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Sharp equivalence between ρ- and τ-mixing coefficients

by

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Abstract. For two σ-algebras $\mathcal{A}$ and $\mathcal{B}$, the ρ-mixing coefficient $\rho(\mathcal{A},\mathcal{B})$ between $\mathcal{A}$ and $\mathcal{B}$ is the supremum correlation between two real random variables X and Y which are $\mathcal{A}$- resp. $\mathcal{B}$-measurable; the $\tau'(\mathcal{A},\mathcal{B})$ coefficient is defined similarly, but restricting to the case where X and Y are indicator functions. It has been known for a long time that the bound $\rho \leq C\tau'(1+|\log\tau'|)$ holds for some constant C; in this article, we show that $C=1$ works and is best possible.

1. Introduction. In this article, we consider two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$, whose correlation level we aim at quantifying. A classical definition for such correlation quantification is the *ρ-mixing coefficient* (also known as "maximal correlation coefficient"):

$$\rho(\mathcal{A},\mathcal{B}) := \sup_{\substack{X\in L^2(\mathcal{A})\\ Y\in L^2(\mathcal{B})}} \frac{|\mathrm{Cov}(X,Y)|}{\mathrm{Var}(X)^{1/2}\,\mathrm{Var}(Y)^{1/2}} \tag{1}$$

(where the supremum is taken only over nonconstant X and Y). This coefficient is 0 if and only if $\mathcal{A}$ and $\mathcal{B}$ are independent; and we consider $\mathcal{A}$ and $\mathcal{B}$ to be as correlated (in the ρ-mixing sense) as $\rho(\mathcal{A},\mathcal{B})$ is large. Note that one always has $\rho(\mathcal{A},\mathcal{B})\leq 1$, because of the Cauchy–Schwarz inequality.

There are other ways to measure dependence between $\mathcal{A}$ and $\mathcal{B}$ (see for instance the review paper [2]): in particular, rather than looking at correlation between $\mathcal{A}$- and $\mathcal{B}$-measurable *random variables*, we can look at correlation between *events*. The most classical measure of dependence in this category is the *α-mixing coefficient*:

$$\alpha(\mathcal{A},\mathcal{B}) := \sup_{\substack{A\in\mathcal{A}\\ B\in\mathcal{B}}} |\mathbb{P}(A\cap B)-\mathbb{P}(A)\,\mathbb{P}(B)|.$$

Still in the same category, the *τ-mixing coefficient* is useful to capture strong

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correlation between small probability events:

$$\tau(\mathcal{A},\mathcal{B}) := \sup_{\substack{A\in\mathcal{A}\\ B\in\mathcal{B}}} \frac{|\mathbb{P}(A\cap B)-\mathbb{P}(A)\,\mathbb{P}(B)|}{\mathbb{P}(A)^{1/2}\,\mathbb{P}(B)^{1/2}}.$$

In this article, we will rather consider a variant of τ:

$$\tau'(\mathcal{A},\mathcal{B}) := \sup_{\substack{A\in\mathcal{A}\\ B\in\mathcal{B}}} \frac{|\mathbb{P}(A\cap B)-\mathbb{P}(A)\,\mathbb{P}(B)|}{\mathbb{P}(A)^{1/2}(1-\mathbb{P}(A))^{1/2}\,\mathbb{P}(B)^{1/2}(1-\mathbb{P}(B))^{1/2}}. \tag{2}$$

The τ'-mixing coefficient is essentially the same as τ, as for all σ-algebras $\mathcal{A},\mathcal{B}$ one has

$$\tau(\mathcal{A},\mathcal{B}) \le \tau'(\mathcal{A},\mathcal{B}) \le 2\tau(\mathcal{A},\mathcal{B});$$

indeed, on the one hand $\tau' \ge \tau$ is obvious, and on the other hand it can always be assumed that $\mathbb{P}(A),\mathbb{P}(B) \le 1/2$ in (2), since the right-hand side of (2) remains unchanged when A or B is replaced by the respective complement.

But (2) is the same definition as (1), except that one restricts X and Y to be indicator functions; so one always has $\tau'(\mathcal{A},\mathcal{B}) \le \rho(\mathcal{A},\mathcal{B})$. Hence, it is a natural question whether some kind of converse link between τ' and ρ also holds, i.e.: can one find some nontrivial ([1]) bound on ρ as a function of τ' (or equivalently of τ)? That question was answered positively by Bradley [1] in 1983.

The next question is: what is the best bound for ρ as a function of τ'? Bradley and Bryc [3, Theorem 1.1(ii)] (and independently Bulinskiĭ [5]) showed that one always has

$$\rho \le C\tau'(1+|\log \tau'|) \tag{3}$$

for some constant C; and Bradley, Bryc and Janson [4, Theorem 3.1] showed that the shape of that bound is sharp, i.e. essentially nothing can be improved in (3) but the value of C. However, the optimal value of C remained unknown.

In this article we will show that $C = 1$ works (Theorem 3.1) and that the corresponding bound is optimal (Theorem 4.1).

2. A first result. In this section we are going to prove a first result on bounding the ρ-mixing coefficient thanks to some condition on events. This result, in addition to being of independent interest, is also interesting for its proof, which involves some ideas which we will re-use in the proof of our main theorem (Theorem 3.1).

([1]) By "nontrivial", I mean that the bound on ρ would tend to 0 as τ' tends to 0.

We first need to define a certain Sobolev space:

Definition 2.1. For a $\mathcal{C}^1$ function $f : (0,1) \to \mathbb{R}$ with compact support (we write $f \in \mathcal{C}_c^1((0,1))$, one defines

$$\|f\|_{\dot{H}_0^1((0,1))} := \Big(\int_0^1 |f'(x)|^2\,dx\Big)^{1/2}. \tag{4}$$

This defines a norm on $\mathcal{C}_c^1((0,1))$; the completion of this set for that norm is denoted by $\dot{H}_0^1((0,1))$—or merely H^1, as no ambiguity can occur.

Some nondifferentiable functions can nevertheless be seen as elements of H^1; in particular, if f is a continuous function defined on $[0,1]$ with $f(0) = f(1) = 0$ and if f is $\mathcal{C}^1$ at all points but a finite number, then (4) remains valid, and f is in H^1 if and only if $\int_0^1 |f'(x)|^2\,dx < \infty$. Conversely, the Sobolev embedding theorem asserts that any element of H^1 can be seen as a continuous function defined on $[0,1]$ and being zero at 0 and 1.

The main result of this section is the following one:

Theorem 2.2. *Take $f, g \in H^1$. Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras such that, for all $A \in \mathcal{A}$, $B \in \mathcal{B}$,*

$$\mathbb{P}(A \cap B) - \mathbb{P}(A)\,\mathbb{P}(B) \le f(\mathbb{P}(A))g(\mathbb{P}(B)). \tag{5}$$

Then

$$\rho(\mathcal{A},\mathcal{B}) \le \|f\|_{H^1}\|g\|_{H^1}.$$

Remark 2.3. Note that taking the absolute value on the left-hand side of (5) is not necessary.

Before proving that theorem, let us record a particular case:

Corollary 2.4. *For $p, q > 1/2$, define*

$$\alpha_{p,q}(\mathcal{A},\mathcal{B}) := \sup_{\substack{A\in\mathcal{A}\\ B\in\mathcal{B}}} \frac{|\mathbb{P}(A\cap B) - \mathbb{P}(A)\,\mathbb{P}(B)|}{\mathbb{P}(A)^p\,\mathbb{P}(B)^q}. \tag{6}$$

Then

$$\rho(\mathcal{A},\mathcal{B}) \le \frac{2^{2-p-q}pq}{(2p-1)^{1/2}(2q-1)^{1/2}}\alpha_{p,q}(\mathcal{A},\mathcal{B}).$$

Proof. Since the numerator on the right-hand side of (6) remains unchanged when A or B is replaced by its complement, the hypotheses of Theorem 2.2 are satisfied with $f(x) = \alpha_{p,q}(\mathcal{A},\mathcal{B})(x^p \wedge (1-x)^p)$ and $g(y) = y^q \wedge (1-y)^q$. Then the conclusion follows from the computation of $\|f\|_{H^1}$ and $\|g\|_{H^1}$. ■

Proof of Theorem 2.2. Since our goal is to bound $\rho(\mathcal{A},\mathcal{B})$ above, let us consider two L^2 real r.v. X and Y which are $\mathcal{A}$- resp. $\mathcal{B}$-measurable; and let us try to bound $|\mathrm{Cov}(X,Y)|$ by some multiple of $\mathrm{Var}(X)^{1/2}\,\mathrm{Var}(Y)^{1/2}$. Actually we will only bound $\mathrm{Cov}(X,Y)$, since then $-\mathrm{Cov}(X,Y)$ will also be bounded by writing it as $\mathrm{Cov}(X,-Y)$.

In order to write the covariance as a function of probabilities of events, we need the following lemma, known as the *Hoeffding identity*:

LEMMA 2.5 (Hoeffding [6]). *Let X,Y be two L^2 real r.v. defined on the same probability space. Then*

$$\mathrm{Cov}(X,Y)=\int_{\mathbb{R}\times\mathbb{R}}(\mathbb{P}(X\le x \text{ and } Y\le y)-\mathbb{P}(X\le x)\,\mathbb{P}(Y\le y))\,dx\,dy. \tag{7}$$

We will give a quick proof of the Hoeffding identity here for the sake of completeness:

Proof of the Hoeffding identity. By a standard approximation argument, we can assume that X and Y are bounded above. Since both sides of (7) remain unchanged when a constant is added to X or to Y, we can even assume that X and Y only take nonpositive values, so that the integral in (7) may actually be taken over $\mathbb{R}_-\times\mathbb{R}_-$.

Now, we start from the formula expressing covariance through expectations:

$$\mathrm{Cov}(X,Y)=\mathbb{E}(XY)-\mathbb{E}(X)\,\mathbb{E}(Y). \tag{8}$$

For a nonpositive r.v. X, one has the classical relation

$$\mathbb{E}(X)=-\int_{\mathbb{R}_-}\mathbb{P}(X\le x)\,dx, \tag{9}$$

which is proved by writing $X(\omega)=-\int_{\mathbb{R}_-}\mathbf{1}_{X(\omega)\le x}\,dx$ and then applying Fubini's theorem. With a similar argument, for nonpositive X and Y one has

$$\mathbb{E}(XY)=\int_{\mathbb{R}_-\times\mathbb{R}_-}\mathbb{P}(X\le x \text{ and } Y\le y)\,dx\,dy. \tag{10}$$

Then, (8) turns into

$$\begin{aligned}\mathrm{Cov}(X,Y)&\\ &=\int_{\mathbb{R}_-\times\mathbb{R}_-}\mathbb{P}(X\le x \text{ and } Y\le y)\,dx\,dy-\int_{\mathbb{R}_-}\mathbb{P}(X\le x)\,dx\int_{\mathbb{R}_-}\mathbb{P}(Y\le y)\,dy\\ &=\int_{\mathbb{R}_-\times\mathbb{R}_-}(\mathbb{P}(X\le x \text{ and } Y\le y)-\mathbb{P}(X\le x)\,\mathbb{P}(Y\le y))\,dx\,dy.\ \blacksquare\end{aligned}$$

Now we go back to the proof of Theorem 2.2. In our case, the hypothesis (5) yields

$$\operatorname{Cov}(X,Y) \le \int_{\mathbb{R}\times\mathbb{R}} f(\mathbb{P}(X\le x))g(\mathbb{P}(Y\le y))\,dx\,dy = \int_{\mathbb{R}} f(\mathbb{P}(X\le x))\,dx \int_{\mathbb{R}} g(\mathbb{P}(Y\le y))\,dy.$$

Thus, it suffices to show that for all r.v. $X\in L^2(\mathbb{P})$,

$$\int_{\mathbb{R}} f(\mathbb{P}(X\le x))\,dx \le \|f\|_{H^1}\operatorname{Var}(X)^{1/2}. \tag{11}$$

To see this, for $p\in(0,1)$, denote

$$\chi(p) := \inf\{x\in\mathbb{R} : \mathbb{P}(X\le x)\ge p\}.$$

By a perturbation argument, we can assume that χ is strictly increasing, so that $\mathbb{P}(X\le\chi(p)) = p$ for all $p\in(0,1)$, and also that $\chi\in\mathcal{C}^1([0,1])$ (2). Then we can perform the change of variables $x=\chi(p)$ to get

$$\int_{\mathbb{R}} f(\mathbb{P}(X\le x))\,dx = \int_0^1 f(p)\chi'(p)\,dp.$$

(This change of variables is legal here because $f(0)=f(1)=0$, so that the integral on the left-hand side of (11) is in fact over $[\chi(0),\chi(1)]$.)

$\operatorname{Var}(X)$ can also be expressed as a function of χ'. Indeed, applying (7) to $Y=X$, one finds

$$\begin{aligned}\operatorname{Var}(X) &= \int_{\mathbb{R}\times\mathbb{R}} (\mathbb{P}(X\le x_1)\wedge\mathbb{P}(X\le x_2) - \mathbb{P}(X\le x_1)\,\mathbb{P}(X\le x_2))\,dx_1\,dx_2\\ &= \int_{(0,1)^2} (p_1\wedge p_2 - p_1p_2)\chi'(p_1)\chi'(p_2)\,dp_1\,dp_2.\end{aligned} \tag{12}$$

In the end, our goal has become the following one: to show that for all $f\in H^1$ and $\varphi\in\mathcal{C}([0,1])$, one has

$$\int_0^1 f(p)\varphi(p)\,dp \le \|f\|_{H^1}\Big(\int_{(0,1)^2} (p_1\wedge p_2 - p_1p_2)\varphi(p_1)\varphi(p_2)\,dp_1dp_2\Big)^{1/2}. \tag{13}$$

Note that, by a density argument, it will be enough to prove (13) only for $f\in\mathcal{C}_c^2((0,1))$.

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(2) When we write χ as a function over the *closed* interval $[0,1]$, the values of $\chi(0)$ and $\chi(1)$ are taken by continuous extension.

Now we have to deal with some bilinear forms. Denote, for $f, g \in \mathcal{C}([0,1])$,

$$\langle f, g\rangle_{L^2} := \int_{(0,1)} f(p)g(p)\,dp,$$

$$\langle f, g\rangle_V := \int_{(0,1)^2} (p_1 \wedge p_2 - p_1p_2) f(p_1)g(p_2)\,dp_1\,dp_2, \tag{14}$$

so that (13) may be written as

$$\langle f, \varphi\rangle_{L^2} \leq \|f\|_{H^1} \langle \varphi, \varphi\rangle_V^{1/2}. \tag{15}$$

When φ is of the form χ', $\langle \varphi, \varphi\rangle_V$ is nonnegative since it corresponds to $\mathrm{Var}(X)$ in (12); and for general φ, drawing our inspiration from the formula

$$\mathrm{Var}(X) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} (x_1 - x_2)^2\,d\mathbb{P}(X = x_1)\,d\mathbb{P}(X = x_2),$$

we find that

$$\langle \varphi, \varphi\rangle_V = \int_{p_1<p_2} \Big(\int_{p_1}^{p_2} \varphi(q)\,dq\Big)^2\,dp_1\,dp_2 \geq 0,$$

which shows that $\langle\cdot,\cdot\rangle_V$ is a scalar product indeed.

Having this scalar product property at hand suggests using the Cauchy–Schwarz inequality to show (15). We define

$$L : \mathcal{C}_c((0,1)) \to \mathcal{C}_0([0,1]), \qquad (Lf)(q) := \int_0^1 (p \wedge q - pq) f(p)\,dp, \tag{16}$$

so that we can write

$$\langle f, g\rangle_V = \langle Lf, g\rangle_{L^2}.$$

Now, if we could find a (right) inverse M for L (i.e. an operator such that $LM = \mathrm{Id}$), we would have

$$\langle f, \varphi\rangle_{L^2} = \langle L(Mf), \varphi\rangle_{L^2} = \langle Mf, \varphi\rangle_V \leq \|Mf\|_V \|\varphi\|_V,$$

which would be a good step towards our goal. Such a right inverse is indeed given by the "minus second derivative" operator, that is, $M : f \in \mathcal{C}_c^2((0,1)) \mapsto -f'' \in \mathcal{C}_c((0,1))$. We compute indeed that, for $q \in (0,1)$,

$$\begin{aligned}(L(-f''))(q) &= -\int_0^1 (p \wedge q - pq) f''(p)\,dp \\ &= -(1-q)\int_0^q p f''(p)\,dp - q\int_q^1 (1-p) f''(p)\,dp\end{aligned}$$

$$= -(1-q)[pf'(p)]_0^q + (1-q)\int_0^q f'(p)\,dp - q[(1-p)f'(p)]_q^1 - q\int_q^1 f'(p)\,dp$$
$$= -q(1-q)f'(q) + (1-q)f(q) + q(1-q)f'(q) + qf(q) = f(q)$$

by integrating by parts (and using that $f'(0) = f'(1) = f(0) = f(1) = 0$).

So, we have

$$\langle f, \varphi\rangle_{L^2} \le \|f''\|_V \|\varphi\|_V.$$

To end the proof, we finally observe that $\|f''\|_V$ is actually equal to $\|f\|_{H^1}$:

$$\|f''\|_V^2 = \langle f'', f''\rangle_V = \langle L(f''), f''\rangle_{L^2} = -\langle f, f''\rangle_{L^2}$$
$$= -\int_0^1 f''(p)f(p)\,dp = \int_0^1 f'(p)^2\,dp = \|f\|_{H^1}^2$$

(where the penultimate equality ensues from integrating by parts). ■

3. $\rho \le \tau'(1 + |\log \tau'|)$

3.1. Statement. The goal of this third section is to prove the statement of its title:

THEOREM 3.1. *For any two σ-algebras $\mathcal{A}, \mathcal{B}$, the coefficient $\rho(\mathcal{A}, \mathcal{B})$ can be bounded using the coefficient $\tau'(\mathcal{A}, \mathcal{B})$, according to the following formula:*

$$\rho \le \tau'(1 - \log \tau') \tag{17}$$

(*where for $\tau' = 0$ we take by continuity $\tau'(1 - \log \tau') = 0$*).

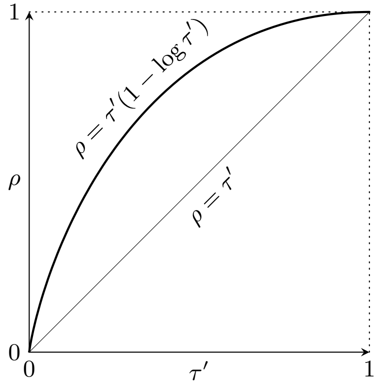

Fig. 1. How we bound ρ as a function of τ'

REMARK 3.2. Since $\tau'(\mathcal{A}, \mathcal{B})$ is always ≤ 1, we can rewrite the right-hand side of (17) as "$\tau'(1 + |\log \tau'|)$", which makes it easier to see that the right-hand side is never less than τ'—it is obvious indeed that one always has $\rho \geq \tau'$.

3.2. Comparison technique

Proof of Theorem 3.1 (first part). We denote $\tau := \tau'(\mathcal{A}, \mathcal{B})$. If $\tau = 0$ or $\tau = 1$ then the conclusion is immediate, since in the first case $\mathcal{A}$ and $\mathcal{B}$ are independent, while in the second case (17) is automatic by the Cauchy–Schwarz inequality. Therefore we will assume that $\tau \in (0,1)$.

So, let X and Y be L^2 real r.v. which are $\mathcal{A}$- resp. $\mathcal{B}$-measurable; our goal is to bound $\mathrm{Cov}(X,Y)$ above (as in the proof of Theorem 2.2, bounding $\mathrm{Cov}(X,Y)$ will actually yield a bound for $|\mathrm{Cov}(X,Y)|$). We start from the formula (7):

$$\mathrm{Cov}(X,Y) = \int_{\mathbb{R}\times\mathbb{R}} \big(\mathbb{P}(X \leq x \text{ and } Y \leq y) - \mathbb{P}(X \leq x)\,\mathbb{P}(Y \leq y)\big)\,dx\,dy. \tag{18}$$

Now we use the τ'-mixing property: for $\mathcal{A}$-, resp. $\mathcal{B}$-measurable events A, B,

$$\mathbb{P}(A \cap B) \leq \mathbb{P}(A)\,\mathbb{P}(B) + \tau\,\mathbb{P}(A)^{1/2}(1-\mathbb{P}(A))^{1/2}\,\mathbb{P}(B)^{1/2}(1-\mathbb{P}(B))^{1/2}. \tag{19}$$

Yet, if we use that formula naively, we shall not get anything better than Theorem 2.2—which in the present case would yield an infinite bound, that is, nothing. The new idea consists in noticing that (19) can be automatically improved to

$$\mathbb{P}(A \cap B) \leq \big(\mathbb{P}(A)\,\mathbb{P}(B) + \tau\,\mathbb{P}(A)^{1/2}(1-\mathbb{P}(A))^{1/2}\,\mathbb{P}(B)^{1/2}(1-\mathbb{P}(B))^{1/2}\big) \wedge \mathbb{P}(A) \wedge \mathbb{P}(B). \tag{20}$$

To alleviate notation, we set

$$Z(p,q) := \big(pq + \tau p^{1/2}(1-p)^{1/2} q^{1/2}(1-q)^{1/2}\big) \wedge p \wedge q$$

(actually Z is also a function of τ, but throughout the proof, τ will be fixed), so that the right-hand side of (20) becomes $Z(\mathbb{P}(A), \mathbb{P}(B))$. So,

$$\mathrm{Cov}(X,Y) \leq \int_{\mathbb{R}\times\mathbb{R}} \big(Z(\mathbb{P}(X \leq x), \mathbb{P}(Y \leq y)) - \mathbb{P}(X \leq x)\,\mathbb{P}(Y \leq y)\big)\,dx\,dy. \tag{21}$$

As in the proof of Theorem 2.2, we now define

$$\chi(p) := \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\} \tag{22}$$

and likewise

$$\upsilon(q) := \inf\{y \in \mathbb{R} : \mathbb{P}(Y \leq y) \geq q\}.$$

Then the change of variables $(x,y) = (\chi(p), \upsilon(q))$ in (21) yields

$$\mathrm{Cov}(X,Y) \leq \int_{(0,1)^2} (Z(p,q) - pq)\chi'(p)\upsilon'(q)\,dp\,dq \tag{23}$$

(using if needed an approximation argument to act as if χ and υ were $\mathcal{C}^1$ and strictly increasing). As in the proof of 2.2 again, one also has

$$\operatorname{Var}(X) = \int_{(0,1)^2} (p_1 \wedge p_2 - p_1 p_2)\chi'(p_1)\chi'(p_2)\, dp_1\, dp_2, \tag{24}$$

$$\operatorname{Var}(Y) = \int_{(0,1)^2} (q_1 \wedge q_2 - q_1 q_2)\upsilon'(q_1)\upsilon'(q_2)\, dq_1\, dq_2. \tag{25}$$

It turns out that the right-hand sides of (23), (24) and (25) can be seen as the covariance and variances of two random variables which we will now introduce. But first, we define a probability law which will play a central role in the following:

DEFINITION 3.3. The *Chogosov law* (3), denoted by Γ, is the probability law on $(0,1)^2$ characterized by

$$\forall p, q \in (0,1)^2 \quad \Gamma((0,p) \times (0,q)) = Z(p,q). \tag{26}$$

(It will be proved in Subsection 3.3 that this law actually exists).

Now, on the space $\{(p,q) \in (0,1)^2\}$ equipped with the Chogosov law, we define the following random variables:

$$X^* := \chi(p), \tag{27}$$

$$Y^* := \upsilon(q). \tag{28}$$

We claim that X^* and Y^* have the same distributions as X resp. Y. Under the Chogosov law indeed, both p and q have a *Uniform*$(0,1)$ distribution (this follows by taking $q = 1$, resp. $p = 1$ in (26)), so that the function χ_{X^*} obtained by replacing X by X^* in (22) coincides with χ (which proves that X and X^* have the same law), and likewise $\upsilon_{Y^*} = \upsilon$. So, the right-hand sides of (24) and (25) are equal to $\operatorname{Var}(X^*)$ resp. $\operatorname{Var}(Y^*)$. Furthermore, by applying (18) to X^* and Y^*, the very definition of the Chogosov law shows that $\operatorname{Cov}(X^*, Y^*)$ is *exactly* the right-hand side of (23). In the end, proving the theorem is tantamount to proving that

$$\operatorname{Cov}(X^*, Y^*) \le \tau(1 - \log\tau) \operatorname{Var}(X^*)^{1/2} \operatorname{Var}(Y^*)^{1/2}. \tag{29}$$

Now, denoting by $\mathcal{A}^*$ the σ-algebra on $(0,1)^2$ spanned by p, and by $\mathcal{B}^*$ the σ-algebra spanned by q, we observe that X^* and Y^* are $\mathcal{A}^*$- resp. $\mathcal{B}^*$-measurable; thus, to show (29), it will be enough to show that

$$\rho(\mathcal{A}^*, \mathcal{B}^*) \le \tau(1 - \log\tau). \tag{30}$$

3.3. The Chogosov law

☛ *To alleviate notation, from now on we will denote* $\bar{p} := 1 - p$ *and* $\hat{p} := p - 1/2$ *(with similar notation for* q*).*

(3) So called in honour of my dear friend M. K. Chogosov.

In this subsection, we make a pause in the proof of Theorem 3.1 to prove the existence of the Chogosov law and to describe its structure. We recall that the Chogosov law Γ is the probability law on $\{(p,q) \in (0,1)^2\}$ defined by

$$\Gamma((0,p) \times (0,q)) = (pq + \tau(p\bar{p})^{1/2}(q\bar{q})^{1/2}) \wedge p \wedge q =: Z(p,q). \tag{31}$$

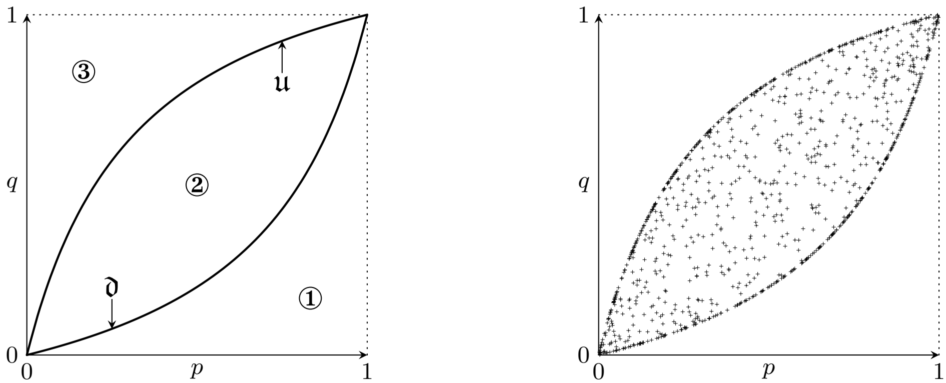

Fig. 2. The Chogosov law Γ. Left: different zones relative to the support of the measure; right: a cloud of 1.024 independent points with law Γ. (The drawings are made for $\tau = 1/2$.)

First we notice that, due to the presence of minimum symbols in the definition of $Z(p,q)$, its analytic expression depends on the zone of $(0,1)^2$ in which (p,q) lies (see Figure 2):

- If $q\bar{p}/(p\bar{q}) < \tau^2$, then $Z(p,q) = q$; in this case we will say that we are in *Zone* 1.
- If $\tau^2 < q\bar{p}/(p\bar{q}) < \tau^{-2}$, then $Z(p,q) = pq + \tau(p\bar{p})^{1/2}(q\bar{q})^{1/2}$; in this case we will say that we are in *Zone* 2.
- If $q\bar{p}/(p\bar{q}) > \tau^{-2}$, then $Z(p,q) = p$; in this case we will say that we are in *Zone* 3.

It will be convenient too to give a name to the boundaries between the different zones: the boundary between Zones 1 and 2 (corresponding to $q\bar{p}/(p\bar{q}) = \tau^2$) will be denoted by $\mathfrak{d}$, and the boundary between Zones 2 and 3 (corresponding to $q\bar{p}/(p\bar{q}) = \tau^{-2}$) by $\mathfrak{u}$. One can parametrize these boundaries by p: $\mathfrak{d}$ is the graph of the function "$q = q_d(p)$" and $\mathfrak{u}$ is the graph of "$q = q_u(p)$", where $q_d(p) := \tau^2 p/(\bar{p} + \tau^2 p)$ and $q_u(p) := p/(\tau^2\bar{p} + p)$.

First, we have to check that the Chogosov law actually exists. In fact, (31) automatically describes a measure on $(0,1)^2$ whose density is $\partial^2 Z/\partial q \partial p$ (in the sense of distributions), but we have to make sure that this measure is nonnegative!

We start by computing $\partial Z/\partial p$:

- In Zone 1, $\partial Z(p,q)/\partial p = 0$.
- In Zone 2, $\partial Z(p,q)/\partial p = q - \tau(q\bar{q})^{1/2}\hat{p}/(p\bar{p})^{1/2}$.
- In Zone 3, $\partial Z(p,q)/\partial p = 1$.

(As Z is continuous at the boundaries $\mathfrak{d}$ and $\mathfrak{u}$, it is not important to know what happens there.) Next we compute $\partial^2 Z/\partial q\partial p$:

- In Zone 1, $\partial^2 Z/\partial q\partial p = 0$.
- At $q = q_d(p)$, $\partial Z(p,\cdot)/\partial p$ has a jump of amplitude
$$q_d(p) - \tau(q_d(p)\bar{q}_d(p))^{1/2}\hat{p}/(p\bar{p})^{1/2};$$
since $\bar{q}_d(p)/\bar{p} = \tau^{-2}q_d(p)/p$, that amplitude can be simplified to $q_d(p) - \hat{p}q_d(p)/p = q_d(p)/(2p)$.
- In Zone 2, $\partial^2 Z/\partial q\partial p = 1 + \tau\hat{p}\hat{q}/((p\bar{p})^{1/2}(q\bar{q})^{1/2})$.
- At $q = q_u(p)$, $\partial Z(p,\cdot)/\partial p$ has a jump of amplitude
$$\bar{q}_u(p) + \tau(q_u(p)\bar{q}_u(p))^{1/2}\hat{p}/(p\bar{p})^{1/2};$$
since $q_u(p)/p = \tau^{-2}\bar{q}_u(p)/\bar{p}$, that amplitude can be simplified to $\bar{q}_u(p) + \hat{p}\bar{q}_u(p)/\bar{p} = \bar{q}_u(p)/(2\bar{p})$.
- Finally, in Zone 3, $\partial^2 Z/\partial q\partial p = 0$.

Now, checking the nonnegativity of Γ is equivalent to verifying that both $\partial^2 Z/\partial q\partial p$ (wherever defined) and the jumps of $\partial Z/\partial p$ are nonnegative. Obviously the only nontrivial case is Zone 2. To show that $1 + \tau \times \hat{p}\hat{q}/((p\bar{p})^{1/2}(q\bar{q})^{1/2})$ is nonnegative on the whole Zone 2, we consider four cases separately:

- If $p \le 1/2$ and $q \le 1/2$, then $\hat{p}\hat{q} \ge 0$, so that the nonnegativity of the density is trivial.
- Likewise, nonnegativity is trivial if $p \ge 1/2$ and $q \ge 1/2$.
- If $p \le 1/2$ and $q \ge 1/2$, we use $q\bar{p}/(p\bar{q}) \le \tau^{-2}$ (since we are in Zone 2) to get $|\tau\hat{p}\hat{q}/((p\bar{p})^{1/2}(q\bar{q})^{1/2})| \le |\hat{p}\hat{q}|/(q\bar{p}) = |\hat{p}/\bar{p}|\,|\hat{q}/q|$; and since $p \le 1/2$ and $q \ge 1/2$, one has $|\hat{p}/\bar{p}|\,|\hat{q}/q| \le 1/2 \times 1/2 \le 1$, which shows that the density is nonnegative.
- Likewise, if $p \ge 1/2$ and $q \le 1/2$, we use $q\bar{p}/(p\bar{q}) \ge \tau^2$ to deduce that $|\tau\hat{p}\hat{q}/((p\bar{p})^{1/2}(q\bar{q})^{1/2})| \le |\hat{p}\hat{q}|/(p\bar{q}) = |\hat{p}/p|\,|\hat{q}/\bar{q}| \le 1/2 \times 1/2 \le 1$.

So we have proved that the Chogosov law actually exists (see also Figure 2). Moreover, the above computations permit a more detailed description of this law:

DEFINITION 3.4. For $p \in (0,1)$, we define the law Γ^p on $(0,1)$ in the following way:

- On $(q_d(p), q_u(p))$, Γ^p has density $1 + \tau\hat{p}\hat{q}/((p\bar{p})^{1/2}(q\bar{q})^{1/2})$ with respect to the Lebesgue measure.

- At $q_d(p)$, Γ^p has an atom of mass $q_d(p)/(2p)$; and at $q_u(p)$, it has an atom of mass $\bar{q}_u(p)/(2\bar{p})$.
- Outside $[q_d(p), q_u(p)]$, Γ^p is zero.

Then we may describe the Chogosov law in the following way:

Proposition 3.5. *(p, q) is distributed according to the Chogosov law Γ if and only if p is uniformly distributed on $(0,1)$ and, conditionally on p, q is distributed according to Γ^p. In other words, for all $A, B \subset (0,1)$,*

$$\Gamma(A \times B) = \int_A \Gamma^p(B)\, dp.$$

3.4. ρ-mixing for the Chogosov law

Proof of Theorem 3.1 (second part). The second and last part of the proof is to show (30). Remember that we are working on the space $\{(p,q) \in (0,1)^2\}$ equipped with the Chogosov law (defined by (31)), and that $\mathcal{A}^*$ is the σ-algebra spanned by p, while $\mathcal{B}^*$ is the σ-algebra spanned by q.

Let us consider random variables X and Y which are $\mathcal{A}^*$- resp. $\mathcal{B}^*$-measurable, that is, $X = f(p)$ and $Y = g(q)$. Our goal will be to bound $|\mathrm{Cov}(X,Y)|$ by some multiple of $\mathrm{Var}(X)^{1/2}\,\mathrm{Var}(Y)^{1/2}$. Subtracting the respective expectations from X and Y (which will not change any of the sides of the inequality to be proved), it will be convenient to assume that X and Y are centered; then, indeed, one will have $\mathrm{Var}(X) = \mathbb{E}(X^2) = \|f\|^2_{L^2((0,1))}$ and likewise $\mathrm{Var}(Y) = \|g\|^2_{L^2((0,1))}$, since both p and q are uniformly distributed on $(0,1)$. Moreover, centering X and Y means that f and g lie in the (closed) subspace of mean zero functions in $L^2((0,1))$: in the rest of the proof, this Hilbert (sub)space will be denoted by H.

Since X and Y are centered, one has $\mathrm{Cov}(X,Y) = \mathbb{E}(XY)$. Thus, we can write $\mathrm{Cov}(X,Y)$ in terms of some linear operator:

Definition 3.6. We define $L : H \to H$ by

$$(Lg)(p) := \mathbb{E}_{\Gamma^p}(g),$$

where we recall that Γ^p is the Chogosov law conditioned on the value of p (cf. Definition 3.4 and Proposition 3.5). In other words, $(Lg)(p)$ is the expectation of $g(q)$ conditioned on p when (p,q) is distributed according to the Chogosov law. (That interpretation ensures that L actually maps H into itself.)

Thus, conditioning on p, one gets

$$\mathrm{Cov}(X,Y) = \langle f, Lg\rangle_H.$$

Therefore, to show (30), it is (necessary and) sufficient to prove that the operator norm $\|L\|_{H\to H}$ is not greater than $\tau(1-\log\tau)$.

It turns out that L has the nice property of being self-adjoint on H. Indeed, we have defined L so that $\langle f, Lg\rangle_H = \mathbb{E}_\Gamma(f(p)g(q))$; but the law Γ is invariant under the permutation $(p,q)\mapsto(q,p)$ (since $Z(p,q)$ is), so that $\langle f, Lg\rangle_H = \mathbb{E}_\Gamma(f(p)g(q)) = \mathbb{E}_\Gamma(f(q)g(p)) = \langle Lf, g\rangle_H$.

We will use the following lemma on self-adjoint operators, whose proof can be found in §3.5:

LEMMA 3.7. *Let L be a self-adjoint operator* (*possibly unbounded*) *on a Hilbert space H. Assume there exists a dense subset $D\subset H$ such that, for some $C<\infty$,*

$$\forall h\in D \qquad \limsup_{k\to\infty} |\langle L^k h, h\rangle_H|^{1/k} \le C.$$

Then $\|L\|_{H\to H}\le C$.

Thanks to Lemma 3.7, we can focus on some dense subset of H on which the work is easier:

DEFINITION 3.8. Let $\varepsilon>0$ be a parameter that we fix for the time being (though in the end we will make it tend to 0). We define formally, for $f\in H$,

$$\|f\|_{\mathrm{Lip}} := \sup_{p\in(0,1)} \frac{|f'(p)|}{(p\bar p)^{-3/2+\varepsilon}},$$

or, in rigorous terms,

$$\|f\|_{\mathrm{Lip}} := \sup_{p_1<p_2} \frac{|f(p_2)-f(p_1)|}{\int_{p_1}^{p_2}(p\bar p)^{-3/2+\varepsilon}\,dp}.$$

We denote by Lip the space of functions of H such that $\|f\|_{\mathrm{Lip}}<\infty$, which we equip with the norm $\|\cdot\|_{\mathrm{Lip}}$.

Obviously Lip is a dense subset of H. Moreover the canonical injection $\mathrm{Lip}\hookrightarrow H$ is continuous: indeed, for $f\in\mathrm{Lip}$, we have $\int_{1/2}^{p} f'(p_1)\,dp_1 = f(p)-f(1/2)$ (here we act as if $f\in\mathcal{C}^1((0,1))$ to alleviate notation, but the reasoning would actually work for all $f\in\mathrm{Lip}$), and since f is orthogonal in $L^2((0,1))$ to the constant functions (for it has zero mean),

$$\Big\|p\mapsto \int_{1/2}^{p} f'(p_1)\,dp_1\Big\|^2_{L^2((0,1))} = \|f\|^2_{L^2((0,1))} + \|f(1/2)\|^2_{L^2((0,1))},$$

whence

$$\begin{aligned}(32)\qquad \|f\|_H = \|f\|_{L^2((0,1))} &\le \Big\|p\mapsto \int_{1/2}^{p} f'(p_1)\,dp_1\Big\|_{L^2((0,1))}\\ &= \Big(\int_0^1\Big(\int_{1/2}^{p} f'(p_1)\,dp_1\Big)^2 dp\Big)^{1/2} \le \Big(\int_0^1\Big(\int_{1/2}^{p} |f'(p_1)|\,dp_1\Big)^2 dp\Big)^{1/2}\\ &\le \|f\|_{\mathrm{Lip}}\times\Big(\int_0^1\Big(\int_{1/2}^{p}(p_1\bar p_1)^{-3/2+\varepsilon}\,dp_1\Big)^2 dp\Big)^{1/2}.\end{aligned}$$

The rightmost factor in (32) being finite because $\varepsilon > 0$, this proves that the injection $\mathrm{Lip} \hookrightarrow H$ is continuous. Denoting by C the norm of this injection, it follows that for all $f \in \mathrm{Lip}$ and $k \in \mathbb{N}$, one has $|\langle L^k f, f\rangle_H| \le \|f\|_H \|L^k f\|_H \le \|f\|_H C \|L^k f\|_{\mathrm{Lip}} \le \|f\|_H C \|L\|_{\mathrm{Lip}\to\mathrm{Lip}}^k \|f\|_{\mathrm{Lip}}$ (note that $C\|f\|_{\mathrm{Lip}}\|f\|_H < \infty$), whence

$$\limsup_{k\to\infty} |\langle L^k f, f\rangle_H|^{1/k} \le \|L\|_{\mathrm{Lip}\to\mathrm{Lip}}.$$

Thus, by Lemma 3.7,

$$\|L\|_{H\to H} \le \|L\|_{\mathrm{Lip}\to\mathrm{Lip}}.$$

As we will see, $\|L\|_{\mathrm{Lip}\to\mathrm{Lip}}$ is easier to bound than $\|L\|_{H\to H}$.

To bound $\|L\|_{\mathrm{Lip}\to\mathrm{Lip}}$, we will use the idea of *monotone coupling* between the Γ^p's. For $\omega \in (0,1)$, let

$$Q(p,\omega) := \inf\{q \in (0,1) : \Gamma^p((0,q]) \ge \omega\}$$

be the inverse repartition function of Γ^p. Then Γ^p is the pushforward of the *Uniform*$(0,1)$ distribution by the map $Q(p,\cdot)$, so that

$$(Lf)(p) = \int_0^1 f(Q(p,\omega))\,d\omega. \tag{33}$$

To alleviate notation, we will act as if f were of class $\mathcal{C}^1$ (treating the general case $f \in \mathrm{Lip}$ would cause no more difficulty but would require heavier formalism). Then, differentiating (33), one finds

$$(Lf)'(p) = \int_0^1 Q'(p,\omega) f'(Q(p,\omega))\,d\omega, \tag{34}$$

where Q' is the derivative of $Q(p,\omega)$ *with respect to* p. (Justification for having differentiated under the integral sign will follow from the upcoming computations on Q'.)

Consequently,

$$|(Lf)'(p)| \le \|f\|_{\mathrm{Lip}} \int_0^1 |Q'(p,\omega)|(Q(p,\omega)\bar{Q}(p,\omega))^{-3/2+\varepsilon}\,d\omega.$$

As that formula is valid for all p and f, it follows that

(35)

$$\|L\|_{\mathrm{Lip}\to\mathrm{Lip}} \le \sup_{p\in(0,1)} \Big\{(p\bar{p})^{3/2-\varepsilon} \int_0^1 |Q'(p,\omega)|(Q(p,\omega)\bar{Q}(p,\omega))^{-3/2+\varepsilon}\,d\omega\Big\},$$

hence

$$\|L\|_{H\to H} \le \text{Right-hand side of (35)}. \tag{36}$$

Before starting with explicit computations, we prove that it is legitimate to take directly $\varepsilon = 0$ in (36) (recall that ε was *a priori* defined to be any *strictly* positive parameter). We first notice that, denoting by $S_p := [q_d(p), q_u(p)]$ the support of Γ^p, one always has $Q(p,\omega) \in S_p$. But for all $q \in S_p$, one has $q\bar{q}/(p\bar{p}) \le \tau^{-2}$: in the case $q \le p$ indeed, having $q \in S_p$ implies that $q\bar{p}/(p\bar{q}) \ge \tau^2$, thus $q\bar{q}/(p\bar{p}) = (q/p)^2/(q\bar{p}/(p\bar{q})) \le 1/\tau^2 = \tau^{-2}$; and there is a similar argument for $q \ge p$. Then, for all p,

$$(p\bar{p})^{3/2-\varepsilon}\int_0^1 |Q'(p,\omega)|(Q(p,\omega)\bar{Q}(p,\omega))^{-3/2+\varepsilon}d\omega$$
$$\le \tau^{-2\varepsilon} \times (p\bar{p})^{3/2}\int_0^1 |Q'(p,\omega)|(Q(p,\omega)\bar{Q}(p,\omega))^{-3/2}\,d\omega.$$

In that formula, the factor $\tau^{-2\varepsilon}$ does not depend on p and tends to 1 as $\varepsilon \to 0$; therefore (36) remains valid for $\varepsilon = 0$.

So, we have to compute the right-hand side of (36) for $\varepsilon = 0$. The first step is to compute Q and Q'. Because of the structure of Γ^p (see Definition 3.4), there are three cases for the analytic expression of $Q(p,\omega)$:

- If $0 < \omega \le q_d(p)/(2p)$, then $Q(p,\omega) = q_d(p)$.
- Likewise, if $1 - \bar{q}_u(p)/(2\bar{p}) \le \omega < 1$, then $Q(p,\omega) = q_u(p)$.
- The case $q_d(p)/(2p) < \omega < 1 - \bar{q}_u(p)/(2\bar{p})$ is more complicated. As Γ^p is the p-conditional law of Γ, the definition (31) of Γ implies that $\Gamma^p((0,q]) = \partial_p Z(p,q)$; thus $Q(p,\omega)$ is the Q such that

$$Q - \tau\hat{p}(Q\bar{Q})^{1/2}/(p\bar{p})^{1/2} = \omega. \tag{37}$$

  (Indeed, remember that in that case one has $Q(p,\omega) \in (q_d(p), q_u(p))$, so that $Z(p,Q) = pQ + \tau(p\bar{p})^{1/2}(Q\bar{Q})^{1/2})$.

From that we get the formula for $Q'(p,\omega)$—recall that Q' is the derivative of Q with respect to p:

- For $\omega < q_d(p)/(2p)$, one has $Q'(p,\omega) = dq_d/dp$. Since q_d is characterized by "$q_d(p)\bar{p} = \tau^2 p\bar{q}_d(p)$", differentiating the latter formula with respect to p yields $Q'(p,\omega) = (q_d(p) + \tau^2\bar{q}_d(p))/(\bar{p} + \tau^2 p)$. Using again that $q_d\bar{p} = \tau^2 p\bar{q}_d$, that expression then simplifies to $Q'(p,\omega) = q_d(p)\bar{q}_d(p)/(p\bar{p})$.
- Likewise, for $\omega > 1 - \bar{q}_u(p)/(2\bar{p})$, one has $Q'(p,\omega) = q_u(p)\bar{q}_u(p)/(p\bar{p})$.
- Finally for $q_d(p)/(2p) < \omega < 1 - \bar{q}_u(p)/(2\bar{p})$, we differentiate (37) to get

$$Q'(p,\omega) = \frac{\tau(Q\bar{Q})^{1/2}}{4(p\bar{p})^{3/2}(1 + \tau\hat{p}\hat{Q}/(p\bar{p}Q\bar{Q})^{1/2})}$$

  where "Q" is shorthand for "$Q(p,\omega)$".

(Note by the way that these computations ensure that $Q'(p,\omega)$ actually exists [for all p, and almost all ω] and that $|Q'|$ is bounded by τ^{-2} [that point, which is nonsubstantial and tedious, is left to the reader], which gives *a posteriori* justification to (34)).

Then we can compute the right-hand side of (36) (for $\varepsilon = 0$):

$$(p\bar{p})^{3/2}\int_0^1 |Q'(p,\omega)|(Q(p,\omega)\bar{Q}(p,\omega))^{-3/2}\,d\omega$$

$$= (p\bar{p})^{3/2}\frac{q_d(p)}{2p}\,\frac{q_d(p)\bar{q}_d(p)}{p\bar{p}}(q_d(p)\bar{q}_d(p))^{-3/2} \tag{38}$$

$$+\,(p\bar{p})^{3/2}\frac{\bar{q}_u(p)}{2\bar{p}}\,\frac{q_u(p)\bar{q}_u(p)}{p\bar{p}}(q_u(p)\bar{q}_u(p))^{-3/2} \tag{39}$$

$$+\,(p\bar{p})^{3/2}\int_{q_d/(2p)}^{1-\bar{q}_u/(2\bar{p})}\frac{\tau(Q(\omega)\bar{Q}(\omega))^{1/2}}{4(p\bar{p})^{3/2}\big(1+\tau\hat{p}\hat{Q}(\omega)/(p\bar{p}Q(\omega)\bar{Q}(\omega))^{1/2}\big)}(Q(\omega)\bar{Q}(\omega))^{-3/2}d\omega \tag{40}$$

where, in (40), q_d, q_u and $Q(\omega)$ are shortcuts for resp. $q_d(p)$, $q_u(p)$ and $Q(p,\omega)$.

Using the formula characterizing $q_d(p)$, (38) simplifies to $(q_d(p)\bar{p}/(p\bar{q}_d(p)))^{1/2}/2 = \tau/2$. Similarly, (39) simplifies to $\tau/2$. To compute (40), we make the change of variables $q = Q(\omega)$. Differentiating (37) with respect to ω, we see that for that change of variables,

$$\left(1+\frac{\tau\hat{p}\hat{Q}(\omega)}{(p\bar{p}Q(\omega)\bar{Q}(\omega))^{1/2}}\right)dq = d\omega,$$

whence

$$(p\bar{p})^{3/2}\int_{q_d/(2p)}^{1-\bar{q}_u/(2\bar{p})}\frac{\tau(Q(\omega)\bar{Q}(\omega))^{1/2}}{4(p\bar{p})^{3/2}\big(1+\tau\hat{p}\hat{Q}(\omega)/(p\bar{p}Q(\omega)\bar{Q}(\omega))^{1/2}\big)}(Q(\omega)\bar{Q}(\omega))^{-3/2}\,d\omega$$

$$= \frac{\tau}{4}\int_{q_d(p)}^{q_u(p)}\frac{1}{q\bar{q}}\,dq = \frac{\tau}{4}\left[\log\frac{q}{\bar{q}}\right]_{q_d(p)}^{q_u(p)} = \frac{\tau}{4}\left(\log\frac{q_u(p)}{\bar{q}_u(p)} - \log\frac{q_d(p)}{\bar{q}_d(p)}\right)$$

which, using the formulas characterizing $q_d(p)$ and $q_u(p)$, is finally equal to

$$\frac{\tau}{4}\left(\log\frac{p}{\tau^2\bar{p}} - \log\frac{p\tau^2}{\bar{p}}\right) = \frac{\tau}{4}\log\frac{1}{\tau^4} = -\tau\log\tau.$$

So, the sum (38)–(40) is equal to $\tau(1-\log\tau)$ for all p, and thus the right-hand side of (36) (for $\varepsilon = 0$) is $\tau(1-\log\tau)$, which proves the theorem. ∎

3.5. Appendix: On the norm of self-adjoint operators. This appendix aims at proving Lemma 3.7.

Proof of Lemma 3.7. By the spectral theorem for self-adjoint operators, up to some isomorphism, we may assume that H is the space $L^2(\mu)$ corresponding to some Radon measure space (X,μ) and that L is a real multiplication operator on that space—i.e. there exists $\lambda \in L^\infty(\mu,\mathbb{R})$ such that

$$\forall x \in X \quad (Lf)(x) = \lambda(x)f(x)$$

(where "$\forall x$" actually means "for μ-almost all x").

Once L is written in that form, we see that for all $f \in L^2(\mu)$,

$$\limsup_{k\to\infty} |\langle L^k f, f\rangle_H|^{1/k} = \sup\bigl\{c > 0 : \mu(\{f \neq 0 \text{ and } |\lambda| > c\}) > 0\bigr\}. \tag{41}$$

(To prove "$\geq$", use that for k even one has $\lambda(x)^k \geq 0\ \forall x$.) But if one had $\mu(\{|\lambda| > C\}) > 0$, the set of $f \in H$ such that $\mu(\{f \neq 0 \text{ and } |\lambda| > C\}) > 0$ would be a nonempty open subset of H, and then (41) would contradict the assumption of the lemma. Therefore $\mu(\{|\lambda| > C\}) = 0$, thus $\|L\|_{H\to H} \leq C$. ∎

4. Optimality of our bound

☛ *Throughout this section, all the sets considered will be tacitly understood to be Borel.*

4.1. Statement of the theorem and outline of the proof. In this section we will prove that our bound (17) cannot be improved. More precisely, we are going to prove the following theorem:

THEOREM 4.1. *Let $\tau \in [0,1]$ and let $\rho < \tau(1-\log\tau)$. Then there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and two σ-algebras $\mathcal{A},\mathcal{B}$ on this space such that*

$$\tau'(\mathcal{A},\mathcal{B}) \leq \tau \quad \textit{and} \quad \rho(\mathcal{A},\mathcal{B}) \geq \rho.$$

Since the map $\tau \mapsto \tau(1-\log\tau)$ is continuous, that theorem is an immediate corollary of the following one:

THEOREM 4.2. *Let $\tau \in [0,1]$ and $\tau_1 > \tau$. Then there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and two σ-algebras $\mathcal{A},\mathcal{B}$ on this space such that*

$$\tau'(\mathcal{A},\mathcal{B}) \leq \tau_1 \quad \textit{and} \quad \rho(\mathcal{A},\mathcal{B}) \geq \tau(1-\log\tau).$$

Note that Theorem 4.2 is immediate for $\tau = 0$ and for $\tau = 1$, so it is enough to prove it for $\tau \in (0,1)$.

To do so, we will have to find a sharp bound on some τ'-mixing coefficient, which is not an easy challenge in general. For that reason, we first focus on some particular measure for which finding this kind of bound is easier. However this measure will not be a probability measure (it will have infinite total mass), so that we will have to use a truncation argument in a second step to get a genuine probability measure.

☛ *In the following, we are considering some fixed $\tau \in (0,1)$, and our goal is to prove 4.2 for that value of τ.*

4.2. The measure Γ_∞

Definition 4.3. We define, for $(p,q) \in (0,\infty)^2$,

$$Z_\infty(p,q) := \tau p^{1/2} q^{1/2} \wedge p \wedge q.$$

We define the measure Γ_∞ on $(0,\infty)^2$ by

$$\forall p,q \in (0,\infty)^2 \quad \Gamma_\infty((0,p)\times(0,q)) = Z_\infty(p,q).$$

(This actually defines a nonnegative measure: the reasoning is similar to—and easier than—the one for the Chogosov law. See also Figure 3.)

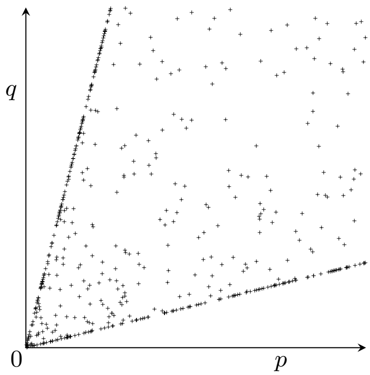

Fig. 3. The measure Γ_∞: a Poisson cloud of points with density Γ_∞. (The scale and density of the cloud are consistent with Figure 2.)

For $p \in (0,\infty)$, we define the measure Γ_∞^p on $(0,\infty)$ in the following way:

- On $(\tau^2 p, \tau^{-2} p)$, Γ_∞^p has density $\tau/(4p^{1/2}q^{1/2})$ with respect to the Lebesgue measure.
- At $\tau^2 p$, Γ_∞^p has an atom of mass $\tau^2/2$; and at $\tau^{-2}p$, it has an atom of mass $1/2$.
- Outside $[\tau^2 p, \tau^{-2} p]$, Γ_∞^p is zero.

(Note that Γ_∞^p is a probability measure).

Just as for the Chogosov law, we can prove the following properties of the measure Γ_∞:

Proposition 4.4.

(1) *Both marginals of Γ_∞ (i.e. its marginals on p and on q) are equal to the Lebesgue measure on $(0,\infty)$.*
(2) *Γ_∞^p is the "p-conditional law" of Γ_∞, in the sense that for all $A, B \subset (0,\infty)$, one has $\Gamma_\infty(A\times B) = \int_A \Gamma_\infty^p(B)\,dp$.*

So, like the Chogosov law, the measure Γ_∞ is made of several components: first, a component with density $\tau/(4p^{1/2}q^{1/2})$ with respect to the Lebesgue measure on the cone $\{(p,q) \in (0,\infty)^2 : \tau^2 p < q < \tau^{-2} p\}$; then, components with a *lineic* density on the half-lines $\{(p, \tau^2 p)\}$ and $\{(p, \tau^{-2} p)\}$. We will give a name to the (*surfacic*) density component:

DEFINITION 4.5. We denote by $\tilde{\Gamma}_\infty$ the absolutely continuous part of the measure Γ_∞ with respect to the Lebesgue measure; in other words, $\tilde{\Gamma}_\infty$ is the measure on $(0,\infty)^2$ defined by

$$d\tilde{\Gamma}_\infty(p,q) = \mathbf{1}_{\tau^2 p < q < \tau^{-2} p} \frac{\tau}{4p^{1/2} q^{1/2}} dp\, dq.$$

We also denote by $\tilde{\Gamma}^p_\infty$ the "p-conditional measure" of $\tilde{\Gamma}_\infty$, that is, the measure on $(0,\infty)$ defined by

$$d\tilde{\Gamma}^p_\infty(q) = \mathbf{1}_{\tau^2 p < q < \tau^{-2} p} \frac{\tau}{4p^{1/2} q^{1/2}} dq,$$

which is such that $\tilde{\Gamma}_\infty(A \times B) = \int_A \tilde{\Gamma}^p_\infty(B)\, dp$. (Beware, $\tilde{\Gamma}^p_\infty$ is not a probability measure).

Now we prove two lemmas essential for the proof of Theorem 4.2:

LEMMA 4.6. *For all $A, B \subset (0,\infty)$ such that $\tau^2 A \subset B$ and $\tau^2 |B| \le |A| < \infty$ ($|A|$ denotes the Lebesgue measure of A),*

$$\tilde{\Gamma}_\infty(A \times B) \le \tau |A|^{1/2} |B|^{1/2} - \tau^2 (|A| + |B|)/2.$$

Proof. We start from the "p-conditional" decomposition of $\tilde{\Gamma}_\infty$:

$$\tilde{\Gamma}_\infty(A \times B) = \int_A \tilde{\Gamma}^p_\infty(B)\, dp. \tag{42}$$

For $x \in (0, |A|)$, we set

$$\pi(x) := \inf\{p \in (0,\infty) : |(0,p] \cap A| \ge x\}, \tag{43}$$

so that the restriction of the Lebesgue measure to A is the pushforward by π of the Lebesgue measure on $(0,|A|)$. Then, changing variables on the right-hand side of (42), one has

$$\tilde{\Gamma}_\infty(A \times B) = \int_0^{|A|} \tilde{\Gamma}^{\pi(x)}_\infty(B)\, dx. \tag{44}$$

Now we are going to bound $\tilde{\Gamma}^{\pi(x)}_\infty(B)$. (In the next computations we abbreviate $\pi(x)$ to π, not to be confused with Archimedes' constant, nowhere involved in this article.) There are three steps:

(1) First, observing that $\tilde{\Gamma}^\pi_\infty$ does not give any mass to $(0, \tau^2 \pi]$, we have $\tilde{\Gamma}^\pi_\infty(B) = \tilde{\Gamma}^\pi_\infty(B \setminus (0, \tau^2 \pi])$. Let us denote $B \setminus (0, \tau^2 \pi] =: B^\times$.

(2) For $y \in (0, |B^\times|)$, let us define

$$\kappa(y) := \inf\{q \in (\tau^2 \pi, \infty) : |(\tau^2 \pi, q] \cap B^\times| \ge y\}, \tag{45}$$

so that the restriction of the Lebesgue measure to $B^\times$ is the pushforward by κ of the Lebesgue measure on $(0,|B^\times|)$. Then, changing variables,

$$(46)\quad \tilde{\varGamma}^\pi_\infty(B) = \int_{B^\times} \mathbf{1}_{q<\tau^{-2}\pi}\frac{\tau}{4\pi^{1/2}q^{1/2}}\,dq = \int_0^{|B^\times|} \mathbf{1}_{\kappa(y)<\tau^{-2}\pi}\frac{\tau}{4\pi^{1/2}\kappa(y)^{1/2}}\,dy.$$

But $\kappa(y) \geq \tau^2\pi + y$ for all y (that is obvious from (45)), so that (46) yields

$$(47)\quad \tilde{\varGamma}^\pi_\infty(B) \leq \int_0^{|B^\times|} \mathbf{1}_{\tau^2\pi+y<\tau^{-2}\pi}\frac{\tau}{4\pi^{1/2}(\tau^2\pi+y)^{1/2}}\,dy$$
$$= \frac{\tau(\tau^2\pi+|B^\times|)^{1/2}}{2\pi^{1/2}} \wedge 1/2 - \tau^2/2 \leq \frac{\tau(\tau^2x+|B^\times|)^{1/2}}{2x^{1/2}} \wedge 1/2 - \tau^2/2,$$

where the last inequality comes from $\pi \geq x$ (because of (43)).

(3) Finally, we claim that $|B^\times| \leq |B| - \tau^2x$: indeed, we have assumed that $B \supset \tau^2 A$, so $|B\cap(0,\tau^2\pi]| \geq |\tau^2A\cap(0,\tau^2\pi]| = \tau^2|A\cap(0,\pi]| = \tau^2x$. Therefore (47) yields

$$(48)\quad \tilde{\varGamma}^\pi_\infty(B) \leq \frac{\tau|B|^{1/2}}{2x^{1/2}} \wedge 1/2 - \tau^2/2.$$

To conclude, we just have to plug (48) into (44) (it is here that the assumption $\tau^2|B| \leq |A|$ is used):

$$\tilde{\varGamma}_\infty(A\times B) \leq \int_0^{|A|}\bigg(\frac{\tau|B|^{1/2}}{2x^{1/2}} \wedge 1/2 - \tau^2/2\bigg)dx$$
$$= \tau^2|B|/2 + \int_{\tau^2|B|}^{|A|} \frac{\tau|B|^{1/2}}{2x^{1/2}}dx - \tau^2|A|/2 = \tau|A|^{1/2}|B|^{1/2} - \tau^2(|A|+|B|)/2. \quad ■$$

Lemma 4.7. *Let $A,B \subset (0,\infty)$ with $|A| \geq \tau^2|B|$. Then there exists $A' \subset (0,\infty)$ such that $|A'| = |A|$, $\varGamma_\infty(A'\times B) \geq \varGamma_\infty(A\times B)$ and $A' \supset \tau^2B$.*

Proof. Denote $A_1 := A\cap\tau^2B$. Since we have assumed that $|A| \geq \tau^2|B|$, we have $|A\setminus A_1| = |A| - |A_1| \geq \tau^2|B| - |A_1| = |\tau^2B\setminus A_1|$, so we can find $A_2 \subset |A\setminus A_1|$ such that $|A_2| = |\tau^2B\setminus A_1|$. Now denote $A_3 := A\setminus A_1\setminus A_2$ and $A_2' := \tau^2B\setminus A_1$, and define $A' := A_1\cup A_2'\cup A_3$. It is clear by construction that $|A'| = |A|$ and $A' \supset \tau^2B$; and still by construction,

$$\varGamma_\infty(A'\times B) = \varGamma_\infty(A\times B) - \varGamma_\infty(A_2\times B) + \varGamma_\infty(A_2'\times B),$$

so it remains to show that $\varGamma_\infty(A_2\times B) \leq \varGamma_\infty(A_2'\times B)$.

Using the "p-conditional" decomposition of Γ_∞, one has

$$\Gamma_\infty(A_2 \times B) = \int_{A_2} \Gamma_\infty^p(B)\,dp, \tag{49}$$

$$\Gamma_\infty(A_2' \times B) = \int_{A_2'} \Gamma_\infty^p(B)\,dp. \tag{50}$$

But, recalling the structure of Γ_∞^p (cf. Definition 4.3), we see that $\Gamma_\infty^p(B) \geq 1/2$ as soon as $\tau^{-2}p \in B$, and thus also $\Gamma_\infty^p(B) \leq 1 - 1/2 = 1/2$ as soon as $\tau^{-2}p \notin B$. Since, by construction, $\tau^{-2}A_2 \cap B = \emptyset$ and $\tau^{-2}A_2' \subset B$, one has

$$\Gamma_\infty(A_2 \times B) = \int_{A_2} \Gamma_\infty^p(B)\,dp \leq |A_2|/2, \tag{51}$$

$$\Gamma_\infty(A_2' \times B) = \int_{A_2'} \Gamma_\infty^p(B)\,dp \geq |A_2'|/2 = |A_2|/2, \tag{52}$$

so that $\Gamma_\infty(A_2 \times B) \leq \Gamma_\infty(A_2' \times B)$, which is the desired result. ■

As the function $Z_\infty(p,q)$ used to define Γ_∞ is invariant under switching p and q, we see that $\Gamma_\infty(A \times B) = \Gamma_\infty(B \times A)$ for all A, B, so that Lemma 4.7 yields the following corollary:

LEMMA 4.8. *Let $A, B \subset (0,\infty)$ with $|B| \geq \tau^2|A|$. Then there exists $B' \subset (0,\infty)$ such that $|B'| = |B|$, $\Gamma_\infty(A \times B') \geq \Gamma_\infty(A \times B)$ and $B' \supset \tau^2 A$.*

Thanks to Lemmas 4.6 and 4.8, we can prove the main result of this subsection:

LEMMA 4.9. *For all $A, B \subset (0,\infty)$,*

$$\Gamma_\infty(A \times B) \leq \tau|A|^{1/2}|B|^{1/2}.$$

Proof. First observe that automatically $\Gamma_\infty(A\times B) \leq |A|\wedge|B|$ (since both marginals of Γ_∞ are equal to the Lebesgue measure), so that the conclusion is immediate if $|A| \leq \tau^2|B|$ or $|B| \leq \tau^2|A|$; we therefore assume that $\tau^2|A| \leq |B| \leq \tau^{-2}|A|$. Then the assumptions of Lemma 4.8 are satisfied, so that up to replacing B by B' we can assume that $\tau^2 A \subset B$, and then apply Lemma 4.6 to get

$$\tilde{\Gamma}_\infty(A \times B) \leq \tau|A|^{1/2}|B|^{1/2} - \tau^2(|A| + |B|)/2. \tag{53}$$

So we have bounded the absolutely continuous component of $\Gamma_\infty(A\times B)$. Now we have to bound the lineic density components. The first of these is

$$\Gamma_\infty(\{(p,q) \in A \times B : q = \tau^2 p\}) \leq \Gamma_\infty(\{(p,q) : p \in A \text{ and } q = \tau^2 p\}), \tag{54}$$

which, using the "p-conditional" decomposition of Γ_∞ and the structure of

Γ^p_∞, is equal to $|A| \times \tau^2/2$. Likewise, the second lineic density component is

$$(55)\quad \Gamma_\infty(\{(p,q) \in A \times B : q = \tau^{-2}p\}) \\ \leq \Gamma_\infty(\{(p,q) : p \in \tau^2 B \text{ and } q = \tau^{-2}p\}) = \tau^2|B| \times 1/2.$$

Summing (53)–(55) yields the desired result. ■

4.3. Proof of optimality. Now that we are equipped with Lemma 4.9, we can at last prove Theorem 4.2. The measurable space we are going to consider is the square $\{(p,q) \in (0,1)^2\}$, on which the σ-algebras $\mathcal{A}$ and $\mathcal{B}$ will be the ones spanned by p resp. q, so that the $\mathcal{A}$-measurable events are those of the form $A \times (0,1)$, while the $\mathcal{B}$-measurable events are those of the form $(0,1) \times B$; and the $\mathcal{A}$-measurable functions are those of the form $f(p)$, while the $\mathcal{B}$-measurable functions are those of the form $g(q)$.

The probability measure $\mathbb{P}$ we are going to build on $(0,1)^2$ will be devised so that both its p- and q-marginals are equal to the Lebesgue measure on $(0,1)$, in order to simplify computations. The principle of the definition we are now going to give is that the probability $\mathbb{P}$—which we will call Γ_b—coincides with Γ_∞ in some neighborhood of $(0,0)$:

Definition 4.10. Take a parameter $\varepsilon \in (0,1)$. On $(0,1)^2$, we define the probability measure Γ_b by

$$(56)\quad \Gamma_b(A \times B) = \begin{cases} \Gamma_\infty(A \times B) & \text{for } A \times B \subset (0,\varepsilon] \times (0,\varepsilon], \\ \dfrac{|B|}{1-\varepsilon}\big(|A| - \Gamma_\infty(A \times (0,\varepsilon])\big) & \text{for } A \times B \subset (0,\varepsilon] \times (\varepsilon,1), \\ \dfrac{|A|}{1-\varepsilon}\big(|B| - \Gamma_\infty((0,\varepsilon] \times B)\big) & \text{for } A \times B \subset (\varepsilon,1) \times (0,\varepsilon], \\ \dfrac{|A|\,|B|}{(1-\varepsilon)^2}\big(1 - 2\varepsilon + \Gamma_\infty((0,\varepsilon)^2)\big) & \text{for } A \times B \subset (\varepsilon,1) \times (\varepsilon,1). \end{cases}$$

(See Figure 4.)

We see that outside $(0,\varepsilon]^2$, the measure Γ_b is absolutely continuous with respect to the Lebesgue measure. For $(p,q) \in (0,\varepsilon] \times (\varepsilon,1)$, we can compute the density of Γ_b to be

$$(57)\quad \frac{d\Gamma_b(p,q)}{dp\,dq} = \begin{cases} 0 & \text{if } p \leq \tau^2\varepsilon, \\ (1 - \tau\varepsilon^{1/2}/(2p^{1/2}))/(1-\varepsilon) & \text{if } p > \tau^2\varepsilon. \end{cases}$$

So, provided ε was chosen so that $\varepsilon \leq \tau/2$—which we will assume from now on—that density is bounded by 1 on the whole set $(0,\varepsilon] \times (\varepsilon,1)$, and thus, using the symmetry of Γ_b under switching p and q, also on $(\varepsilon,1) \times (0,\varepsilon]$. Moreover, for $(p,q) \in (\varepsilon,1) \times (\varepsilon,1)$ we compute that the density of Γ_b at

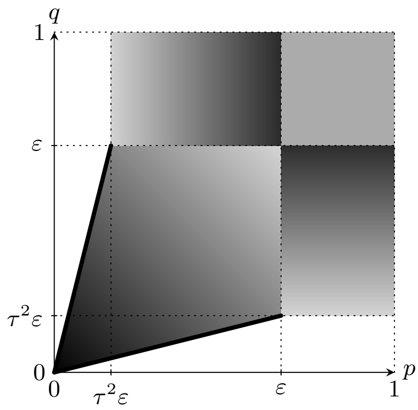

Fig. 4. A schematic representation of the measure Γ_b

(p,q) is

$$\frac{d\Gamma_b(p,q)}{dp\,dq} = \frac{1-2\varepsilon+\tau\varepsilon}{(1-\varepsilon)^2}. \tag{58}$$

Now let us consider $A, B \subset (0,1)$. We denote $A_1 := A \cap (0,\varepsilon]$ and $A_2 := A \cap (\varepsilon, 1)$, $B_1 := B \cap (0,\varepsilon], B_2 := B \cap (\varepsilon,1)$. Using the above density computations, we obtain $\Gamma_b(A_1 \times B_2) \le |A_1|\,|B_2|$ and $\Gamma_b(A_2 \times B_1) \le |A_2|\,|B_1|$. Thus, splitting $A \times B$ into $A_1 \times B_1 \cup A_1 \times B_2 \cup A_2 \times B_1 \cup A_2 \times B_2$, we get

(59)

$$\begin{aligned}\Gamma_b(A\times B) - |A|\,|B| &\le \Gamma_b(A_1 \times B_1) - |A_1|\,|B_1| + \Gamma_b(A_2\times B_2) - |A_2|\,|B_2| \\ &\le \tau |A_1|^{1/2}|B_1|^{1/2} + \frac{\tau\varepsilon-\varepsilon^2}{(1-\varepsilon)^2}|A_2|\,|B_2|,\end{aligned}$$

where the second inequality using simultaneously Lemma 4.9, nonnegativity of $|A_1|\,|B_1|$, and the value of the density of Γ_b on $(\varepsilon,1)^2$.

Our goal is to prove that for $\mathbb{P} = \Gamma_b$, one has $\tau'(\mathcal{A},\mathcal{B}) \le \tau_1$ (where $\tau_1 > \tau$ is the arbitrary number which was fixed in the statement of Theorem 4.2). In other words, we want to show that

(60)

$$\forall A, B \subset (0,1) \quad \bigl|\Gamma_b(A\times B) - |A|\,|B|\bigr| \le \tau_1 |A|^{1/2}|B|^{1/2}(1-|A|)^{1/2}(1-|B|)^{1/2}.$$

First, we notice that it suffices to prove (60) with no absolute value on the left-hand side:

(61)

$$\forall A, B \subset (0,1) \quad \Gamma_b(A\times B) - |A|\,|B| \le \tau_1 |A|^{1/2}|B|^{1/2}(1-|A|)^{1/2}(1-|B|)^{1/2};$$

indeed, if one replaces B by its complement $B^c := (0,1) \setminus B$, the left-hand side of (61) just changes sign while the right-hand side remains unchanged.

It is even sufficient to prove (61) only for $|A| \leq 1/2$, since none of the sides of (61) changes when one replaces simultaneously A by A^c and B by B^c. Therefore, from now on we will assume that $|A| \leq 1/2$. But then (61) is automatic for $|B| \geq 1/(1+\tau_1^2)$, since in that case

$$\begin{aligned}\Gamma_b(A\times B) - |A|\,|B| \leq \Gamma_b(A\times(0,1)) - |A|\,|B| &= |A|(1-|B|)\\ &\leq |A|^{1/2}(1-|A|)^{1/2} \times \tau_1|B|^{1/2}(1-|B|)^{1/2}.\end{aligned}$$

So, it will be enough to prove (61) for $|A| \leq 1/2$ and $|B| \leq 1/(1+\tau_1^2)$.

We start from (59):

$$\Gamma_b(A\times B) - |A|\,|B| \leq \tau|A_1|^{1/2}|B_1|^{1/2} + \frac{\tau\varepsilon-\varepsilon^2}{(1-\varepsilon)^2}|A_2|\,|B_2|. \tag{62}$$

Our goal is to bound above the right-hand side of (62) by some multiple of $|A|^{1/2}|B|^{1/2}(1-|A|)^{1/2}(1-|B|)^{1/2}$. Recall that $A_1 := A\cap(0,\varepsilon]$ and $B_1 := B\cap(0,\varepsilon]$, so that $|A_1|, |B_1| \leq \varepsilon$. First we have

$$|A_1|^{1/2} \leq |A_1|^{1/2}\frac{(1-|A_1|)^{1/2}}{(1-\varepsilon)^{1/2}} \leq (1-\varepsilon)^{-1/2}|A|^{1/2}(1-|A|)^{1/2},$$

where the second inequality comes from the fact that $|A_1| \leq |A|$ and that $p \mapsto p^{1/2}(1-p)^{1/2}$ is increasing on $[0,1/2]$. Similarly, provided ε was chosen small enough,

$$\begin{aligned}|B_1|^{1/2} \leq |B_1|^{1/2}\frac{(1-|B_1|)^{1/2}}{(1-\varepsilon)^{1/2}} &\leq (1-\varepsilon)^{-1/2}(|B|\wedge\varepsilon)^{1/2}(1-|B|\wedge\varepsilon)^{1/2}\\ &\leq (1-\varepsilon)^{-1/2}|B|^{1/2}(1-|B|)^{1/2},\end{aligned}$$

where the last inequality is a consequence of the fact that $\varepsilon^{1/2}(1-\varepsilon)^{1/2} \leq q^{1/2}(1-q)^{1/2}$ for all $q \in [\varepsilon, 1-\varepsilon]$, hence for all $q \in [\varepsilon, 1/(1+\tau_1^2)]$ provided $\varepsilon \leq \tau_1^2/(1+\tau_1^2)$ (which we will assume from now on). Next,

$$|A_2| \leq |A| \leq |A|^{1/2}(1-|A|)^{1/2}$$

(using again that $|A| \leq 1/2$), and similarly

$$|B_2| \leq |B| \leq \tau_1^{-1}|B|^{1/2}(1-|B|)^{1/2}.$$

Putting the previous bounds into (62), we find that for all A, B such that $|A| \leq 1/2, |B| \leq 1/(1+\tau_1^2)$ (and provided ε was chosen small enough),

$$\Gamma_b(A\times B) - |A|\,|B| \leq \left(\frac{\tau}{1-\varepsilon} + \frac{\varepsilon\tau-\varepsilon^2}{\tau_1(1-\varepsilon)^2}\right)|A|^{1/2}|B|^{1/2}(1-|A|)^{1/2}(1-|B|)^{1/2}. \tag{63}$$

The numerical factor on the right-hand side of (63) tends to τ as $\varepsilon \searrow 0$, so it is actually $\leq \tau_1$ provided ε was chosen small enough. In the end we have proved that for $\mathbb{P} = \Gamma_b$, one has $\tau'(\mathcal{A},\mathcal{B}) \leq \tau_1$.

To end the proof of Theorem 4.2, it remains to show that $\rho(\mathcal{A},\mathcal{B}) \geq \tau(1-\log\tau)$. That will be easier, as it suffices to find $\mathcal{A}$- resp. $\mathcal{B}$-measurable L^2 r.v. X and Y such that $|\mathrm{Cov}(X,Y)|/\mathrm{Var}(X)^{1/2}\,\mathrm{Var}(Y)^{1/2}$ is arbitrarily close to $\tau(1-\log\tau)$. To do that, we take $l \in (0,\varepsilon)$ and we set

$$X := \mathbf{1}_{l\leq p\leq\varepsilon}p^{-1/2}, \qquad Y := \mathbf{1}_{l\leq q\leq\varepsilon}q^{-1/2}.$$

Since both p- and q-marginals of $\varGamma_b$ are equal to the Lebesgue measure, we have $\mathbb{E}(X) = \mathbb{E}(Y) = 2\varepsilon^{1/2} - 2l^{1/2}$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2) = \log\varepsilon - \log l$, whence $\mathrm{Var}(X), \mathrm{Var}(Y) \sim |\log l|$ as $l \searrow 0$. On the other hand, XY is zero outside $(0,\varepsilon]^2$, so by the structure of $\varGamma_b$ we have

$$\mathbb{E}(XY) = \int \mathbf{1}_{l\leq p,q\leq\varepsilon}p^{-1/2}q^{-1/2}\,d\varGamma_\infty(p,q).$$

According to the structure of $\varGamma_\infty$, we compute that quantity to be equal to

$$\begin{aligned}\mathbb{E}(XY) &= \tau(\log\tau\log l - \log\tau\log\varepsilon - \log^2\tau - \log l + 2\log\tau + \log\varepsilon)\\ &\sim_{l\searrow 0} \tau(1-\log\tau)|\log l|.\end{aligned}$$

(In our computation we assumed that $l \leq \tau^4\varepsilon$.) So, when $l \searrow 0$, the Pearson correlation between X and Y tends to $\tau(1-\log\tau)$. This shows that $\rho(\mathcal{A},\mathcal{B}) \geq \tau(1-\log\tau)$, thus ending the proof of Theorem 4.2.

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