An interplay between the weak form of Peano's theorem and structural aspects of Banach spaces

by

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Abstract. We establish some results that concern the Cauchy–Peano problem in Banach spaces. We first prove that a Banach space contains a nontrivial separable quotient iff its dual admits a weak*-transfinite Schauder frame. We then use this to recover some previous results on quotient spaces. In particular, by applying a recent result of Hájek–Johanis, we find a new perspective for proving the failure of the weak form of Peano's theorem in general Banach spaces. Next, we study a kind of algebraic genericity for the weak form of Peano's theorem in Banach spaces E having complemented subspaces with unconditional Schauder basis. Let $\mathcal{K}(E)$ denote the family of all continuous vector fields $f \colon E \to E$ for which $u' = f(u)$ has no solutions at any time. It is proved that $\mathcal{K}(E) \cup \{0\}$ is spaceable in the sense that it contains a closed infinite-dimensional subspace of $C(E)$, the locally convex space of all continuous vector fields on E with the linear topology of uniform convergence on bounded sets. This yields a generalization of a recent result proved for the space c_0. We also introduce and study a natural notion of weak-approximate solutions for the nonautonomous Cauchy–Peano problem in Banach spaces. It is proved that the absence of ℓ_1-isomorphs inside the underlying space is equivalent to the existence of such approximate solutions.

1. Introduction. The classical Peano theorem [P] states that if E is an n-dimensional Euclidean space and $f \colon E \to E$ is a continuous vector field, then the ODE $u' = f(u)$ has a solution. It is natural to ask if the same result can be proved when E is infinite-dimensional. The answer in general is no, and the first negative result was obtained by Dieudonné [D] in 1954 when he exhibited a counterexample for the space c_0. This gave rise to a fascinating research line and several researchers have provided invaluable information on this topic (see, for instance, [A, C, G, L, Sz] and references therein). In 1974 Godunov [G] settled the question when Peano's theorem is valid for general Banach spaces. More precisely, he proved that Peano's theorem remains true

for a Banach space E iff E is finite-dimensional. Generalizations of this result in the context of locally convex and nonnormable Fréchet spaces were given by Astala [A], Lobanov [L], Shkarin [Sh1] and others. Nonetheless, some existence results have been derived by using nonnormable linear topologies (cf. [Sz, T]). A little more than ten years ago, a new trend appeared which aims at studying relationships between geometric and structural aspects of Banach spaces and the weak form of Peano's theorem (cf. [HJ, Sh2]). The weak form of Peano's theorem (WFPT, for short) states that if E is finite-dimensional and $f : \mathbb{R} \times E \to E$ is continuous, then $u' = f(t, u)$ has a solution in some open interval. In [Sh2], Shkarin proved that if E is a Banach space containing a complemented subspace with an unconditional Schauder basis, then WFPT fails to be true. Hájek and Johanis [HJ] extended this result to the class of Banach spaces with an infinite-dimensional separable quotient. In fact, they showed more: There exist continuous vector fields $f : E \to E$ such that the differential equation $u' = f(u)$ has no solutions at any point (cf. [HJ, Theorem 8]). They also raised the question of whether this result remains true in any Banach space.

Henceforth, E will denote a real infinite-dimensional Banach space. The first two main results of the present research concern WFPT. The first focuses on relationships between separable quotients and weak*-transfinite Schauder frames.

DEFINITION 1.1. Let ξ be an ordinal number. A transfinite sequence $(f_\alpha)_{\alpha < \xi}$ in E^* is called a *weak*-transfinite Schauder frame* in E^* if:

(i) For every $y^* \in \overline{\mathrm{span}}^{w^*}\{f_\alpha : \alpha < \xi\}$ there exists a transfinite sequence of scalars $(a_\alpha(y^*))_{\alpha < \xi} \in \ell_\infty(\xi)$ such that $y^* = \sum_{\alpha < \xi} a_\alpha(y^*) f_\alpha$, with convergence in the weak* topology $\sigma(E^*, E)$, i.e., for each $x \in E$ we have
$$\lim_{\alpha \to \xi} \Big\langle \sum_{\gamma = 0}^{\alpha} a_\gamma(y^*) f_\gamma, x \Big\rangle = \langle y^*, x \rangle.$$

(ii) $(f_\alpha)_{\alpha < \xi}$ admits a biorthogonal transfinite sequence $(e_\alpha)_{\alpha < \xi}$ in E.

It is easy to see that weak*-transfinite Schauder frames include weak*-Schauder basic sequences (as in [JR, Definition II.1]). Thus, in this case, both definitions are consistent. It is also worth noticing that a weak*-transfinite Schauder frame need not be a weak*-Schauder basic sequence (cf. [Sin, Example 14.1]).

THEOREM 1.2. *E has a nontrivial separable quotient iff E^* has a weak*-transfinite Schauder frame.*

This result was first proved by Johnson–Rosenthal [JR] in the context of weak*-Schauder basic sequences—and, in this case, it is also a consequence

of Theorem 3 in [Slw]. The proof will be given in Section 2 and makes use of some techniques related to barrelled spaces and the ℓ_1-lifting property. After getting a glimpse of some related topics, we will also discuss some consequences and we will revisit known results. As a general comment, Theorem 1.2 and [HJ, Theorem 8] together yield the following corollary which seems to bring at the very least a new slant on Hájek–Johanis's problem.

COROLLARY 1.3. *Assume E^* has a weak*-transfinite Schauder frame. Then there exists a continuous vector field $f: E \to E$ such that $u' = f(u)$ has no solutions at any point.*

Our second main result concerns the analysis of a kind of genericity of solutions for the weak form of Peano's theorem. Let $C(E)$ be the locally convex space of all continuous vector fields on E endowed with the linear topology $\mathcal{T}_{\mathrm{uc}}$ of uniform convergence on bounded sets. Denote by $\mathcal{K}(E)$ the family of all vector fields f in $C(E)$ for which $u' = f(u)$ does not have solutions at any time. The central question we address here is: Does $\mathcal{K}(E) \cup \{0\}$ have an infinite-dimensional $\mathcal{T}_{\mathrm{uc}}$-closed vector space? This leads to the following definition of algebraic genericity.

DEFINITION 1.4. A property (P) is said to be *algebraically generic* for $\mathcal{K}(E)$ if $\mathcal{K}(E) \cup \{0\}$ contains an infinite-dimensional $\mathcal{T}_{\mathrm{uc}}$-closed vector space L such that (P) holds for all nonzero vector fields in L.

We obtain the following.

THEOREM 1.5. *Assume that E contains a complemented subspace with an unconditional Schauder basis. Then nonvalidity of WFPT is algebraically generic for $\mathcal{K}(E)$.*

Inspiration for this result comes from an idea contained in several works highlighting the usefulness of techniques involving lineability and spaceability (see, for instance, [AGS, APPS, BBFP] as well as references therein). As a matter of fact, a result of the same genre was previously obtained in [BBFP, Theorem 2.1] for $E = c_0$. Theorem 1.5 provides a generalization of that result. The proof given here borrows some ideas from [BBFP], but the details are realized in a different and rather simple way. The basic task is to obtain a countable family $\{f_n\}$ of linearly independent vector fields on E such that the system $u' = f_n(u)$, treated as an uncoupled system of ODEs in E, has no solution at any time. In the process, a special role is played by a result of Shkarin [Sh2] concerning Osgood's Theorem in Banach spaces having complemented subspaces with unconditional basis. It further furnishes the uniform continuity of the family $\{f_n\}$, a crucial stepping stone to fully proving the result.

For our third aim, we consider the nonautonomous Cauchy–Peano problem

(1.1) $u' = f(t, u), \quad t \in I, \quad \text{and} \quad u(0) = u_0 \in E,$

where $I = [0, T]$. Here $f \colon [0, T] \times E \to E$ is a *Carathéodory vector field*, i.e., one with the following properties:

(f$_1$) for all $t \in I$, $f(t, \cdot) \colon E \to E$ is continuous,
(f$_2$) for all $x \in E$, $f(\cdot, x) \colon I \to E$ is measurable.

We introduce a special notion of weak-approximate solutions (WAS) to the problem (1.1) (see Definition 1.6). Our main result on this topic provides, for a large class of Carathéodory vector fields, a characterization of the existence of WAS in terms of ℓ_1-containments. In what follows, we describe the class of fields that will be addressed here. Let $I \subset \mathbb{R}$ be as above and denote by $\mathfrak{X}(I, E)$ the family of all Carathéodory vector fields $f \colon I \times E \to E$ fulfilling the following growth condition:

($\star$) $\|f(s, x)\|_E \le \alpha(s)\varphi(\|x\|_E)$ for a.e. $s \in I$ and every $x \in E$, where α and φ have the following properties:

  (a) $\alpha \in L_1[0, T]$;
  (b) $\varphi \colon [0, \infty) \to (0, \infty)$ is a nondecreasing continuous function such that
  $$\int_0^T \alpha(s)\, ds < \int_0^\infty \frac{ds}{\varphi(s)}.$$

DEFINITION 1.6 (WAS). Let $f \in \mathfrak{X}(I, E)$ be given. We say a sequence $(u_n) \subset C(I, E)$ of continuous E-valued functions on I is a *weak-approximate solution* of (1.1) if the following holds:

  (i) each u_n is almost everywhere strongly differentiable in I,
  (ii) both (u_n) and (u'_n) are bounded sequences in $C(I, E)$,
  (iii) (u_n) is weakly Cauchy and $u_n(t) - \int_0^t f(s, u_n(s))\, ds \rightharpoonup u_0$ in $C(I, E)$ for all $t \in I$,
  (iv) $u_n(t) - u_0 \in \overline{\text{span}}(f(I \times E))$ for all $t \in I$ and $n \in \mathbb{N}$.

We can now state our third main result.

THEOREM 1.7. *Problem* (1.1) *always has WAS for any* $f \in \mathfrak{X}(I, E)$ *if and only if* E *contains no subspace isomorphic to* ℓ_1.

This is a kind of generalization of [Bar1, Theorem 4.2]. The proof given here relies on three important results from functional analysis: (1) a fundamental characterization of weak compactness in $L_\infty(\mu, E)$ due to Schlüchtermann [Sch]; (2) a characterization of reflexivity due to Cellina [C], and (3) Rosenthal's famous ℓ_1-theorem.

1.1. Preliminary notation and definitions. Before starting the proofs, we fix some basic notations and recall the necessary definitions. All of the Banach spaces we consider are over the reals. Given a Banach space

X with norm $\|\cdot\|_X$, we denote the closed unit ball of X by B_X. If $A \subset X$ is nonempty, $\overline{A}$ and $\operatorname{span} A$ stand for the closure and the linear span of A, respectively. Subspaces of Banach spaces are understood to be closed infinite-dimensional subspaces. As usual, we will write $X \approx Y$ to indicate that X and Y are linearly isomorphic spaces. A Banach space X is said to *have an isomorphic copy of* ℓ_1 if it contains a basic sequence (x_n) spanning a subspace isomorphic to ℓ_1. This is in turn equivalent to the existence of constants $C_1, C_2 > 0$ such that for all scalars $t_1, \ldots, t_n$,

$$C_1^{-1} \sum_{i=1}^n |t_i| \leq \Big\| \sum_{i=1}^n t_i x_i \Big\|_X \leq C_2 \sum_{i=1}^n |t_i|.$$

Recall that a series $\sum x_n$ in a Banach space X is *unconditionally convergent* if $\sum \epsilon_n x_n$ converges for all choices of signs $\epsilon_n = \pm 1$. A basic sequence (x_n) is said to be *unconditional* if for every $x \in \overline{\operatorname{span}}\{x_n \colon n \in \mathbb{N}\}$, its expansion $x = \sum a_n x_n$ converges unconditionally. The reader is referred to [F–Z] for more background in Banach space theory.

We will need the following well-known result (cf. [F–Z, Proposition 5.10]):

PROPOSITION 1.8. *ℓ_1 has the lifting property.*

Let us recall that a Banach space X is said to have the *lifting property* if for all Banach spaces Y, Z such that there is an onto bounded linear operator $S \colon Y \to Z$ and for every bounded linear operator $T \colon X \to Z$, there is another bounded linear operator $\tilde{T} \colon X \to Y$ such that $T = S \circ \tilde{T}$.

To end this section, we recall a few known definitions from topological vector space theory. Let (X, τ) be a Hausdorff locally convex space. A subset U of X is called a *barrel* in X if it is closed, absolutely convex and absorbing in X. Further, X is said to be a *barrelled space* if every barrel in X is a neighborhood of 0 in X. For example, it is well-known that every Baire locally convex space is barrelled (cf. [J, p. 220]). A linear map $S \colon X \to Y$ between topological vector spaces is called a *closed linear map* if it has closed graph in $X \times Y$ (cf. [J, p. 92]).

2. Proof of Theorem 1.2. Two important ingredients will be helpful in the proof. The first one is the following result whose proof can be found in [PB, Theorem 4.1.10] (see also [J, Theorem 8, p. 221]).

THEOREM 2.1 (Closed graph theorem for barrelled spaces). *Let X be a barrelled space, Y a Fréchet space and $S \colon X \to Y$ a linear map with closed graph in $X \times Y$. Then S is continuous.*

The second one deals with a characterization of spaces having separable quotient in terms of barrelled subspaces (cf. [Mu, Theorem 3.2, p. 313] and [W, Theorem 1, p. 255]).

THEOREM 2.2. *Let $(X, \|\cdot\|)$ be a Banach space. Then $(X, \|\cdot\|)$ has an infinite-dimensional separable quotient if and only if $(X, \|\cdot\|)$ has a nonbar-relled proper dense subspace.*

These results will be used below to prove a preliminary lemma concerning separable quotients and ℓ_1-fundamental biorthogonal systems.

DEFINITION 2.3. A biorthogonal system $\{x_\alpha; x_\alpha^*\}_{\alpha \in \Gamma} \subset E \times E^*$ is called ℓ_1-*fundamental* if the linear space $\{x \in E \colon \sum_{\alpha \in \Gamma} |x_\alpha^*(x)| \, \|x_\alpha\| < \infty\}$ is norm-dense in E.

It is clear that every fundamental biorthogonal system is ℓ_1-fundamental.

LEMMA 2.4. *If $\{x_\alpha; x_\alpha^*\}_{\alpha \in \Gamma}$ is an ℓ_1-fundamental biorthogonal system in $E \times E^*$, then E has a nontrivial separable quotient.*

Proof. Let $\|\cdot\|$ denote the norm of E. For contradiction, suppose that $(E, \|\cdot\|)$ does not have any nontrivial infinite-dimensional separable quotient. Then two conclusions can be drawn. First, by [Mu, Theorem 4.1, p. 317], ℓ_1 is not isomorphic to a subspace of $(E, \|\cdot\|)$. Second, from Theorem 2.2 we conclude that every proper dense subspace of $(E, \|\cdot\|)$ is barrelled. Let now Z denote the subspace of $(E, \|\cdot\|)$ formed by those $x \in E$ for which

$$\sum_{\alpha \in \Gamma} |x_\alpha^*(x)| \, \|x_\alpha\| < \infty.$$

By assumption, $(Z, \|\cdot\|)$ is dense in $(E, \|\cdot\|)$. Of course, we may assume that $Z \neq E$. Then by the second conclusion above, $(Z, \|\cdot\|)$ is barrelled. We will use this fact to show that the linear mapping $S \colon (Z, \|\cdot\|) \to (\ell_1(\Gamma), \|\cdot\|_{\ell_1(\Gamma)})$ given, for each $x \in Z$, by

$$S(x) = (x_\alpha^*(x)\|x_\alpha\|)_{\alpha \in \Gamma},$$

is continuous. Suppose that $u_k \to u$ in $(Z, \|\cdot\|)$ and $S(u_k) \to v$ in $(\ell_1(\Gamma), \|\cdot\|_{\ell_1(\Gamma)})$. Write $v = (v_\alpha)_{\alpha \in \Gamma}$ and fix $\gamma \in \Gamma$. Notice that for all $k \in \mathbb{N}$,

$$\left| x_\gamma^*(u_k)\|x_\gamma\| - v_\gamma \right| \leq \sum_{\alpha \in \Gamma} \left| x_\alpha^*(u_k)\|x_\alpha\| - v_\alpha \right| = \|S(u_k) - v\|_{\ell_1(\Gamma)}.$$

Thus, letting $k \to \infty$ we obtain $x_\gamma^*(u)\|x_\gamma\| = v_\gamma$. Since γ was arbitrary, this implies that $S(u) = v$, so S has closed graph in $(Z, \|\cdot\|) \times (\ell_1(\Gamma), \|\cdot\|_{\ell_1(\Gamma)})$; hence, by Theorem 2.1, it is continuous, as desired.

As S is bounded, it can be linearly extended to the whole space E. This extension will be denoted by S too. Since S is a noncompact operator, it is standard to check that $S(B_E)$ contains a seminormalized sequence (x_j) in $\ell_1(\Gamma)$ which is equivalent to the unit basis of ℓ_1. By the lifting property of ℓ_1 (cf. Proposition 1.8), the formal inverse S^{-1} from the span of $\{x_j\}$ back to E is bounded, and so it is really the inverse linear operator. Clearly, the image $S^{-1}(x_j)$ is a sequence equivalent to the unit basis of ℓ_1 again. Thus E

contains an isomorphic copy of ℓ_1. But this contradicts the first conclusion stated at the beginning of the proof, and hence the proof of the lemma is finished. ∎

Proof of Theorem 1.2. If E has a nontrivial separable quotient, then by [Slw, Theorem 3] we conclude that E^* contains a weak* basic sequence. Conversely, assume $(f_\alpha)_{\alpha < \xi}$ is a weak*-transfinite Schauder frame in E^*. Let

$$X := \mathtt{span}\{e_\alpha : \alpha < \xi\}, \quad Y := \overline{\mathtt{span}}^{w^*}\{f_\alpha : \alpha < \xi\}.$$

From Definition 1.1, it is straightforward to see that $X^\top \cap Y = \{0\}$. Since $(X + Y_\perp)^\top = (X \cup Y_\perp)^\top = X^\top \cap Y$, we infer that $X + Y_\perp$ is norm-dense in E. In particular,

$$Z = \left\{ x \in E : \sum_{\alpha < \xi} |f_\alpha(x)| \, \|e_\alpha\| < \infty \right\}$$

is norm-dense in E. The result now follows from Lemma 2.4. ∎

2.1. Some remarks and known results on separable quotients. The Separable Quotient Problem (SQP) asks whether every Banach space E has a nontrivial separable quotient, i.e. a closed infinite-dimensional subspace M so that E/M is linearly isomorphic to a separable infinite-dimensional Banach space. This still open problem was formulated at different times by Banach and Pełczyński. Many special spaces (e.g., separable or reflexive spaces) are known to have nontrivial separable quotients. In fact, there are a number of important works on this subject, including that of Johnson–Rosenthal [JR], Hagler–Johnson [HaJ], Rosenthal [R1] and Argyros–Dodos–Kanellopoulos [ADK]. We also refer the reader to Mujica's article [Mu] for a rich and systematic survey on the matter.

Quite recently, Śliwa [Slw] characterized Banach spaces which have separable quotients as those whose duals have weak*-Schauder basic sequences, or equivalently, those which admit strongly normal sequences. Let us recall that a normalized sequence $(x_n^*) \subset E^*$ is called *strongly normal* if the linear space $\{x \in E : \sum_{n=1}^\infty |x_n^*(x)| < \infty\}$ is norm-dense in E. Using the techniques developed in [JR], Śliwa proved that strongly normal sequences admit weak* basic subsequences (cf. [Slw, Theorem 1]). Furthermore, he established the existence of strongly normal sequences in the dual space of every WCG Banach space.

Given the bond between WFPT and SQP, it may be important to highlight some facts, including known results, concerning the existence of separable quotients, as well as the existence of weak* basic sequences. We start with the following simple consequence of Lemma 2.4, which is due to Plichko [Pl2, Pl3].

COROLLARY 2.5. *Every real Banach space having a fundamental biortho-gonal system has a nontrivial separable quotient.*

REMARK. It is worth noting that a wide class of Banach spaces admit fundamental biorthogonal systems, including $\ell_\infty(\Gamma)$, WLD, WCD and WCG-spaces (cf. [Z]).

PROPOSITION 2.6. *Let E be a Banach space whose dual contains a subspace with separable dual. Then E^* has a weak* basic sequence.*

Proof. Let Y be a subspace of E^* with separable dual Y^*. Then there exists a normalized sequence $(u_n^*) \subset Y \subset E^*$ which is weakly null. Since Y^* is separable, by a result of Johnson–Rosenthal [JR, Theorem III.3], we conclude that (u_n^*) contains a (shrinking) weak* basic subsequence. ∎

REMARK 2.7. Observe that E^* has a subspace with separable dual provided that E^* is Asplund, contains a reflexive subspace or an isomorphic copy of c_0. Proposition 2.6 is a well-known partial answer to SQP (see [JR, Remark III.3]). It seems to be nearly best possible, for if E^* has a subspace isomorphic to ℓ_1 then E contains a quotient isomorphic to c_0 or ℓ_2 (cf. [Mu, Theorem 4.2]).

The following provides quantitative information on spaces whose duals have unconditional basic sequences.

PROPOSITION 2.8. *Let E be a Banach space whose dual has an unconditional basic sequence. Then one of the following holds:*

(i) *E contains a subspace isomorphic to ℓ_1.*
(ii) *E^* contains a subspace with separable dual or an isomorphic copy of ℓ_1.*

Proof. This follows from a classical result of James [F–Z, Corollary 6.36]. Indeed, let (u_n) be an unconditional basic sequence in E^*. Let $R := \overline{\text{span}}\{u_n : n \in \mathbb{N}\}$. Then either R is reflexive, contains a subspace isomorphic to ℓ_1, or contains a subspace isomorphic to c_0. Assume that (i) does not hold. Then, by a result of Bessaga–Pełczyński [HaJ, Theorem 4], E^* does not contain an isomorphic copy of c_0. Hence, either E^* contains a separable, reflexive space whose dual is separable, or it contains an isomorphic copy of ℓ_1. ∎

The following direct consequence of the above proposition was originally proved by Hagler and Johnson [HaJ]. An alternative proof was given in [ADK].

COROLLARY 2.9. *If E^* has an unconditional basic sequence, then E has a nontrivial separable quotient.*

We mention that Argyros, Dodos and Kanellopoulos [ADK] showed that the bi-dual of a separable Banach space with nonseparable dual admits an unconditional basic sequence. Using this fact and the above results, we are also able to offer a slightly different proof of the following result first proved in [ADK]:

COROLLARY 2.10. *E^* has a nontrivial separable quotient.*

Proof. If E^* has the Radon–Nikodym property, then E is Asplund. Thus E^* has a fundamental biorthogonal system (cf. [Z, Theorem 7.13]). By Lemma 2.4, E^* has a separable quotient. Assume that E^* does not have the Radon–Nikodym property. Then E^{**} contains an uncondtional basic sequence. By Corollary 2.9, E^* has a nontrivial separable quotient. ∎

REMARK 2.11. Note that if a continuous linear map maps E onto a dual Banach space then, by Corollary 2.10, E has a nontrivial separable quotient.

The above leads to the following problem.

PROBLEM 2.12. *Does the assertion of Theorem* 1.1 *still hold if condition* (ii) *of Definition* 1.1 *is dropped?*

3. The algebraic genericity of the failure of WFPT. In this section we establish the algebraic genericity of the failure of the weak form of Peano's theorem in Banach spaces having complemented subspaces with unconditional Schauder basis.

3.1. Proof of Theorem 1.5. By assumption, there exists a complemented subspace X of E having an unconditional Schauder basis $\{e_n; e_n^*\}_{n=1}^\infty$. As X is complemented, there exists a bounded linear projection P of E onto X (cf. [F–Z]). Split $\mathbb{N}$ into $\mathbb{N} = \bigcup_{i \geq 1} \mathbb{N}_i$, where each $\mathbb{N}_i$ is infinite and $\mathbb{N}_i \cap \mathbb{N}_j = \emptyset$ if $i \neq j$. We use the convention $\mathbb{N}_0 = \mathbb{N}$. Let $X_i = \overline{\text{span}}\{e_n : n \in \mathbb{N}_i\}$. We then define, for each $i \in \mathbb{N}$, the ith projection π_i from X into X_i by

$$(3.1) \qquad \pi_i(x) = \sum_{n=1}^\infty (e_{\mathbb{N}_i}^*)_n(x) e_n, \quad x \in X$$

where $(e_{\mathbb{N}_i}^*)_n(x) = e_n^*(x)$ if $n \in \mathbb{N}_i$, and 0 otherwise. Since $\{e_n : n \in \mathbb{N}_i\}$ is an unconditional Schauder basis for X_i (see [F–Z, Proposition 6.31]), $\pi_i(x)$ is well-defined and $\sup_{i \in \mathbb{N}} \|\pi_i\| < \infty$. Moreover, $X_i = \pi_i(X)$.

The following lemma, whose proof can be found in [Sh2, Corollary 1.5], will be the crucial ingredient in the proof of the algebraic genericity of the failure of the weak form of Peano's theorem on E.

LEMMA 3.1. *Let X be a Banach space with a complemented subspace which has an unconditional basis. Then for any $\alpha \in (0,1)$ and $\epsilon > 0$, there exists $f : X \to X$ such that*

  (i) $\|f(x)\| \leq 2$ *for all* $x \in X$,

  (ii) $\|f(x) - f(y)\| \leq \epsilon \|x - y\|^{\alpha}$ *for all* $x, y \in X$,

 (iii) *the equation* $u' = f(u)$ *has no solutions in any interval of the real line.*

Fix $\alpha \in (0, 1)$ and let $\epsilon = 1$. Now according to Lemma 3.1, for each $i \in \mathbb{N}$ there exists a vector field $f_i \colon X_i \to X_i$ with properties (i)–(iii) above. Let $h_i \colon E \to E$ be the continuous vector field given by

$$h_i(x) = f_i(\pi_i(Px)), \quad x \in E.$$

For each $(a_n) \in \ell_1$ we define another vector field $f_{(a_n)}$ on E by putting

$$f_{(a_n)}(x) = \sum_{i=1}^{\infty} a_i h_i(x), \quad x \in E.$$

A direct computation shows that $f_{(a_n)} \equiv 0$ iff $(a_n) = 0$, and

$$\|f_{(a_n)}(x) - f_{(a_n)}(y)\| \leq \left(\sup_{i \in \mathbb{N}} \|\pi_i\| \, \|P\| \right)^{\alpha} \|(a_n)\|_{\ell_1} \|x - y\|^{\alpha}, \quad \forall x, y \in E.$$

Using the same reasoning we can also prove the following proposition.

PROPOSITION 3.2. *The operator* $T \colon \ell_1 \to C(E)$ *given by* $T((a_n)) = f_{(a_n)}$ *is well-defined and is an injective continuous linear map.*

We conclude that $T(\ell_1)$ is algebraically isomorphic to ℓ_1. We now claim that

$$(3.2) \qquad\qquad\qquad T(\ell_1) \subset \mathcal{K}(E) \cup \{0\}.$$

Indeed, let $(a_n) \in \ell_1 \setminus \{0\}$. Then $a_m \neq 0$ for some $m \geq 1$. Assume to the contrary that $T((a_n)) \notin \mathcal{K}(E)$. Hence we can find an open interval $I \subset \mathbb{R}$ so that

$$(3.3) \qquad\qquad\qquad u'(t) = f_{(a_n)}(u(t))$$

has a solution on I, say u. Define $v \colon I \to X_m$ by

$$v(t) = (\pi_m \circ P)(u(t/a_m)), \quad t \in I.$$

Taking into account that (3.3) is an infinite uncoupled system of ODEs, after projecting and calculating the derivatives, we get

$$v'(t) = f_m(v(t)) \quad \forall t \in I,$$

which contradicts the fact that for the field f_m the WFPT fails in X_m. This concludes the proof of (3.2).

It remains to prove the finer inclusion

$$\overline{T(\ell_1)}^{\,\mathcal{J}_{uc}} \subset \mathcal{K}(E) \cup \{0\}.$$

To see this, let $h \in \overline{T(\ell_1)}^{\,\mathcal{J}_{uc}}$. Then there is a sequence

$$\{x_k\}_{k \in \mathbb{N}} := \{(a_n^k)_{n=1}^{\infty}\}_{k \in \mathbb{N}}$$

in ℓ_1 such that the sequence

$$\left\{ \sum_{n=1}^{\infty} a_n^k f_n(\pi_n(Px)) \right\}_{k \in \mathbb{N}}$$

converges to h uniformly on B_E as $k \to \infty$. From Lemma 3.1(iii), we conclude that $f_i(0) \neq 0$ for all $i \in \mathbb{N}$. In addition, as a simple computation shows, we have $a_i^k \to \pi_i Ph(0)/f_i(0)$ for all i. Let $a_i = \pi_i Ph(0)/f_i(0)$. Then after taking the limit over k, we obtain

$$\pi_i Ph(x) = a_i f_i(\pi_i Px)$$

for all $x \in E$ and every i.

Now suppose for contradiction that some solution of $u'(t) = h(u(t))$ is known at some time t. If we fix any integer $i \geq 1$, then by defining $v(t) = \pi_i P(u(t/a_i))$ we readily obtain $v'(t) = f_i(v(t))$. This contradicts Lemma 3.1, and hence concludes the proof. $\square$

4. Characterization of Banach spaces containing ℓ_1 in terms of WAS for the Cauchy–Peano problem. The celebrated dichotomy theorem of H. Rosenthal [R2] states that every bounded sequence in a Banach space E either has a weak Cauchy subsequence, or contains a subsequence equivalent to the unit vector basis of ℓ_1. In this section, we will use this result, together with a characterization of weak compactness in $L_\infty(\mu, E)$, due to Schlüchtermann [Sch], to prove Theorem 1.7.

4.1. Proof of Theorem 1.7. *Necessity.* Suppose X is a subspace of E which is isomorphic to ℓ_1. Then X is not reflexive and, by a result of Cellina [C], there exist a continuous linear functional $\vartheta \in B_{X^*}$ with $\|\vartheta\| = 1$ and a continuous fixed-point-free map $g \colon B_X \to B_X$ satisfying the equality

$$\langle \vartheta, g(x) \rangle = \tfrac{1}{2}(\langle \vartheta, x \rangle + 1), \quad \forall x \in B_X.$$

Let $G \colon E \to B_X$ be a continuous extension of g to the whole E, with range in B_X. It exists by the classical result of Dugundji [Dug] stating that convex sets in locally convex spaces are absolute retracts.

By following the arguments of [C], we can define a continuous vector field $f_G \colon \mathbb{R} \times E \to E$ by

$$f_G(t, x) = \begin{cases} 2tG(x/t^2), & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Notice that

(4.1) $$\|f_G(t, x)\|_E \leq 2|t|, \quad \forall t \in \mathbb{R}, \forall x \in E.$$

Thus f_G belongs to the class $\mathfrak{X}(\mathbb{R}, E)$ with α, φ given as follows: $\alpha(t) = 2|t|$ and $\varphi \equiv 1$. In [C], Cellina proved that there is no solution for the Cauchy problem

$$(4.2) \qquad \begin{cases} u'(t) = f_G(t, u(t)), \\ u(0) = 0. \end{cases}$$

That is, the integral equation

$$(4.3) \qquad u(t) = \int_0^t f_G(s, u(s)) \, ds$$

has no solutions.

CLAIM 1. (4.2) *does not have a WAS.*

Indeed, suppose to the contrary that (4.2) admits a WAS. Then for some bounded interval $I \subset \mathbb{R}$ containing 0, there is a sequence $(u_n) \subset C(I, E)$ satisfying conditions (i)–(iv) from Definition 1.6. First, since $f_G(I \times E) \subset X$, we see from Definition 1.6(v) that $u_n(t) \in X$ for all $n \in \mathbb{N}$ and $t \in I$. Furthermore, (iii) implies that

$$u_n(t) - \int_0^t f_G(s, u_n(s)) \, ds \rightharpoonup 0 \quad \text{in } X, \ \forall t \in I.$$

Now as $X \approx \ell_1$, it follows that X has Schur's property and is $\sigma(X, X^*)$-sequentially complete. Hence for each t,

$$(4.4) \qquad u_n(t) - \int_0^t f_G(s, u_n(s)) \, ds \to 0 \quad \text{in } X.$$

On the other hand, it is easily seen from Definition 1.6(iv) that each sequence $(u_n(t))_n$ is weakly Cauchy in X. Fix $t \in I$. As X is $\sigma(X, X^*)$-sequentially complete, $(u_n(t))_n$ converges weakly (and so strongly) to some $u(t) \in X$. By the estimate in (4.1) and the Lebesgue Dominated Convergence Theorem, (4.3) implies

$$u(t) = \int_0^t f_G(s, u(s)) \, ds, \quad \forall t \in I.$$

So u belongs to $C(I, E)$ and is a solution of (4.3), a contradiction.

Sufficiency. Informally, the strategy we employ is to show that the mapping $F \colon C(I, E) \to C(I, E)$ given by

$$F(u)(t) = u_0 + \int_0^t f(s, u(s)) \, ds, \quad t \in I,$$

has a weak-approximate fixed point sequence, that is, a sequence (u_n) such that $u_n - F(u_n) \rightharpoonup 0$ in $C(I, E)$. Such a sequence will be a WAS of (1.1). In

order to check this, consider the sets

$$A = \{u \in C(I, E) : \|u(t)\|_E \le b(t) \text{ for a.e. } t \in I\},$$
$$B = \{v \in C(I, E) : v(I) \subset W, \|v(t)\|_E \le \alpha(t)\varphi(b(t)) \text{ for a.e. } t \in I\},$$
$$C = \Big\{u \in A : u(t) = u_0 + \int_0^t \hat{u}(s)\, ds \text{ for a.e. } t \in I \text{ and some } \hat{u} \in B\Big\},$$

where $W = \overline{\text{span}}(f(I \times E))$ and $b \colon [0, \infty) \to \mathbb{R}$ is defined by

$$b(t) = J^{-1}\Big(\int_0^t \alpha(s)\, ds\Big), \quad t \ge 0,$$

where $J(z) = \int_{\|u_0\|_E}^z ds/\varphi(s)$. Clearly both A and B are closed convex subsets of $C(I, E)$, while C is bounded and convex. Moreover, it is easy to see that $F(C) \subset C$. The rest of the proof is divided into three steps.

STEP A. *F is demicontinuous.*

Indeed, suppose that $u_n \to u$ in C. We have to show that $F(u_n) \rightharpoonup F(u)$ in $C(I, E)$. By assumption, $u_n(t) \to u(t)$ in E for all $t \in I$. Since f is Carathéodory, this implies $f(t, u_n(t)) \to f(t, u(t))$ in E for a.e. $t \in I$. On the other hand, condition (f_2) shows that $\|f(t, u_n(t))\|_E \le b'(t)$ for all $t \in I$. So by Lebesgue's Dominated Convergence Theorem,

$$\int_0^t \|f(s, u_n(s)) - f(s, u(s))\|_E\, ds \to 0, \quad \forall t \in I.$$

In particular, $F(u_n)(t) \to F(u)(t)$ for all $t \in I$. The next claim is a key point to conclude the demicontinuity of F.

CLAIM 2. $K = \{F(u_n) : n \in \mathbb{N}\}$ *is relatively weakly compact.*

In order to prove this, we need the following characterization of weak compactness in $L_\infty(I, E)$ (see [Sch, Theorem 2.7] for its full statement):

THEOREM 4.1. *Let (Ω, Σ, μ) be a positive and finite measure space. For a bounded subset $K \subset L_\infty(\mu, E)$ the following are equivalent:*

(a) *K is relatively weakly compact.*
(b) *For any sequence $(v_n) \subset K$ there exist a subsequence (w_i) of (v_n), a function $w \in L_\infty(\mu, E)$ and a set $N \subset \Omega$ with $\mu(N) = 0$ such that:*

  (i) *$w_i(t) \to w(t)$ weakly in E for all $t \in \Omega \setminus N$,*
  (ii) *for any sequences $(x_j^*) \subset B_{E^*}$ and $(t_j) \subset \Omega \setminus N$, there exist subsequences $(x_{j_k}^*)$, (t_{j_k}) such that*

$$\lim_{i \to \infty} \lim_{k \to \infty} \langle x_{j_k}^*, (w_i - w)(t_{j_k}) \rangle = 0.$$

Let (v_i) be any subsequence of $(F(u_n))$ and (w_i) any subsequence of (v_i). Set $w = F(u)$, and pick any set $N \subset I$ with $|N| = 0$. We have already proved that $w_i(t) \to w(t)$ for all $t \in I \setminus N$. Hence, condition (b)(i) in Theorem 4.1 is fulfilled. Let now (x_j^*) be any sequence in the unit ball B_{E^*} of E^*, and let (t_j) be any sequence of real numbers in $I \setminus N$. It is easy to check that

$$|\langle x_j^*, (w_i - w)(t_j)\rangle| \leq \int_0^T \|f(s, u_{m_i}(s)) - f(s, u(s))\|_E \, ds$$

where we are assuming that $w_i = F(u_{m_i})$ for all $i \in \mathbb{N}$. Using again Lebesgue's Dominated Convergence Theorem, it is easy to verify that

$$\lim_{i \to \infty} \int_0^T \|f(s, u_{m_i}(s)) - f(s, u(s))\|_E \, ds = 0,$$

which implies that

$$\lim_{i \to \infty} \lim_{j \to \infty} |\langle x_j^*, (w_i - w)(t_j)\rangle| = 0.$$

Thus we get for free the assumption (b)(ii) in Theorem 4.1, given the arbitrariness of the sequences (x_j^*) and (t_j). Hence K is relatively weakly compact in $L_\infty(I, E)$, and the proof of Claim 2 is complete.

As is well known, Claim 2 implies that for some subsequence (u_{n_k}) of (u_n) the sequence $(F(u_{n_k}))$ converges weakly to some v in $L_\infty(I, E)$. Then, by Theorem 2.11 in [T], $F(u_{n_k})(t) \rightharpoonup v(t)$ in E for almost every $t \in I$. Since the weak topology is Hausdorff, it follows that $v \equiv F(u)$ a.e. in I. It is not hard to deduce that $F(u_n)$ converges weakly to $F(u)$ in $C(I, E)$. This proves that F is demicontinuous.

STEP B. *F has a weak-approximate fixed point sequence in C.*

To prove this, we need the following result as a crucial tool.

LEMMA 4.2. *Let X be a Banach space, $C \subset X$ a bounded convex set and $F \colon C \to \overline{C}$ a demicontinuous map. Assume that C does not contain any isomorphic copy of ℓ_1. Then there exists a sequence (u_n) in C so that $u_n - F(u_n) \rightharpoonup 0$ in X.*

Proof. This is a direct consequence of Proposition 3.7 in [BKL]. ∎

Let us turn our attention to the proof of Theorem 1.2. Since I is not scattered and X contains no isomorphic copy of ℓ_1, a result of Cembranos [Ce] shows that $C(I, X)$ has no isomorphic copy of ℓ_1 either. Thus, by Lemma 4.2, there is a sequence (u_n) in C so that $u_n - F(u_n) \rightharpoonup 0$ in $C(I, E)$. This concludes the proof of Step B.

STEP C. *The sequence (u_n) obtained in Step B is a WAS for problem (1.1).*

Indeed, conditions (i)–(ii) and (iv) of Definition 1.6 follow easily from the fact that (u_n) belongs to C. Finally, after passing to a subsequence if needed, item (iii) is a direct consequence of Rosenthal's ℓ_1-theorem [R2] and the fact that $u_n - F(u_n) \rightharpoonup 0$ in $C(I, E)$. This concludes the proof of Theorem 1.2. ∎

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