < 2\varepsilon$ for all $m \in \mathbb{N}$ with $u_k \le m < u_{k-1}$ where $u_0 := \infty$. Consequently, the set

$$\bar{S}_\varepsilon := M_\varepsilon \cup \{u_1, \ldots, u_N\} \tag{6.5}$$

is a 2ε-net of $(\mathbb{N}, \bar{d})$.

Next we want to apply the preceding construction to build suitable ε-nets on trees. Recall that $R(n)$ denotes the number of elements in the nth generation of a tree.

PROPOSITION 6.1. *Let T be a tree such that $R(n) \le \rho(n)$ for a certain continuous, non-decreasing function ρ on $[0,\infty)$. Furthermore, suppose that the weights α and σ on T satisfy* (6.1) *for a certain $q \ge 1$ and some function*

φ as before. Define the metric d as in (3.1) *with α, σ and q. Then*

$$\tilde{N}(T,d,\varepsilon) \le \int_0^{\varphi^{-1}(\varepsilon^q/2)+1} \rho(x)\,dx + \rho(\Phi^{-1}(\varepsilon^q/2)) + 2\varepsilon^{-q} \int_{\varphi^{-1}(\varepsilon^q/2)}^{\Phi^{-1}(\varepsilon^q/2)} \rho(y)\varphi(y)\,dy$$

where Φ is as in (6.2).

Proof. Assuming (6.1) it follows (recall that σ is non-increasing) that for all $t \preceq s$ in T,

$$\begin{aligned} d(t,s) &= \max_{t \prec v \preceq s} \Big\{ \Big(\sum_{t \prec r \preceq v} \alpha(r)^q \Big)^{1/q} \sigma(v) \Big\} \le \Big(\sum_{t \prec r \preceq s} (\alpha(r)\sigma(r))^q \Big)^{1/q} \\ &\le \Big(\sum_{t \prec r \preceq s} \varphi(|r|) \Big)^{1/q} = \Big(\sum_{|t|<k\le|s|} \varphi(k) \Big)^{1/q} \le \Big(\int_{|t|}^{|s|} \varphi(x)\,dx \Big)^{1/q} \\ &= \bar{d}(|t|,|s|)^{1/q}. \end{aligned}$$

Hence, if $\bar{S}_\varepsilon = M_\varepsilon \cup \{u_1,\dots,u_N\}$ is defined as in (6.5), setting

$$S_\varepsilon := \{t \in T : |t| \in \bar{S}_{\varepsilon^q}\} \tag{6.6}$$

we obtain an order $2^{1/q}\varepsilon$-net for (T,d).

To proceed further we have to estimate $\#S_\varepsilon$ suitably. In view of $R(n) \le \rho(n)$ we get

$$\begin{aligned} \#S_\varepsilon &\le \sum_{n \le \varphi^{-1}(\varepsilon^q)} \rho(n) + \sum_{k=1}^N \rho(u_k) \le \sum_{n \le \varphi^{-1}(\varepsilon^q)} \rho(n) + \sum_{k=1}^N \rho(\tilde{u}_k) \\ &= \sum_{n \le \varphi^{-1}(\varepsilon^q)} \rho(n) + \sum_{k=1}^N \rho(\Phi^{-1}(k\varepsilon^q)). \end{aligned}$$

Since ρ is non-decreasing and $\rho \circ \Phi^{-1}$ non-increasing, this leads to

$$\begin{aligned} \#S_\varepsilon &\le \int_0^{\varphi^{-1}(\varepsilon^q)+1} \rho(x)\,dx + \rho(\Phi^{-1}(\varepsilon^q)) + \int_1^{\Phi(\varphi^{-1}(\varepsilon^q))/\varepsilon^q} \rho(\Phi^{-1}(x\varepsilon^q))\,dx \\ &= \int_0^{\varphi^{-1}(\varepsilon^q)+1} \rho(x)\,dx + \rho(\Phi^{-1}(\varepsilon^q)) + \varepsilon^{-q} \int_{\varphi^{-1}(\varepsilon^q)}^{\Phi^{-1}(\varepsilon^q)} \rho(y)\varphi(y)\,dy. \end{aligned}$$

Finally, we use $\tilde{N}(T,d,2^{1/q}\varepsilon) \le \#S_\varepsilon$ and replace ε^q by $\varepsilon^q/2$. ∎

One can slightly simplify the bound for subsequent use as follows.

Corollary 6.2.

(1) (Convergent case) *Suppose that* $\int_1^\infty \rho(y)\varphi(y)\,dy < \infty$. *Then*

$$\tilde N(T,d,\varepsilon) \le \int\limits_0^{\varphi^{-1}(\varepsilon^q/2)+1} \rho(x)\,dx + \rho(\Phi^{-1}(\varepsilon^q/2)) + 2\varepsilon^{-q} \int\limits_{\varphi^{-1}(\varepsilon^q/2)}^{\infty} \rho(y)\varphi(y)\,dy. \tag{6.7}$$

(2) (Divergent case) *If* $\int_1^\infty \rho(y)\varphi(y)\,dy = \infty$, *then*

$$\tilde N(T,d,\varepsilon) \le \int\limits_0^{\varphi^{-1}(\varepsilon^q/2)+1} \rho(x)\,dx + \rho(\Phi^{-1}(\varepsilon^q/2)) + 2\varepsilon^{-q} \int\limits_{1}^{\Phi^{-1}(\varepsilon^q/2)} \rho(y)\varphi(y)\,dy. \tag{6.8}$$

Let us give a first application of Proposition 6.1 in the case of moderate trees, i.e. those where the number of elements in the generations increases at most polynomially.

Proposition 6.3. *Let T be a tree such that $R(n) \le cn^H$ for some $H \ge 0$. Suppose, furthermore, that*

$$\alpha(t)\sigma(t) \le c|t|^{-\gamma/q}, \quad t \in T,$$

for some $\gamma > 1$. Then

$$\tilde N(T,d,\varepsilon) \le c \begin{cases} \varepsilon^{-qH/(\gamma-1)}, & \gamma < H+1, \\ \varepsilon^{-q}\log(1/\varepsilon), & \gamma = H+1, \\ \varepsilon^{-q(H+1)/\gamma}, & \gamma > H+1. \end{cases}$$

Proof. First we note that in all three cases the first and the second terms in (6.7) and (6.8) behave like $\varepsilon^{-q(H+1)/\gamma}$ and $\varepsilon^{-qH/(\gamma-1)}$, respectively. Only the third term behaves differently in each of the three cases.

Thus let us start with the investigation of this third term in the convergent case, i.e., if $\gamma > H+1$. We use (6.7) and observe that the third term behaves as the first term, i.e., as

$$c\varepsilon^{-q}[\varphi^{-1}(\varepsilon^q/2)]^{H-\gamma+1} \le c\varepsilon^{-q(H+1)/\gamma}.$$

Since here $H/(\gamma-1) < (H+1)/\gamma$, the second term in (6.7) is of smaller order and we obtain

$$\tilde N(T,d,\varepsilon) \le c\varepsilon^{-q(H+1)/\gamma}$$

as asserted.

Next assume $\gamma = H+1$. This is a kind of divergent case and the third term in (6.8) is of order $\varepsilon^{-q}\log(1/\varepsilon)$, while the first and second terms are of lower order ε^{-q}, and we get

$$\tilde{N}(T,d,\varepsilon) \le c\varepsilon^{-q}\log(1/\varepsilon)$$

as claimed above.

Finally, suppose $\gamma < H+1$. This is again a divergent case and the third term in (6.8) behaves like

$$\varepsilon^{-q}[\Phi^{-1}(\varepsilon^q/2)]^{H-\gamma+1} \le c\varepsilon^{-qH/(\gamma-1)},$$

thus the second and third terms are of the same order. Since for $\gamma < H+1$ we have $H/(\gamma-1) > (H+1)/\gamma$, the first term that behaves like $\varepsilon^{-q(H+1)/\gamma}$ is of smaller order. Thus it follows that

$$\tilde{N}(T,d,\varepsilon) \le c\varepsilon^{-qH/(\gamma-1)}. \blacksquare$$

An application of Theorem 4.4 to the above estimates leads to the following.

THEOREM 6.4. *Suppose* $1 < q < \infty$ *and let as before* $p := \min\{2,q\}$. *Suppose that the tree* T *satisfies* $R(n) \le cn^H$ *for a certain* $H \ge 0$ *and that*

$$\alpha(t)\sigma(t) \le c|t|^{-\gamma/q}, \quad t \in T,$$

for a certain $\gamma > 1$. *Then*

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \le c \begin{cases} n^{-\frac{\gamma-1}{qH}-\frac{1}{p'}}, & \gamma < H+1, \\ n^{-\frac{1}{q}-\frac{1}{p'}}(\log n)^{1/q}, & \gamma = H+1, \\ n^{-\frac{\gamma}{q(H+1)}-\frac{1}{p'}}, & \gamma > H+1. \end{cases}$$

REMARK. Note that $p = q$ for $1 < q \le 2$. In particular, in that case

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \le c \begin{cases} n^{-\frac{\gamma-1}{qH}-\frac{1}{q'}}, & \gamma < H+1, \\ n^{-1}(\log n)^{1/q}, & \gamma = H+1, \\ n^{-\frac{\gamma}{q(H+1)}-\frac{1}{q'}}, & \gamma > H+1. \end{cases}$$

Our next objective is to investigate weighted summation operators on binary trees. Here we have $\rho(x) = 2^x$. Let us first suppose that the weights decay polynomially, i.e.,

$$\alpha(t)\sigma(t) \le c|t|^{-\gamma/q}, \quad t \in T,$$

for some $\gamma > 1$. Of course, in order to estimate $\tilde{N}(T,d,\varepsilon)$ we have to use the divergent case of Corollary 6.2. Then we get

$$\log \int_0^{\varphi^{-1}(\varepsilon^q/2)+1} \rho(x)\,dx \approx \varepsilon^{-q/\gamma}, \quad \log\rho(\Phi^{-1}(\varepsilon^q/2)) \approx \varepsilon^{-q/(\gamma-1)}.$$

Furthermore, as can be easily seen, the logarithm of the third term in (6.8) behaves like $\varepsilon^{-q/(\gamma-1)}$ as well.

Summing up, it follows that

$$\log \tilde{N}(T,d,\varepsilon) \le c\varepsilon^{-q/(\gamma-1)}.$$

Hence we see that the critical case appears if $q/(\gamma-1) = p'$ (recall that $p = \min\{2,q\}$), i.e.,

$$\gamma = \begin{cases} q & \text{if } 1 < q \le 2, \\ q/2+1 & \text{if } 2 \le q < \infty. \end{cases}$$

In the non-critical cases we get the following.

Theorem 6.5. *Let T be a binary tree and suppose that*

$$\alpha(t)\sigma(t) \le c|t|^{-\gamma/q}, \quad t \in T,$$

for a certain $\gamma > 1$ with $\gamma \ne q/p' + 1$. Then

(a) *for $1 < q \le 2$,*

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \le c \begin{cases} n^{-1/q'}(\log n)^{1-\gamma/q}, & \gamma > q, \\ n^{-(\gamma-1)/q}, & \gamma < q; \end{cases}$$

(b) *for $2 \le q < \infty$,*

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \le c \begin{cases} n^{-1/2}(\log n)^{1/2-(\gamma-1)/q}, & \gamma > q/2+1, \\ n^{-(\gamma-1)/q}, & \gamma < q/2+1. \end{cases}$$

Remark. For one-weight operators, i.e., if $\sigma(t) = 1$, $t \in T$, and for $q = 2$ the preceding result was also proved in [L]. Moreover, it was shown there that the above estimates are sharp. But the main result in [L] is the investigation of the critical case $\gamma = 2$ if $q = 2$. As mentioned above, the general results for the entropy of the convex hull in [CSt] lead only to

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \le cn^{-(\gamma-1)/q} \log n$$

in the critical case $\gamma = q/p' + 1$.

Let us briefly mention a third example. Again we take a binary tree T, but this time the weights decrease exponentially, i.e., we assume

$$\alpha(t)\sigma(t) \le c2^{-\frac{\gamma}{q}|t|}, \quad t \in T,$$

for some $\gamma > 0$. Hence we have $\varphi(x) = 2^{-\gamma x}$ and

$$\varphi^{-1}(\varepsilon^q) \sim \Phi^{-1}(\varepsilon^q) \sim \frac{q}{\gamma} \log_2(1/\varepsilon).$$

Thus all terms in (6.7) and (6.8) are of the same order $\varepsilon^{-q/\gamma}$ and under these assumptions

$$\tilde{N}(T,d,\varepsilon) \le c\varepsilon^{-q/\gamma}.$$

Thus,

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \le cn^{-\gamma/q-1/p'} \tag{6.9}$$

in that case. In completely different probabilistic language, this example was studied in [AL].

7. Examples of lower entropy estimates. In Section 6 we proved upper estimates for $N(T,d,\varepsilon)$ under certain growth assumptions for the weights and for $R(n)$, the number of elements in the nth generation of T. The aim of this section is to prove in a similar way lower estimates for $N(T,d,\varepsilon)$ or $e_n(V_{\alpha,\sigma})$, respectively, assuming lower growth estimates. Thus we investigate weights satisfying

$$(\alpha(t)\sigma(t))^q \geq \varphi(|t|), \quad t \in T, \tag{7.1}$$

for a function φ as in Section 6 and, furthermore, we assume

$$R(n) \geq \rho(n), \quad n \in \mathbb{N}_0, \tag{7.2}$$

where ρ is as before non-increasing and continuous with $\rho(0) = 1$.

Under these assumptions we get the following.

PROPOSITION 7.1. *Assume* (7.1) *and* (7.2). *Then*

$$N(T,d,\varepsilon/2) \geq \int_1^{\varphi^{-1}(\varepsilon^q)-1} \rho(x)\,dx.$$

Proof. Fix $\varepsilon > 0$ and set

$$T_\varepsilon := \{t \in T : 0 \leq |t| \leq \varphi^{-1}(\varepsilon^q)\}.$$

Given $s \in T$, $s \neq \mathbf{0}$, let s' be the parent element of s, i.e., s is an offspring of s'. Then (7.1) implies

$$d(s',s) = \alpha(s)\sigma(s) \geq \varphi(|s|)^{1/q} \geq \varepsilon$$

provided that $s \in T_\varepsilon$. Let now $t, s \in T_\varepsilon$ with $t \neq s$. If $t \prec s$, then $t \preceq s' \prec s$, hence $d(t,s) \geq d(s',s) \geq \varepsilon$. Otherwise, i.e. if t and s are incomparable, by the same argument we get

$$d(t,s) \geq d(t \wedge s, s) \geq \varepsilon$$

as well. Consequently, T_ε is an ε-separated subset of T, which implies

$$N(T,d,\varepsilon/2) \geq \#T_\varepsilon.$$

Thus, it suffices to estimate $\#T_\varepsilon$ suitably. Here we use (7.2) to obtain

$$\#T_\varepsilon = \sum_{0 \leq n \leq \varphi^{-1}(\varepsilon^q)} R(n) \geq \sum_{0 \leq n \leq \varphi^{-1}(\varepsilon^q)} \rho(n) \geq \int_1^{\varphi^{-1}(\varepsilon^q)-1} \rho(x)\,dx. \quad \blacksquare$$

COROLLARY 7.2. *Suppose that*

$$\int_1^{\varphi^{-1}(\varepsilon^q)-1} \rho(x)\,dx \geq c\varepsilon^{-a}|\log\varepsilon|^b$$

for certain $a > 0$ and $b \geq 0$. Then

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \geq \tilde{c} n^{-1/a-1/q'} (\log n)^{b/a}. \tag{7.3}$$

Proof. Using Proposition 7.1 the assumption leads to

$$N(T, d, \varepsilon) \geq c' \varepsilon^{-a} |\log \varepsilon|^b.$$

Consequently, Theorem 5.3 applies and proves (7.3). ∎

Let us apply the preceding corollary to concrete functions φ and ρ. We start with the investigation of moderate trees and polynomial weights, i.e., ρ is of polynomial growth and $\varphi(x) \geq c x^{-\gamma}$ for a certain $\gamma > 1$. Here we get

Proposition 7.3. *Suppose that T is a tree with $R(n) \geq c n^H$ for some $H \geq 0$. Furthermore assume*

$$\alpha(t)\sigma(t) \geq c|t|^{-\gamma/q}, \quad t \in T,$$

for some $\gamma > 1$. Then

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \geq \tilde{c} n^{-\frac{\gamma}{q(H+1)} - \frac{1}{q'}}. \tag{7.4}$$

Proof. This follows directly from Corollary 7.2 by evaluating the integral. ∎

Remark. Suppose $1 < q \leq 2$. Then the preceding proposition shows that the estimates in Theorem 6.4 are sharp provided that $\gamma > H + 1$. We will see later on that this is no longer always true if $1 < \gamma \leq H + 1$.

Another application of Proposition 7.1 leading to sharp lower estimates is as follows.

Proposition 7.4. *Let T be a binary tree and suppose that*

$$\alpha(t)\sigma(t) \geq c 2^{-\frac{\gamma}{q}|t|}, \quad t \in T,$$

for some $\gamma > 0$. Then

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \geq c n^{-\gamma/q-1/q'}.$$

Proof. Again this is a direct consequence of Corollary 7.2 and the fact that

$$\varphi^{-1}(\varepsilon^q) = \frac{q}{\gamma} \log_2(1/\varepsilon) + \frac{\log_2 c}{\gamma}.$$

Recall that ρ may be chosen as $\rho(x) = 2^x$ in that case. ∎

Remark. Combining the preceding proposition with (6.9) gives the following: Let T be a binary tree and suppose $1 < q \leq 2$. If

$$\alpha(t)\sigma(t) \approx 2^{-\frac{\gamma}{q}|t|}, \quad t \in T,$$

then

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_q(T)) \approx n^{-\gamma/q-1/q'}.$$

As we said above, Proposition 7.3 does not always lead to sharp lower estimates, even if $1 < q \le 2$. The reason is that here the structure of the underlying tree plays a role. We assume now that $\xi(t) \ge 1$ for each $t \in T$. In other words, we suppose that each element in T has at least one offspring. Furthermore we restrict ourselves to one-weight operators defined as follows. We write V_α instead of $V_{\alpha,\sigma}$ if $\sigma \equiv 1$, i.e.,

$$(V_\alpha x)(t) := \alpha(t) \sum_{s \succeq t} x(s), \quad t \in T. \tag{7.5}$$

Condition (7.1) now reads

$$\alpha(t) \ge \varphi(|t|)^{1/q}, \quad t \in T. \tag{7.6}$$

To proceed further we have to exclude functions φ decreasing too fast. Thus we assume that there is a constant $\kappa \ge 1$ such that

$$\varphi(x) \le \kappa \varphi(2x), \quad x \ge x_0. \tag{7.7}$$

Let Φ be defined as in (6.2). For later use we mention that (7.7) implies

$$\frac{\Phi(x)}{\varphi(x)} \ge x \frac{\varphi(2x)}{\varphi(x)} \ge \kappa^{-1} x, \quad x \ge x_0,$$

hence

$$\frac{\Phi(\varphi^{-1}(y))}{y} \ge \kappa^{-1} \varphi^{-1}(y) \to \infty \quad \text{as } y \to 0. \tag{7.8}$$

Under these assumptions we get the following general lower estimate.

PROPOSITION 7.5. *Let T be a tree with $\xi(t) \ge 1$ for each $t \in T$ and suppose* (7.6) *and* (7.7) *hold. Then there is an $\varepsilon_0 > 0$ such that*

$$N(T, d, \varepsilon/2) \ge 4^{-1} \varepsilon^{-q} \int_{\varphi^{-1}(\varepsilon^q)}^{\Phi^{-1}(8\varepsilon^q)} \rho(y) \varphi(y)\, dy \tag{7.9}$$

whenever $0 < \varepsilon < \varepsilon_0$.

Proof. First note that for one-weight operators the metric d reduces to

$$d(t,s) = \Big(\sum_{t \prec r \preceq s} \alpha(r)^q \Big)^{1/q}$$

whenever $t \preceq s$. By (7.6) this implies

$$d(t,s) \ge \Big(\sum_{k=|t|+1}^{|s|} \varphi(k) \Big)^{1/q} \ge \bar{d}(|t|+1, |s|)^{1/q} \tag{7.10}$$

with $\bar{d}$ defined in (6.3).

We now construct positive numbers $\tilde{u}_1 > \cdots > \tilde{u}_N$ as in (6.4) but this time directly with ε^q instead of ε, i.e.,

$$\tilde{u}_k := \Phi^{-1}(k\varepsilon^q), \quad 1 \le k \le N,$$

where N satisfies

$$N \le \frac{\Phi(\varphi^{-1}(\varepsilon^q))}{\varepsilon^q} < N + 1.$$

Next set

$$v_k := [\tilde{u}_{3k}] + 1, \quad 1 \le k \le m,$$

where

$$m = \left[\frac{N}{3}\right] \ge \frac{\Phi(\varphi^{-1}(\varepsilon^q))}{3\varepsilon^q} - 2.$$

Using (7.8) this implies

$$m \ge \frac{\Phi(\varphi^{-1}(\varepsilon^q))}{4\varepsilon^q} \tag{7.11}$$

provided that $\varepsilon < \varepsilon_0$ for a certain ε_0 depending on φ.

Since $\tilde{u}_k - \tilde{u}_{k-1} > 1$ we get $[\tilde{u}_{3k-2}, \tilde{u}_{3k-3}] \subseteq [v_k + 1, v_{k-1}]$, hence

$$\bar{d}(v_k + 1, v_{k-1}) \ge \varepsilon^q. \tag{7.12}$$

Let us now construct an ε-separated subset $S_\varepsilon \subseteq T$ as follows. For $1 \le k \le m$ set

$$T_k := \{t \in T : |t| = v_k\}$$

and given $t \in T_k$ with $2 \le k \le m$ we choose exactly one $s_{k-1}(t) \in T_{k-1}$ satisfying $s_{k-1}(t) \succ t$. Those $s_{k-1}(t)$ exist because we assumed $\xi(t) \ge 1$ for all $t \in T$. Finally, define

$$S_\varepsilon := \bigcup_{k=2}^{m} \{s_{k-1}(t) : t \in T_k\}.$$

Because of (7.10) and (7.12) the points in S_ε are ε-separated and since $\tilde{u}_{3k} \le v_k$ the properties of ρ yield

$$\#S_\varepsilon = \sum_{k=2}^{m} \#T_k = \sum_{k=2}^{m} R(v_k) \ge \sum_{k=2}^{m} \rho(v_k) \ge \sum_{k=2}^{m} \rho(\tilde{u}_{3k}) = \sum_{k=2}^{m} \rho(\Phi^{-1}(3k\varepsilon^q)).$$

Clearly this implies

$$N(T, d, \varepsilon/2) \ge \sum_{k=2}^{m} \rho(\Phi^{-1}(3k\varepsilon^q)).$$

Observe that $\rho \circ \Phi^{-1}$ is decreasing and recall (7.11). Then we get

$$N(T,d,\varepsilon/2) \ge \int_2^{\frac{\Phi(\varphi^{-1}(\varepsilon^q))}{4\varepsilon^q}} \rho(\Phi^{-1}(3x\varepsilon^q))\,dx \ge \int_2^{\frac{\Phi(\varphi^{-1}(\varepsilon^q))}{4\varepsilon^q}} \rho(\Phi^{-1}(4x\varepsilon^q))\,dx$$

$$= 4^{-1}\varepsilon^{-q} \int_{\varphi^{-1}(\varepsilon^q)}^{\Phi^{-1}(8\varepsilon^q)} \rho(y)\varphi(y)\,dy. \ \blacksquare$$

REMARK. Unfortunately, we do not know whether or not an estimate similar to (7.9) remains valid in the case of two weights α and σ satisfying (7.1). The crucial point is that in this case estimate (7.10) is no longer valid. For example, take $\varphi(x) = x^{-\gamma}$ for some $\gamma > 1$ and choose the weights to be $\alpha(t) = 2^{|t|/q}$ and $\sigma(t) = |t|^{-\gamma/q}2^{-|t|/q}$ to see that (7.10) is not satisfied in general.

A first application is to moderate trees with polynomial decay of the weight α. It shows that the estimates in Theorem 6.4 are also sharp (at least for one-weight operators and $1 < q \le 2$) for $1 < \gamma \le H + 1$, provided we have the additional assumption $\xi(t) \ge 1$ for $t \in T$.

PROPOSITION 7.6. *Let T be a tree with $\xi(t) \ge 1$ for $t \in T$ such that $R(n) \ge cn^H$. Given $\gamma > 1$ let $\alpha(t) \ge c|t|^{-\gamma/q}$. If $\gamma < H + 1$, then*

$$N(T,d,\varepsilon) \ge c\varepsilon^{-qH/(\gamma-1)} \tag{7.13}$$

Similarly, if $\gamma = H + 1$, then

$$N(T,d,\varepsilon) \ge c\varepsilon^{-q}\log(1/\varepsilon). \tag{7.14}$$

For the operator V_α we have

$$e_n(V_\alpha : \ell_1(T) \to \ell_q(T)) \ge c \begin{cases} n^{-\frac{\gamma-1}{qH}-\frac{1}{q'}}, & \gamma < H+1, \\ n^{-1}(\log n)^{1/q}, & \gamma = H+1. \end{cases}$$

Proof. Of course, $\varphi(x) = c^q x^{-\gamma}$ satisfies condition (7.7). Thus Proposition 7.5 applies and the lower estimates in (7.13) and (7.14) are direct consequences of $\varphi^{-1}(\varepsilon^q) \approx \varepsilon^{-q/\gamma}$ as well as of $\Phi^{-1}(\varepsilon^q) \approx \varepsilon^{-q/(\gamma-1)}$. The estimates for $e_n(V_\alpha)$ now follow from Theorem 5.3. $\blacksquare$

Another application of Proposition 7.5 is to binary trees and polynomial decay of α.

PROPOSITION 7.7. *Let T be a binary tree and suppose $\alpha(t) \ge c|t|^{-\gamma/q}$ for some $\gamma > 1$. Then*

$$\log N(T,d,\varepsilon) \ge c\varepsilon^{-q/(\gamma-1)}.$$

Proof. Again φ satisfies (7.7), hence Proposition 7.5 applies as well and the assertion easily follows from

$$\log \int\limits_{\varphi^{-1}(\varepsilon^q)}^{\Phi^{-1}(8\varepsilon^q)} \rho(y)\varphi(y)\,dy \approx \Phi^{-1}(8\varepsilon^q) \approx \varepsilon^{-q/(\gamma-1)}. \ \blacksquare$$

Combining Proposition 7.7 with Theorem 5.4 leads to the following.

PROPOSITION 7.8. *Let T be a binary tree and suppose that $\alpha(t) \geq c|t|^{-\gamma/q}$ for a certain $\gamma > 1$. Then*

$$e_n(V_\alpha : \ell_1(T) \to \ell_q(T)) \geq c \begin{cases} n^{-1/q'}(\log n)^{1-\gamma/q}, & \gamma > q, \\ n^{-(\gamma-1)/q}, & \gamma \leq q. \end{cases}$$

8. Biased trees. We will now test the sharpness of our bounds on an interesting class of trees whose branches, unlike the case of Proposition 7.6, die out quickly. Let $H \geq 1$. We define a biased tree of order H as follows. Take a binary tree, draw it on the plane so that it grows from the bottom to the top, and for any level $n \geq 0$ keep only the $R(n)$ rightmost nodes where

$$R(n) := \begin{cases} 2^n, & n \leq 2H, \\ n^H, & n > 2H. \end{cases}$$

The set of nodes we have kept is a tree since

$$R(n+1) \leq 2R(n), \qquad n \geq 0.$$

We call this tree a *biased tree* (because it is really biased to the right) of order H and denote it by T_H. Since the size of its nth level for large n is n^H, the biased tree satisfies both the upper and lower size bounds

$$cn^H \leq R(n) \leq Cn^H, \tag{8.1}$$

as in Theorem 6.4 and in Proposition 7.3, respectively. At the same time the nodes situated on large levels die out pretty quickly, which enables more efficient covering constructions than in the general case.

On T_H we will consider the usual one-weight operator V_α defined in (7.5). Recall that $\tilde{N}(T_H, d, \varepsilon)$ stands for the order covering numbers of the tree T_H defined in (3.2). Our main result for biased trees is the following.

PROPOSITION 8.1. *Let V_α be the one-weight operator on T_H with the weight $\alpha(t) = |t|^{-\gamma/q}$, $\gamma > 1$, and let d be the metric on T_H corresponding to this weight. For the related order covering numbers we have*

$$c\varepsilon^{-q(H+1)/\gamma} \leq \tilde{N}(T_H, d, \varepsilon) \leq C\varepsilon^{-q(H+1)/\gamma}. \tag{8.2}$$

For the entropy numbers of V_α we have

$$cn^{-\gamma/q(H+1)-1/q'} \leq e_n(V_\alpha) \leq Cn^{-\gamma/q(H+1)-1/q'}. \tag{8.3}$$

This shows that the lower bound (7.4) of Section 7 cannot be improved in general, unless we make some extra assumptions about the tree, as in Proposition 7.6.

We also see from this bound that the upper estimates for order covering numbers obtained in Proposition 6.3 and those for entropy numbers obtained in Theorem 6.4 are not sharp for certain trees in the convergent case $\gamma < H+1$ and in the intermediate case $\gamma = H+1$, while the results of Sections 6 and 7 show that in the divergent case ($\gamma > H+1$) the estimate for the entropy numbers is sharp for *any* tree satisfying (8.1).

Proof of Proposition 8.1. The construction will be based on the same set of levels as in (6.5) but we specify it for our situation. Let

$$\varPhi(y) = \int_y^\infty x^{-\gamma}\,dx = cy^{-(\gamma-1)}. \tag{8.4}$$

We will use the following elementary property: for any positive integers $n < m$,

$$\sum_{k=n+1}^{m} k^{-\gamma} \le \int_n^m x^{-\gamma}\,dx = \varPhi(n) - \varPhi(m). \tag{8.5}$$

Given $\varepsilon \in (0,1)$, let $J = [\varepsilon^{-q/\gamma}]$ and define a decreasing sequence $(n_j)_{1\le j\le J}$ of integers by

$$n_j := \inf\{n \in \mathbb{N} : \varPhi(n) \le j\varepsilon^q\}.$$

We also let $n_0 := +\infty$ for uniformity of further writing. By (8.4) we have

$$n_j \le C(j\varepsilon^q)^{-1/(\gamma-1)}.$$

In particular, we have

$$n_J \le C\varepsilon^{-q/\gamma}. \tag{8.6}$$

Now we define our order net to be $S_\varepsilon := S_\varepsilon^1 \cup S_\varepsilon^2$, where $S_\varepsilon^1 := \{s : |s| < n_J\}$ and

$$S_\varepsilon^2 := \bigcup_{j=1}^{J} S_{\varepsilon,j},$$

where $S_{\varepsilon,j}$ consists of the first $\nu_j := \min\{c_*\varepsilon^{-qH/\gamma}, R(n_j)\}$ rightmost nodes of the level n_j. The large constant c_* will be specified later. Recall that in the construction used to prove Proposition 6.1 we included the entire levels in the net (see (6.6)). Due to the structure of the biased tree, only a small part of the level suffices, thus the net is more efficient.

The size of the net is bounded by

$$\#S_\varepsilon \le \sum_{n=1}^{n_J} R(n) + \sum_{j=1}^{J} \nu_j \le C \sum_{n=1}^{n_J} n^H + J \cdot c_* \varepsilon^{-qH/\gamma}$$
$$\le C n_J^{H+1} + J \cdot c_* \varepsilon^{-qH/\gamma} \le C \varepsilon^{-q(H+1)/\gamma}$$

in view of (8.6) and by the definition of J.

In order to evaluate the precision of the net, we will use the following structural property of the biased tree.

LEMMA 8.2. *Let $j \le J$ and let $s \in T$ be such that $|s| \ge n_j$. Then there exists a $t \in S_{\varepsilon,j+1}$ such that $t \prec s$.*

Proof. First of all, notice that it is enough to consider the case $|s| = n_j$. Indeed, for any s with $|s| \ge n_j$ we find s' satisfying $|s'| = n_j$ and $s' \preceq s$. Once the lemma is proved for s', we find an appropriate $t \in S_{\varepsilon,j+1}$ for s' and conclude from $t \prec s' \preceq s$ that $t \prec s$.

So assume that $|s| = n_j$. Look at $\nu_{j+1} = \min\{c_*\varepsilon^{-qH/\gamma}, R(n_{j+1})\}$. If $\nu_{j+1} = R(n_{j+1})$, this means that $S_{\varepsilon,j+1}$ coincides with the entire n_{j+1}th level of T_H. Then of course there exists $t \in S_{\varepsilon,j+1}$ such that $t \prec s$.

On the other hand, if $\nu_{j+1} = c_*\varepsilon^{-qH/\gamma}$, then our statement reduces to the numerical inequality

$$c_*\varepsilon^{-qH/\gamma} \cdot 2^{n_j - n_{j+1}} \ge \tilde{C} n_j^H. \tag{8.7}$$

Here the left hand side is the total number of offsprings of elements in $S_{\varepsilon,j+1}$ counted on the n_jth level of the initial binary tree, and the right hand side is an upper bound for the size $R(n_j)$ of the n_jth level in T_H.

It follows from the definition of n_j that $n_j \sim c(j\varepsilon^q)^{-1/(\gamma-1)}$, hence

$$n_{j+1} - n_j \ge c\varepsilon^{-\frac{q}{\gamma-1}} j^{-(1+\frac{1}{\gamma-1})} \ge c\varepsilon^{-\frac{q}{\gamma-1}} (n_j^{\gamma-1}\varepsilon^q)^{1+\frac{1}{\gamma-1}} := c_1 n_j^\gamma \varepsilon^q.$$

If c_* is large enough, then for any $x \ge 0$ we have $2^{c_1 x} \ge c_*^{-1}\tilde{C} x^{H/\gamma}$. By letting here $x = n_j^\gamma \varepsilon^q$ we obtain

$$2^{c_1 n_j^\gamma \varepsilon^q} \ge \tilde{C} c_*^{-1} n_j^H \varepsilon^{qH/\gamma},$$

hence

$$c_*\varepsilon^{-qH/\gamma} \cdot 2^{n_j - n_{j+1}} \ge c_*\varepsilon^{-qH/\gamma} \cdot 2^{c_1 n_j^\gamma \varepsilon^2} \ge c_*\varepsilon^{-qH/\gamma} \cdot \tilde{C} c_*^{-1} n_j^H \varepsilon^{qH/\gamma} = \tilde{C} n_j^H,$$

and (8.7) follows. ∎

Now the precision of the net is easy to establish. Recall that if $t \prec s$ then

$$d(t,s)^q = \sum_{t \prec\, r \preceq s} \alpha(r)^q = \sum_{|t|<k\le|s|} k^{-\gamma}. \tag{8.8}$$

Next, if $s \notin S_\varepsilon$, only the following two cases are possible.

1) $|s| > n_1$. Apply Lemma 8.2 with $j = 1$. We find $t \in S_{\varepsilon,2}$ such that $t \prec s$. Then by (8.8) and (8.5) we have

$$d(t,s)^q \le \sum_{k=n_2+1}^{\infty} k^{-\gamma} \le \Phi(n_2) \le 2\varepsilon^q.$$

2) $n_j < |s| < n_{j-1}$ for some $2 \le j \le J$. Apply Lemma 8.2 with j. We find a $t \in S_{\varepsilon,j+1}$ such that $t \prec s$. Then by (8.8) and (8.5) we have

$$d(t,s)^q \le \sum_{k=n_{j+1}+1}^{n_{j-1}-1} k^{-\gamma} \le \Phi(n_{j+1}) - \Phi(n_{j-1}-1) \le 2\varepsilon^q.$$

Therefore for any $s \in T_H$ we have a $t \in S_\varepsilon$ such that $t \preceq s$ and $d(t,s) \le 2^{1/q}\varepsilon$. Taking into account the bound for $\#S_\varepsilon$, we see that $\tilde{N}(T_H,d,\varepsilon) \le C\varepsilon^{-q(H+1)/\gamma}$.

For the lower bound, take any distinct $s,t \in S^1_\varepsilon$, that is, $|s| < n_J$, $|t| < n_J$. Then

$$d(s,t) \ge \max\{\alpha(s),\alpha(t)\} \ge (n_J)^{-\gamma/q} \ge c\varepsilon,$$

while the number of points we consider is bounded from below by

$$\#S^1_\varepsilon = \sum_{n=1}^{n_J-1} R(n) \ge c \sum_{n=1}^{n_J-1} n^H \ge c(n_J-1)^{H+1} \ge c\varepsilon^{-q(H+1)/\gamma}.$$

It follows that

$$\tilde{N}(T_H,d,\varepsilon) \ge N(T_H,d,\varepsilon) \ge c\varepsilon^{-q(H+1)/\gamma},$$

as required in (8.2). For the entropy numbers, the upper bound in (8.3) follows from the upper bound in (8.2) via Theorem 4.4, while the lower bound in (8.3) was proved in the more general context of Proposition 7.3 (see (7.4)). ■

9. A probabilistic application. Due to the well known relations between the entropy of operators on Hilbert spaces and small deviation probabilities of Gaussian random functions, our results have immediate probabilistic consequences. Thus regard $V_{\alpha,\sigma}$ as an operator from $\ell_1(T)$ into $\ell_2(T)$. Its dual $V^*_{\alpha,\sigma}$ maps $\ell_2(T)$ into $\ell_\infty(T)$, hence it generates a Gaussian random function $X = (X_t)_{t\in T}$ by

$$X_t := \sum_{r\in T} \xi_r (V^*_{\alpha,\sigma}\delta_r)(t) = \sigma(t) \sum_{r\preceq t} \alpha(r)\xi_r$$

where $\{\xi_r : r \in T\}$ is a family of independent $\mathcal{N}(0,1)$-distributed random variables. The covariance structure of X is given by

$$\mathbb{E}X_tX_s = \sigma(t)\sigma(s) \sum_{r\preceq t\wedge s} \alpha(r)^2, \quad t,s \in T.$$

Such summation schemes on trees are extensively studied and applied: see e.g. the literature on Derrida random energy model [BK] or displacements in random branching walks [Pe], to mention just a few.

As a consequence of our results we get the following.

PROPOSITION 9.1. *Suppose $N(T,d,\varepsilon) \approx \varepsilon^{-a}|\log \varepsilon|^b$, for some $a > 0$, $b \ge 0$. Then*

$$-\log \mathbb{P}\Big(\sup_{t\in T} |X_t| < \varepsilon\Big) \approx \varepsilon^{-a}|\log \varepsilon|^b.$$

Proof. An application of Theorem 5.3 implies

$$e_n(V_{\alpha,\sigma} : \ell_1(T) \to \ell_2(T)) \approx n^{-1/a-1/2}(\log n)^{b/a}.$$

Next, duality results for entropy numbers (cf. [TJ]) lead to

$$e_n(V^*_{\alpha,\sigma} : \ell_2(T) \to \ell_\infty(T)) \approx n^{-1/a-1/2}(\log n)^{b/a}$$

as well. Recall that $V^*_{\alpha,\sigma}$ generates X, hence we may apply the classical Kuelbs–Li result (see [KL] or [LiL]) to obtain

$$-\log \mathbb{P}\Big(\sup_{t\in T} |X_t| < \varepsilon\Big) \approx \varepsilon^{-a}|\log \varepsilon|^b. \ \blacksquare$$

REMARK. By the same methods one shows that $N(T,d,\varepsilon) \le c\varepsilon^{-a}|\log \varepsilon|^b$ yields

$$-\log \mathbb{P}\Big(\sup_{t\in T} |X_t| < \varepsilon\Big) \le c\varepsilon^{-a}|\log \varepsilon|^b.$$

Surprisingly, this looks exactly as a special case of a general small deviation result due to M. Talagrand (cf. [Ta] or [Le]). Yet the main difference is that in the cited result one uses the covering numbers with respect to the so-called Dudley distance d_X while our results are based on the metric d defined in (3.1). This suggests that there is maybe some relation between d_X and d. Even if this is the case, it is not obvious.

10. Concluding remarks and open problems. The study of summation operators on trees we merely initiated here is far from being complete. For example, many of our estimates are proven to be sharp only in the range $q \le 2$ while there are gaps for $q > 2$. One possibility to fill these gaps would be to apply the technique used in [C, pp. 91 and 92]. To this end one has to modify the proof of Proposition 5.1 for those q by inscribing Walsh matrices into $V_{\alpha,\sigma}$ instead of the identity Id_m from ℓ_1^m into ℓ_q^m. But at the moment we do not see how to accomplish this.

Moreover, it would also be quite natural to consider operators acting from $\ell_p(T)$ into $\ell_q(T)$ with general $p, q \in [1,\infty]$. However, the technique of convex hulls that we refer to in Section 4 is not appropriate anymore and other tools are needed.

In this context let us mention the following related open question: Given $1 < p, q < \infty$ and a tree T, for which weights α and σ is $V_{\alpha,\sigma}$ a bounded operator from $\ell_p(T)$ into $\ell_q(T)$? To our knowledge this is unknown even if T is a binary tree. Let us briefly recall the answer in the case $T = \mathbb{N}_0$ (cf. [CL] where it was derived from the classical Maz'ya–Rosin Theorem for weighted integration operators). We formulate it only in the case $1 \le p \le q \le \infty$ although the answer is known for all $p, q \in [1, \infty]$.

PROPOSITION 10.1. *If $1 \le p \le q \le \infty$, then $V_{\alpha,\sigma}$ is bounded from $\ell_p(\mathbb{N}_0)$ into $\ell_q(\mathbb{N}_0)$ if and only if*

$$\sup_{v \in \mathbb{N}_0} \|\alpha \mathbf{1}_{[0,v]}\|_q \|\sigma \mathbf{1}_{[v,\infty)}\|_{p'} < \infty.$$

We briefly mention two other problems related to this topic.

(1) Throughout the paper we always assumed σ to be non-increasing. This property was used at several places. For example, it played an important role in the proofs of Propositions 3.1 and 5.1. If σ is not necessarily non-increasing, then the distance d has surely to be modified as

$$\hat{d}(t, s) := \max_{t \prec v \preceq s} \|\alpha \mathbf{1}_{(t,v]}\|_q \|\sigma \mathbf{1}_{[v,s]}\|_\infty$$

whenever $t \preceq s$. Unfortunately, then, in general, $\hat{d}$ can no longer be extended to a metric on T. Nevertheless we believe that some covering properties of T with respect to $\hat{d}$ are closely connected with compactness properties of $V_{\alpha,\sigma}$. At least this is suggested by the known results on compactness and approximation properties of weighted integration operators as proved in [EEH1] or [LL].

(2) A challenging problem is the critical case as treated in Theorem 4.4. Some related partial results are known. For example, in [L] the problem is solved for the binary tree provided that $q = 2$ and $\sigma(t) \equiv 1$. Other results in the critical case we are aware of are based on [CE] and will be handled in a separate publication.

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