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Analytic semigroups on vector valued noncommutative L^p-spaces

by

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Abstract. We give sufficient conditions on an operator space E and on a semigroup of operators on a von Neumann algebra M to obtain a bounded analytic or R-analytic semigroup $(T_t \otimes \mathrm{Id}_E)_{t\geq 0}$ on the vector valued noncommutative L^p-space $L^p(M,E)$. Moreover, we give applications to the $H^\infty(\Sigma_\theta)$ functional calculus of the generators of these semigroups, generalizing some earlier work of M. Junge, C. Le Merdy and Q. Xu.

1. Introduction. It is shown in [JMX] that whenever $(T_t)_{t\geq 0}$ is a noncommutative diffusion semigroup on a von Neumann algebra M equipped with a faithful normal state such that each T_t has the Rota dilation property, then the negative generator of its L^p-realization $(1 < p < \infty)$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $0 < \theta < \pi/2$ where $\Sigma_\theta = \{z \in \mathbb{C}^* : |\arg z| < \theta\}$ is the open sector of angle 2θ around the positive real axis $(0,+\infty)$. Our first principal result is an extension of this theorem to the vector valued case. We use a different approach using R-analyticity instead of square functions.

In order to describe our result, we need several definitions.

DEFINITION 1.1. Let (M,ϕ) and (N,ψ) be von Neumann algebras equipped with normal faithful states ϕ and ψ respectively. A linear map $T: M \to N$ is called a (ϕ,ψ)-*Markov map* if

(1) T is completely positive,
(2) T is unital,
(3) $\psi \circ T = \phi$,
(4) $T \circ \sigma_t^\phi = \sigma_t^\psi \circ T$ for all $t \in \mathbb{R}$, where $(\sigma_t^\phi)_{t\in\mathbb{R}}$ and $(\sigma_t^\psi)_{t\in\mathbb{R}}$ denote the automorphism groups of the states ϕ and ψ respectively.

In particular, when $(M,\phi) = (N,\psi)$, we say that T is a ϕ-*Markov map*.

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A linear map $T\colon M \to N$ satisfying conditions (1)–(3) above is normal. If, moreover, condition (4) is satisfied, then it is known that there exists a unique completely positive, unital map $T^*\colon N \to M$ such that

$$\phi(T^*(y)x) = \psi(yT(x)), \qquad x \in M,\, y \in N.$$

The next definition is a variation of the one of [AD] (see also [R] and [HM]).

Definition 1.2. A (ϕ, ψ)-Markov map $T\colon M \to N$ is called *QWEP-factorizable* if there exist a von Neumann algebra P with QWEP equipped with a faithful normal state χ, and $*$-monomorphisms $J_0\colon N \to P$ and $J_1\colon M \to P$ such that J_0 is (ϕ, χ)-Markov and J_1 is (ψ, χ)-Markov, satisfying, moreover, $T = J_0^* \circ J_1$. We say that T is *hyper-factorizable* if the same property is true with a hyperfinite von Neumann algebra P.

Now, we introduce the following definition (compare [HM, Definition 4.1] and [A1, Property 4.10])

Definition 1.3. Let M be a von Neumann algebra equipped with a normal faithful state ϕ. Let $(T_t)_{t\geq 0}$ be a w^*-continuous semigroup of ϕ-Markov maps on M. We say that the semigroup is *QWEP-dilatable* if there exist a von Neumann algebra N with QWEP equipped with a normal faithful state ψ, a w^*-continuous group $(U_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of N, a $*$-monomorphism $J\colon M \to N$ such that each U_t is ϕ-Markov and J is (ϕ, ψ)-Markov satisfying

$$T_t = \mathbb{E} \circ U_t \circ J, \qquad t \geq 0,$$

where $\mathbb{E} = J^*\colon N \to M$ is the canonical faithful normal conditional expectation preserving the states associated with J.

Let $C^*(\mathbb{F}_\infty)$ be the full group C^*-algebra of the free group $\mathbb{F}_\infty$. We say that an operator space E is *locally-*$C^*(\mathbb{F}_\infty)$ if

$$d_f(E) = \sup_{F\subset E,\,\text{finite-dimensional}} \inf\{d_{cb}(F,G) : G \subset C^*(\mathbb{F}_\infty)\} < \infty.$$

This property is stable under duality and complex interpolation. All natural examples satisfy $d_f(E) = 1$ (see [Pi6, Chapter 21] and [Har] for more information on this class of operator spaces). If M is a von Neumann algebra with QWEP equipped with a normal faithful state and if E is locally-$C^*(\mathbb{F}_\infty)$, then the vector valued non commutative L^p-space $L^p(M,E)$ is well-defined and generalizes the classical construction for hyperfinite von Neumann algebras (see Section 2 for more information).

For any index set I, we denote by $\mathrm{OH}(I)$ the associated operator Hilbert space introduced by G. Pisier (see [Pi4] and [Pi6] for the details). Recall that OUMD_p is the operator space analogue of the *Unconditional Martingale Differences* (*UMD*) property of Banach spaces (see Section 2 for more information). We also use a similar property OUMD'_p for QWEP von Neumann algebras. The definition is given in Section 2. Our main result is the following theorem.

THEOREM 1.4. *Let M be a von Neumann algebra with QWEP equipped with a normal faithful state. Let $(T_t)_{t\geq 0}$ be a QWEP-dilatable semigroup of operators on M. Suppose $1 < p, q < \infty$ and $0 < \alpha < 1$. Let E be an operator space such that $E = (\mathrm{OH}(I), F)_\alpha$ for some index set I and for some operator space F with $1/p = (1-\alpha)/2 + \alpha/q$. Assume one of the following conditions holds:*

1. *Each T_t is hyper-factorizable and F is $OUMD_q$.*
2. *Each T_t is QWEP-factorizable and F is $OUMD'_q$.*

Let $-A_p$ be the generator of the strongly continuous semigroup $(T_t \otimes \mathrm{Id}_E)_{t\geq 0}$ on $L^p(M, E)$. Then for some $0 < \theta < \pi/2$, the operator A_p has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus.

This result can be used for the noncommutative Poisson semigroup or the q-Ornstein–Uhlenbeck semigroup and for example $E = \mathrm{OH}(I)$ for any index set I. A version of this result for semigroups of Schur multipliers is also given.

A famous theorem of G. Pisier [Pi2] says that a Banach space X is K-convex (i.e. does not contain ℓ^1_n's uniformly) if and only if the vectorial Rademacher projection $P \otimes \mathrm{Id}_X$ is bounded on the Bochner space $L^2(\Omega, X)$ where Ω is a probability space. In his proof, Pisier showed that if X is K-convex then any w^*-continuous semigroup $(T_t)_{t\geq 0}$ of positive unital self-adjoint Fourier multipliers on a locally compact abelian group G induces a bounded analytic semigroup $(T_t \otimes \mathrm{Id}_X)_{t\geq 0}$ on the Bochner space $L^p(G, X)$ where $1 < p < \infty$. Moreover, he proved a similar result for general w^*-continuous semigroups of positive unital contractions on a measure space if X does not contain, for some $\lambda > 1$, any subspace λ-isomorphic to ℓ^1_2. We give noncommutative analogues of these results. They are crucial steps in the proof of our Theorem 1.4.

We say that an operator space E is *OK-convex* if the vector valued Schatten space $S^2(E)$ is K-convex. This notion was introduced in [JP]. Using the preservation of K-convexity under complex interpolation (see [Pi1]), it is easy to see that it is equivalent to the K-convexity of the Banach space $S^p(E)$ for some (equivalently all) $1 < p < \infty$.

Our second principal result is the following theorem:

THEOREM 1.5. *Suppose that G is an amenable discrete group or that G is the free group $\mathbb{F}_n$ with n generators $(1 \leq n \leq \infty)$. Let $(T_t)_{t\geq 0}$ be a w^*-continuous semigroup of self-adjoint completely positive unital Fourier multipliers on the group von Neumann algebra $\mathrm{VN}(G)$ preserving the canonical trace. Let E be an OK-convex operator space. If $G = \mathbb{F}_n$, suppose that E*

is locally-$C^*(\mathbb{F}_\infty)$ *with* $d_f(E)=1$. *Consider* $1<p<\infty$. *Then* $(T_t \otimes \mathrm{Id}_E)_{t\geq 0}$ *defines a strongly continuous bounded analytic semigroup on the Banach space* $L^p(\mathrm{VN}(G),E)$.

We will show that this result can be used, for example, in the case where E is a Schatten space S^q or a commutative L^q-space with $1<q<\infty$.

The next theorem is an R-analytic version of Theorem 1.5 and it is our last principal result.

THEOREM 1.6. *Let* M *be a von Neumann algebra with QWEP equipped with a normal faithful state. Let* $(T_t)_{t\geq 0}$ *a* w^**-continuous semigroup of operators on* M. *Suppose* $1<p,q<\infty$ *and* $0<\alpha<1$. *Let* E *be an operator space such that* $E=(\mathrm{OH}(I),F)_\alpha$ *for some index set* I *and for some operator space* F *with* $1/p=1-\alpha/2+\alpha/q$. *Assume one of the following conditions holds:*

1. *Each* T_t *is hyper-factorizable and* F *is* $OUMD_q$.
2. *Each* T_t *is QWEP-factorizable and* F *is* $OUMD'_q$.

Then $(T_t \otimes \mathrm{Id}_E)_{t\geq 0}$ *defines a strongly continuous* R*-analytic semigroup on the Banach space* $L^p(M,E)$.

Finally, we also give a version of these two theorems for semigroups of Schur multipliers.

The paper is organized as follows. Section 2 gives a brief presentation of vector valued noncommutative L^p-spaces and we introduce some notions which are relevant to our paper. Section 3 contains the proof of Theorem 1.5. Section 4 is devoted to proving Theorem 1.6. In Section 5, we give applications to functional calculus. In particular, we prove Theorem 1.4. Finally, we present some natural examples to which the results of this paper can be applied.

2. Preliminaries. The readers are referred to [ER], [Pa] and [Pi6] for details on operator spaces and completely bounded maps and to the survey [PX] for noncommutative L^p-spaces.

The theory of vector valued noncommutative L^p-spaces was initiated by G. Pisier [Pi5] for the case where the underlying von Neumann algebra is hyperfinite and equipped with a faithful normal semifinite trace. Suppose $1\leq p<\infty$. For an operator space E and a hyperfinite von Neumann algebra M equipped with a faithful normal semifinite trace, we define by complex interpolation

$$L^p(M,E)=(M\otimes_{\min} E, L^1(M)\,\widehat{\otimes}\, E)_{1/p}, \tag{2.1}$$

where $\otimes_{\min}$ and $\widehat{\otimes}$ denote the injective and the projective tensor products of operator spaces. In [J1] and [J2], M. Junge extended this theory to the case where the underlying von Neumann algebra satisfies QWEP using the following characterization of QWEP von Neumann algebras. It is unknown whether every von Neumann algebra has this property. See the survey [O] for more information on this notion.

PROPOSITION 2.1. *A von Neumann algebra M is QWEP if and only if there exist an index set I, a free ultrafilter $\mathcal{U}$ on an index set L, a normal $*$-monomorphism*

$$\pi : M \to B(\ell_I^2)^{\mathcal{U}}$$

and a normal conditional expectation

$$\mathbb{E}\colon B(\ell_I^2)^{\mathcal{U}} \to \pi(M)$$

where $B(\ell_I^2)^{\mathcal{U}}$ denotes the ultrapower of $B(\ell_I^2)$ associated with $\mathcal{U}$.

Note that, in general, $\mathbb{E}$ is not faithful.

Now, we introduce the vector valued noncommutative L^p-spaces associated to a von Neumann algebra with QWEP equipped with a normal faithful state. Let M and $\beta = (\pi, \mathbb{E}, \mathcal{U}, I)$ be as in Proposition 2.1. Suppose $1 < p < \infty$. Let E be an operator space. Recall that the vector valued noncommutative L^p-space $L^p(M, \beta, E)$ is a closed subspace of the ultrapower $S_I^p(E)^{\mathcal{U}}$ which depends on the choice of β (see [J2] for a precise definition). However, when E is locally-$C^*(\mathbb{F}_\infty)$, then $L^p(M, \beta, E)$'s are all equivalent allowing the constant $d_f(E)^2$. Thus, we can still say that $L^p(M, \beta, E)$ does not depend on the choice of β, and therefore we will use the more convenient notation $L^p(M, E)$ instead of $L^p(M, \beta, E)$.

Note the following vector valued extension property of completely positive maps between noncommutative L^p-spaces (see [Pi3] and [J2]).

PROPOSITION 2.2. *Suppose $1 < p < \infty$.*

1. *Let M and N be hyperfinite von Neumann algebras equipped with normal faithful semifinite traces and let E be an operator space. Let $T\colon M \to N$ be a trace preserving unital normal completely positive map. Then the operator $T \otimes \mathrm{Id}_E$ extends to a bounded operator from $L^p(M, E)$ into $L^p(N, E)$ and*

$$\|T \otimes \mathrm{Id}_E\|_{L^p(M,E) \to L^p(N,E)} \leq \|T\|_{L^p(M) \to L^p(N)}.$$

2. *Let M and N be von Neumann algebras with QWEP equipped with normal faithful states ϕ and ψ respectively and let E be a locally-$C^*(\mathbb{F}_\infty)$ operator space. Let $T\colon M \to N$ be a (ϕ, ψ)-Markov map. Then the operator $T \otimes \mathrm{Id}_E$ extends to a bounded operator from $L^p(M, E)$ into $L^p(N, E)$ and*

$$\|T \otimes \mathrm{Id}_E\|_{L^p(M,E) \to L^p(N,E)} \leq d_f(E) \|T\|_{L^p(M) \to L^p(N)}.$$

Suppose that G is a discrete group. We denote by e_G the neutral element of G. We denote by $\lambda_g\colon \ell^2_G \to \ell^2_G$ the unitary operator of left translation by g, and by $\mathrm{VN}(G)$ the von Neumann algebra of G spanned by the λ_g where $g \in G$. It is a finite algebra with normalized trace given by

$$\tau_G(x) = \langle \epsilon_{e_G}, x(\epsilon_{e_G})\rangle_{\ell^2_G}$$

where $(\epsilon_g)_{g\in G}$ is the canonical basis of ℓ^2_G and $x \in \mathrm{VN}(G)$. For any $g \in G$, note that

$$\tau_G(\lambda_g) = \delta_{g,e_G}. \tag{2.2}$$

Recall that the von Neumann algebra $\mathrm{VN}(G)$ is hyperfinite if and only if G is amenable [SS, Theorem 3.8.2]. A *Fourier multiplier* is a normal linear map $T\colon \mathrm{VN}(G) \to \mathrm{VN}(G)$ such that there exists a function $\varphi\colon G \to \mathbb{C}$ such that $T(\lambda_g) = \varphi_g\lambda_g$ for any $g \in G$. In this case, we denote T by

$$M_\varphi : \mathrm{VN}(G) \to \mathrm{VN}(G), \qquad \lambda_g \mapsto \varphi_g\lambda_g.$$

Let M be a von Neumann algebra equipped with a semifinite normal faithful trace τ. Suppose that $T\colon M \to M$ is a normal contraction. We say that T is *selfadjoint* if for all $x, y \in M \cap L^1(M)$ we have

$$\tau(T(x)y^*) = \tau(x(T(y))^*).$$

In this case, it is not hard to show that the restriction $T|M \cap L^1(M)$ extends to a contraction $T\colon L^1(M) \to L^1(M)$. By complex interpolation, for any $1 \le p \le \infty$, we obtain a contractive map $T\colon L^p(M) \to L^p(M)$. Moreover, the operator $T\colon L^2(M) \to L^2(M)$ is selfadjoint. If $T\colon M \to M$ is actually a normal selfadjoint complete contraction, it is easy to see that the map $T\colon L^p(M) \to L^p(M)$ is completely contractive for any $1 \le p \le \infty$. It is not difficult to show that a contractive Fourier multiplier $M_\varphi\colon \mathrm{VN}(G) \to \mathrm{VN}(G)$ is selfadjoint if and only if $\varphi\colon G \to \mathbb{C}$ is a real function. Finally, one can prove that a contractive Schur multiplier $M_A\colon B(\ell^2_I) \to B(\ell^2_I)$ associated with a matrix A is selfadjoint if and only if all entries of A are real.

Now, we introduce the operator space version of the Banach space property UMD. Let E be an operator space and $1 < p < \infty$. We say that E is OUMD_p if there exists a positive constant C such that for any positive integer n, any choice of signs $\varepsilon_k = \pm 1$ and any martingale difference sequence $(dx_k)_{k=1}^n \subset L^p(M, E)$ relative to a filtration $(M_k)_{k\ge1}$ of a hyperfinite von Neumann algebra M equipped with a normal faithful finite trace we have

$$\Big\|\sum_{k=1}^n \varepsilon_k dx_k\Big\|_{L^p(M,E)} \le C\Big\|\sum_{k=1}^n dx_k\Big\|_{L^p(M,E)}. \tag{2.3}$$

See [M] and [Q] for more information on this property. We also need a variant of this property for QWEP von Neumann algebras.

DEFINITION 2.3. Suppose $1 < p < \infty$. Let E be a locally-$C^*(\mathbb{F}_\infty)$ operator space with $d_f(E) = 1$. We say that E is OUMD'_p if there exists a positive constant C such that for any positive integer n, any choice of signs $\varepsilon_k = \pm 1$ and any martingale difference sequence $(dx_k)_{k=1}^n \subset L^p(M, E)$ relative to a filtration $(M_k)_{k\geq 1}$ of a QWEP von Neumann algebra M equipped with a normal faithful state we have the inequality (2.3).

Suppose $1 < p < \infty$. Any OUMD'_p-operator space E is OUMD_p. Let $1 < q, r < \infty$ and $0 < \theta < 1$ be such that $1/p = (1-\alpha)/q + \alpha/r$. If (E_0, E_1) is a compatible couple of operator spaces, where E_0 is OUMD_q and E_1 is OUMD_r, then the complex interpolation operator space $E_\theta = (E_0, E_1)_\alpha$ is OUMD_p. For any index set I and any $1 < p < \infty$, the operator Hilbert space $\mathrm{OH}(I)$ is OUMD_p. If E is OUMD_p then the Banach space $S^p(E)$ is UMD (hence K-convex). It is easy to see that the same properties are valid for OUMD'_p operator spaces (with the same proofs).

3. Analyticity. Let X be a Banach space. A strongly continuous semigroup $(T_t)_{t\geq 0}$ is called *bounded analytic* if there exist $0 < \theta < \pi/2$ and a bounded holomorphic extension

$$\Sigma_\theta \to B(X), \qquad z \mapsto T_z.$$

See [KW] and [Haa] for more information on this notion.

We need the following theorem which is a corollary [Pi2, Lemma 4] of a result of Beurling [Be] (see also [F] and [Hin]).

THEOREM 3.1. *Let X be a Banach space. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of contractions on X. Suppose that there exists some integer $n \geq 1$ such that for any $t > 0$,*

$$\|(\mathrm{Id}_X - T_t)^n\|_{X\to X} < 2^n.$$

Then the semigroup $(T_t)_{t\geq 0}$ is bounded analytic.

Moreover, we will use the following lemma [Pi2, Lemma 1.5].

LEMMA 3.2. *Let $n \geq 1$ be an integer. Suppose that X is a real Banach space such that, for some $\lambda > 1$, X does not contain any subspace λ-isomorphic to ℓ^1_{n+1}. Then there exists a real number $0 < \rho < 2$ such that if $P_1, \ldots, P_n$ is any finite collection of mutually commuting norm one projections on X, then*

$$\Big\| \prod_{1\leq k\leq n} (\mathrm{Id}_X - P_k) \Big\|_{X\to X} \leq \rho^n.$$

Note that it is known that a complex Banach space contains ℓ^1_n's uniformly if and only if the underlying real Banach space has the same property.

Finally, we recall the following result (see [DH, Lemma 1] and [K, Theorem 2.14] for a complete proof).

LEMMA 3.3. *Let M and N be two von Neumann algebras equipped with faithful normal finite traces, and let $(x_i)_{i\in I} \in M$ and $(y_i)_{i\in I} \in N$ be families of elements of M and N respectively that have the same $*$-distribution. Then the von Neumann algebras generated respectively by the x_i's and the y_i's are isomorphic, with a normal $*$-isomorphism sending x_i to y_i and preserving the trace.*

We will use the result below which is a variant of Fell's well-known absorption principle (see [Pi6, Proposition 8.1]). This proposition is a substitute for the trick of G. Pisier [Pi2, Lemma 1.6].

PROPOSITION 3.4. *Let G be a discrete group and $1 \leq p < \infty$. Let E be an operator space. If G is non-amenable, we assume that $\mathrm{VN}(G)$ has QWEP and that E is locally-$C^*(\mathbb{F}_\infty)$ with $d_f(E) = 1$. For any function $a\colon G \to E$ finitely supported on G, we have*

$$(3.1)\qquad \Big\|\sum_{g\in G} \lambda_g \otimes \overset{(n)}{\cdots} \otimes \lambda_g \otimes a_g\Big\|_{L^p(\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G),E)} = \Big\|\sum_{g\in G} \lambda_g \otimes a_g\Big\|_{L^p(\mathrm{VN}(G),E)}.$$

Moreover, for any completely positive unital Fourier multiplier M_φ on the von Neumann algebra $\mathrm{VN}(G)$ preserving the canonical trace and any positive integer n we have

$$(3.2)\qquad \|M_\varphi^n \otimes \mathrm{Id}_E\|_{L^p(\mathrm{VN}(G),E)\to L^p(\mathrm{VN}(G),E)} \leq \|M_\varphi^{\otimes n} \otimes \mathrm{Id}_E\|_{L^p(\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G),E)\to L^p(\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G),E)}.$$

Proof. Suppose that m is a positive integer and $g_1,\ldots,g_m \in G$. Let $\eta_1,\ldots,\eta_m \in \{*,1\}$ and $\varepsilon_1,\ldots,\varepsilon_m \in \{-1,1\}$ be the associated signs (i.e. $\varepsilon_i = -1$ if and only if $\eta_i = *$). For any integer n, using (2.2), we see that

$$\begin{aligned}
&\tau_G^{\otimes n}\big((\lambda_{g_{r_1}} \otimes \overset{(n)}{\cdots} \otimes \lambda_{g_{r_1}})^{\eta_1} \cdots (\lambda_{g_{r_m}} \otimes \overset{(n)}{\cdots} \otimes \lambda_{g_{r_m}})^{\eta_m}\big)\\
&\qquad = \tau_G^{\otimes n}\big((\lambda_{g_{r_1}}^{\eta_1} \otimes \overset{(n)}{\cdots} \otimes \lambda_{g_{r_1}}^{\eta_1}) \cdots (\lambda_{g_{r_m}}^{\eta_m} \otimes \overset{(n)}{\cdots} \otimes \lambda_{g_{r_m}}^{\eta_m})\big)\\
&\qquad = \tau_G(\lambda_{g_{r_1}^{\varepsilon_1}\cdots g_{r_m}^{\varepsilon_m}})^n = \tau_G(\lambda_{g_{r_1}^{\varepsilon_1}\cdots g_{r_m}^{\varepsilon_m}}) = \tau_G(\lambda_{g_{r_1}}^{\eta_1} \cdots \lambda_{g_{r_m}}^{\eta_m}).
\end{aligned}$$

We infer that the families $(\lambda_g \otimes \cdots \otimes \lambda_g)_{g\in G}$ and $(\lambda_g)_{g\in G}$ have the same $*$-distribution with respect to the von Neumann algebras $\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G)$ equipped with $\tau_G^{\otimes n}$ and $\mathrm{VN}(G)$ equipped with τ_G. We conclude by using Lemma 3.3 and Proposition 2.2.

Now, we prove the "moreover" part. Using (3.1) twice, for any positive integer n and any function $a\colon G \to E$ finitely supported on G, we have

$$\begin{aligned}
&\Big\|(M_\varphi^n \otimes \mathrm{Id}_E)\Big(\sum_{g\in G} \lambda_g \otimes a_g\Big)\Big\|_{L^p(\mathrm{VN}(G),E)} = \Big\|\sum_{g\in G} \varphi_g^n \lambda_g \otimes a_g\Big\|_{L^p(\mathrm{VN}(G),E)}\\
&= \Big\|\sum_{g\in G} \varphi_g^n\, \lambda_g \otimes \overset{(n)}{\cdots} \otimes \lambda_g \otimes a_g\Big\|_{L^p(\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G),E)}\\
&= \Big\|(M_\varphi \otimes \overset{(n)}{\cdots} \otimes M_\varphi \otimes \mathrm{Id}_E)\Big(\sum_{g\in G} \lambda_g \otimes \overset{(n)}{\cdots} \otimes \lambda_g \otimes a_g\Big)\Big\|_{L^p(\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G),E)}\\
&\leq \|M_\varphi^{\otimes n} \otimes \mathrm{Id}_E\| \Big\|\sum_{g\in G} \lambda_g \otimes \overset{(n)}{\cdots} \otimes \lambda_g \otimes a_g\Big\|_{L^p(\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G),E)}\\
&= \|M_\varphi^{\otimes n} \otimes \mathrm{Id}_E\| \Big\|\sum_{g\in G} \lambda_g \otimes a_g\Big\|_{L^p(\mathrm{VN}(G),E)}.
\end{aligned}$$

Hence (3.2) follows. ■

PROPOSITION 3.5. *Let M be a von Neumann algebra. Let E be an OK-convex operator space and $1 < p < \infty$. Suppose that one of the following assertions is true.*

1. *M is hyperfinite and equipped with a normal faithful semifinite trace.*
2. *M has QWEP and is equipped with a normal faithful state and E is locally-$C^*(\mathbb{F}_\infty)$.*

Then the Banach space $L^p(M,E)$ is K-convex.

Proof. We begin with the second case. By definition, the Banach space $L^p(M,E)$ is a closed subspace of an ultrapower $S_I^p(E)^{\mathcal{U}}$ of the vector valued Schatten space $S_I^p(E)$ for some index set I. The Banach space $S^p(E)$ is K-convex. Hence the Banach space $S_I^p(E)$ is also K-convex. Recall that K-convexity is a super-property [DJT, p. 261], i.e. passes from a Banach space to all closed subspaces of its ultrapowers. We conclude that the Banach space $L^p(M,E)$ is K-convex. The first case is similar using [Pi5, Theorem 3.4] instead of ultraproducts. ■

Now, we are ready to give the proof of Theorem 1.5.

Proof of Theorem 1.5. By [R], for any $t \geq 0$, the operator $T_t = (T_{t/2})^2$ admits a Rota dilation (see [HM, Definition 5.1] and [JMX, p. 124] for a precise definition)

$$T_t^k = Q\mathbb{E}_k\pi, \qquad k \geq 1$$

where $Q\colon M \to \mathrm{VN}(G)$ and $\pi\colon \mathrm{VN}(G) \to M$ and where we use a crossed product

$$M = \Gamma_{-1}(\ell^{2,T} \otimes_2 \ell^2) \rtimes_\alpha G$$

equipped with its canonical trace. We infer that

$$\mathrm{Id}_{L^p(\mathrm{VN}(G))} - T_t = Q(\mathrm{Id}_{L^p(M)} - \mathbb{E}_1)\pi.$$

Let n be a positive integer. We deduce that

$$(\mathrm{Id}_{L^p(\mathrm{VN}(G))} - T_t)^{\otimes n} = Q(\mathrm{Id}_{L^p(M)} - \mathbb{E}_1)^{\otimes n}\pi.$$

For any integer $1 \leq j \leq n$, we let

$$\Pi_j = \mathrm{Id}_{L^p(M)} \otimes \overset{(j-1)}{\cdots} \otimes \mathrm{Id}_{L^p(M)} \otimes \mathbb{E}_1 \otimes \mathrm{Id}_{L^p(M)} \otimes \overset{(n-j)}{\cdots} \otimes \mathrm{Id}_{L^p(M)}.$$

Recall that the fermion algebra $\Gamma_{-1}(\ell^{2,T} \otimes_2 \ell^2)$ is $*$-isomorphic to the hyperfinite factor of type II_1. Moreover, if G is amenable then by [C, Proposition 6.8], M is hyperfinite. If $G = \mathbb{F}_n$, by [A1, Proposition 4.8], the von Neumann algebra M has QWEP. By Proposition 2.2, we deduce that the $\Pi_j \otimes \mathrm{Id}_E$'s are well-defined and form a family of mutually commuting contractive projections on $L^p(M, E)$. Moreover, we have

$$(\mathrm{Id}_{L^p(M)} - \mathbb{E}_1)^{\otimes n} \otimes \mathrm{Id}_E = \prod_{1 \leq j \leq n} (\mathrm{Id}_{L^p(M,E)} - (\Pi_j \otimes \mathrm{Id}_E)).$$

Using (3.2), Proposition 2.2 and Lemma 3.2, we obtain

$$\begin{aligned}
&\|(\mathrm{Id}_{L^p(\mathrm{VN}(G))} - T_t)^n \otimes \mathrm{Id}_E\|_{L^p(\mathrm{VN}(G),E) \to L^p(\mathrm{VN}(G),E)} \\
&\quad \leq \|(\mathrm{Id}_{L^p(\mathrm{VN}(G))} - T_t)^{\otimes n} \otimes \mathrm{Id}_E\|_{L^p(\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G),E) \to L^p(\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G),E)} \\
&\quad = \|Q(\mathrm{Id}_{L^p(M)} - \mathbb{E}_1)^{\otimes n}\pi \otimes \mathrm{Id}_E\|_{L^p(\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G),E) \to L^p(\overline{\bigotimes}_{i=1}^n \mathrm{VN}(G),E)} \\
&\quad \leq \|(\mathrm{Id}_{L^p(M)} - \mathbb{E}_1)^{\otimes n} \otimes \mathrm{Id}_E\|_{L^p(M,E) \to L^p(M,E)} \\
&\quad = \Big\| \prod_{1 \leq j \leq n} (\mathrm{Id}_{L^p(M,E)} - (\Pi_j \otimes \mathrm{Id}_E)) \Big\|_{L^p(M,E) \to L^p(M,E)} \leq \rho^n.
\end{aligned}$$

We conclude by applying Theorem 3.1. ∎

Now, we pass to general semigroups on QWEP von Neumann algebras. Note that the class of operator spaces considered in the following theorem is included in the class of OK-convex operator spaces.

THEOREM 3.6. *Let M be a von Neumann algebra with QWEP equipped with a normal faithful state. Let $(T_t)_{t \geq 0}$ be a w^*-continuous semigroup of QWEP-factorizable maps on M. Suppose $1 < p < \infty$. Let E be a locally-$C^*(\mathbb{F}_\infty)$ operator space with $d_f(E) = 1$ such that, for some $\lambda > 1$, $S^p(E)$ does not contain any subspace λ-isomorphic to ℓ_1^2. Then $(T_t \otimes \mathrm{Id}_E)_{t \geq 0}$ defines a strongly continuous bounded analytic semigroup on the Banach space $L^p(M, E)$.*

Proof. Using the fact that each T_t is QWEP-factorizable and [HM, Theorem 5.3], it is easy to see that each T_t admits a Rota dilation

$$T_t^k = Q\mathbb{E}_k\pi, \qquad k \geq 1,$$

where $Q \colon N \to M$ and $\pi \colon M \to N$ where N is a QWEP von Neumann algebra. The rest of the proof is similar to the one of Theorem 1.5. The only difference is that we do not use Proposition 3.4. We apply Lemma 3.2 with $n = 1$. ∎

For instance, if $S^p(E)$ (where $1 < p < \infty$) is uniformly convex then for some $\lambda > 1$, $S^p(E)$ does not contain any subspace λ-isomorphic to ℓ^1_2.

Suppose $1 < p, q < \infty$. Note that if an operator space E is OUMD_q then the Banach space $S^q(E)$ is UMD, hence uniformly convex. Now, we can write $1/p = (1-\alpha)/q + \alpha/r$ for some $0 < \alpha < 1$ and some $1 < r < \infty$. Then we have $S^p(E) = (S^q(E), S^r(E))_\alpha$. Now recall that, by [CR], if (X_0, X_1) is a compatible couple of Banach spaces, one of which is uniformly convex, then for any $0 < \alpha < 1$ the complex interpolation space $(X_0, X_1)_\alpha$ is also uniformly convex. We deduce that the Banach space $S^p(E)$ is uniformly convex. Hence, we can use in Theorem 3.6 (and also in Theorem 1.5) any operator space E which is OUMD_q for *some* $1 < q < \infty$ and locally-$C^*(\mathbb{F}_\infty)$ with $d_f(E) = 1$. This large class contains in particular Schatten spaces S^q and commutative L^q-spaces for any $1 < q < \infty$.

Finally, we deal with semigroups of Schur multipliers. Note that the construction in [R] of the Rota dilation for Schur multipliers on finite-dimensional $B(\ell^2_I)$ is actually true for any index set I. Moreover, the von Neumann algebra $\Gamma^e_{-1}(\ell^{2,T})$ of [R] is hyperfinite. Hence, the von Neumann algebra

$$M = B(\ell^2_I) \mathbin{\overline{\otimes}} \Big(\overline{\bigotimes_{n \in \mathbb{N}}} (\Gamma^e_{-1}(\ell^{2,T}), \tau) \Big).$$

of the Rota dilation of [R] is also hyperfinite. Using the above ideas, one can prove the following theorem.

Theorem 3.7. *Let $(T_t)_{t \geq 0}$ be a w^*-continuous semigroup of completely positive unital selfadjoint Schur multipliers on $B(\ell^2_I)$. Suppose $1 < p < \infty$. Let E be an operator space such that, for some $\lambda > 1$, $S^p(E)$ does not contain any subspace λ-isomorphic to ℓ^2_1. Then $(T_t \otimes \mathrm{Id}_E)_{t \geq 0}$ defines a strongly continuous bounded analytic semigroup on the Banach space $S^p_I(E)$.*

Remark 3.8. In [A1, Proposition 5.4], the author gives a concrete description of the semigroups of the above theorem.

4. R-analyticity. Let $(\varepsilon_k)_{k \geq 1}$ be a sequence of independent Rademacher variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We let $\mathrm{Rad}(X) \subset L^2(\Omega, X)$ be the closure of $\mathrm{span}\{\varepsilon_k \otimes x : k \geq 1,\ x \in X\}$ in the Bochner space $L^2(\Omega, X)$. Thus for any finite family $x_1, \dots, x_n$ in X, we have

$$\Big\| \sum_{k=1}^n \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(X)} = \Big(\int_\Omega \Big\| \sum_{k=1}^n \varepsilon_k(\omega) x_k \Big\|_X^2 \, d\omega \Big)^{1/2}.$$

We say that a set $F \subset B(X)$ is *R-bounded* provided that there is a constant $C \geq 0$ such that for any finite families $T_1, \dots, T_n$ in F and $x_1, \dots, x_n$ in X, we have

$$\Big\|\sum_{k=1}^{n} \varepsilon_k \otimes T_k(x_k)\Big\|_{\mathrm{Rad}(X)} \leq C\Big\|\sum_{k=1}^{n} \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)}.$$

In this case, we let $R(F)$ denote the smallest possible C, which is called the *R-bound* of F. Note that any singleton $\{T\}$ is automatically R-bounded with $R(\{T\}) = \|T\|_{X \to X}$. If a set F is R-bounded then the strong closure of F is R-bounded (with the same R-bound). Recall that if H is a Hilbert space, a subset of $B(H)$ is bounded if and only if it is R-bounded.

R-boundedness was introduced in [BG] and then developed in the fundamental paper [CPSW]. We refer to the latter paper and [KW, Section 2] for a detailed presentation.

The next result is a noncommutative version of the classical result of Bourgain [Bo] which itself is a vector valued generalization of a result of Stein [S].

Proposition 4.1. *Suppose $1 < p < \infty$. Let E be an operator space and let M be a von Neumann algebra. Assume one of the following condition holds:*

1. *M is hyperfinite and equipped with a normal faithful semifinite trace and E is OUMD_p.*
2. *M has QWEP and is equipped with a normal faithful state and E is OUMD'_p.*

Consider an increasing (or decreasing) sequence $(\mathbb{E}(\cdot|M_i))_{i\in\mathbb{N}}$ of (canonical) conditional expectations on some von Neumann subalgebras M_i of M. Then the set $\{\mathbb{E}(\cdot|M_i) \otimes \mathrm{Id}_E : i \in \mathbb{N}\}$ of conditional expectation operators is R-bounded on $L^p(M, E)$.

Proof. We only prove the decreasing case $M_1 \supset M_2 \supset \cdots$ and the hyperfinite case. The proofs of the other statements are similar. First suppose that the trace is finite. Let n be a positive integer and fix some positive integers $i_1 > \cdots > i_n$. We define the σ-algebra $\mathcal{F}_k = \sigma(\varepsilon_1, \dots, \varepsilon_k)$, $k = 1, \dots, n$, and $\mathcal{F}_0 = \emptyset$. We define the family $(N_m)_{m=1}^{2n}$ of von Neumann subalgebras of $L^\infty(\Omega) \otimes M$ by

$$N_{2k-1} = L^\infty(\Omega, \mathcal{F}_{k-1}, \mathbb{P}) \otimes M_{i_k}, \quad k = 1, \dots, n,$$
$$N_{2k} = L^\infty(\Omega, \mathcal{F}_k, \mathbb{P}) \otimes M_{i_k}, \quad k = 1, \dots, n.$$

These subalgebras form an increasing sequence $N_1 \subset \cdots \subset N_{2n}$. For an element $y \in L^p(L^\infty(\Omega) \otimes M, E)$, we define the martingale $(y_m)_{1 \leq m \leq 2n}$ by

$$y_m = (\mathbb{E}(\cdot|N_m) \otimes \mathrm{Id}_E)(y), \quad m = 1, \dots, 2n.$$

Let $(d_m)_{1\leq m\leq 2n}$ be the associated martingale difference sequence. Then, by the OUMD_p-property of E, we have

$$\Big\|\sum_{k=1}^{n} d_{2k}\Big\|_{L^p(L^\infty(\Omega)\otimes M,E)} = \frac{1}{2}\Big\|\sum_{m=1}^{2n} d_m + \sum_{m=1}^{2n}(-1)^m d_m\Big\|_{L^p(L^\infty(\Omega)\otimes M,E)}$$
$$\leq \frac{1}{2}\Big(\Big\|\sum_{m=1}^{2n} d_m\Big\|_{L^p(L^\infty(\Omega)\otimes M,E)} + \Big\|\sum_{m=1}^{2n}(-1)^m d_m\Big\|_{L^p(L^\infty(\Omega)\otimes M,E)}\Big)$$
$$\lesssim \Big\|\sum_{m=1}^{2n} d_m\Big\|_{L^p(L^\infty(\Omega)\otimes M,E)}$$

(here and below, we use $\lesssim$ to indicate an inequality up to a constant). Now fix $x_1,\ldots,x_n \in L^p(M,E)$ and put $y=\sum_{l=1}^n \varepsilon_l\otimes x_l$. For this choice of y and any integer $1\leq k\leq n$, we have

$$y_{2k-1} = (\mathbb{E}(\cdot|N_{2k-1})\otimes \mathrm{Id}_E)(y)$$
$$= \sum_{l=1}^{n}\big(\mathbb{E}(\cdot|L^\infty(\Omega,\mathcal{F}_{k-1},\mathbb{P})\otimes M_{i_k})\otimes \mathrm{Id}_E\big)(\varepsilon_l\otimes x_l)$$
$$= \sum_{l=1}^{n}\mathbb{E}(\varepsilon_l|L^\infty(\Omega,\mathcal{F}_{k-1},\mathbb{P}))\otimes(\mathbb{E}(\cdot|M_{i_k})\otimes \mathrm{Id}_E)(x_l)$$
$$= \sum_{l=1}^{k-1}\varepsilon_l\otimes(\mathbb{E}(\cdot|M_{i_k})\otimes \mathrm{Id}_E)(x_l)$$

and similarly for any integer $1\leq k\leq n$ we have

$$y_{2k} = \sum_{l=1}^{k}\varepsilon_l\otimes(\mathbb{E}(\cdot|M_{i_k})\otimes \mathrm{Id}_E)(x_l).$$

Therefore, for any $1\leq k\leq n$,

$$d_{2k} = y_{2k}-y_{2k-1}$$
$$= \sum_{l=1}^{k}\varepsilon_l\otimes(\mathbb{E}(\cdot|M_{i_k})\otimes \mathrm{Id}_E)(x_l) - \sum_{k=1}^{k-1}\varepsilon_l\otimes(\mathbb{E}(\cdot|M_{i_k})\otimes \mathrm{Id}_E)(x_l)$$
$$= \varepsilon_k\otimes(\mathbb{E}(\cdot|M_{i_k})\otimes \mathrm{Id}_E)(x_k)$$

and similarly $d_{2k-1}=0$ for any $1\leq k\leq n$. Finally, we obtain

$$\Big\|\sum_{k=1}^{n}\varepsilon_k\otimes(\mathbb{E}(\cdot|M_{i_k})\otimes \mathrm{Id}_E)(x_k)\Big\|_{L^p(L^\infty(\Omega)\otimes M,E)} = \Big\|\sum_{k=1}^{n} d_{2k}\Big\|_{L^p(L^\infty(\Omega)\otimes M,E)}$$
$$\lesssim \Big\|\sum_{m=1}^{2n} d_m\Big\|_{L^p(L^\infty(\Omega)\otimes M,E)} = \|y_{2m}\|_{L^p(L^\infty(\Omega)\otimes M,E)}$$

$$= \|(\mathbb{E}(\cdot|N_{2n}) \otimes \mathrm{Id}_E)(y)\|_{L^p(L^\infty(\Omega)\otimes M,E)}$$
$$\lesssim \|y\|_{L^p(L^\infty(\Omega)\otimes M,E)} = \Big\|\sum_{k=1}^n \varepsilon_k \otimes x_k\Big\|_{L^p(L^\infty(\Omega)\otimes M,E)}.$$

By the Khintchine–Kahane inequalities (e.g. [DJT, p. 211]), we conclude that

$$\Big\|\sum_{k=1}^n \varepsilon_k \otimes (\mathbb{E}(\cdot|M_{i_k}) \otimes \mathrm{Id}_E)(x_k)\Big\|_{\mathrm{Rad}(L^p(M,E))} \lesssim \Big\|\sum_{k=1}^n \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(L^p(M,E))}.$$

We deduce the general case of von Neumann algebras equipped with faithful normal semifinite traces by a straightforward application of the well-known reduction method of Haagerup [HJX]. ■

Let X be a Banach space. A strongly continuous semigroup $(T_t)_{t\geq 0}$ is called *R-analytic* if there exist $0 < \theta < \pi/2$ and a holomorphic extension

$$\Sigma_\theta \to B(X), \quad z \mapsto T_z,$$

with $R(\{T(z) : z \in \Sigma_\theta\}) < \infty$. See [KW] for more information and for applications to maximal regularity.

The following result is a particular case of [F, Theorem 6.1].

THEOREM 4.2. *Let $(T_{1,t})_{t\geq 0}$ and $(T_{2,t})_{t\geq 0}$ be two consistent semigroups given on an interpolation couple (X_1, X_2) of K-convex Banach spaces. Suppose $0 < \theta < 1$. Assume that $(T_{1,t})_{t\geq 0}$ is strongly continuous and R-analytic and that $R(\{T_{2,t} : 0 < t < 1\}) < \infty$. Then there exists a unique strongly continuous R-analytic semigroup $(T_t)_{t\geq 0}$ on $(X_1, X_2)_\theta$ which is consistent with $(T_{1,t})_{t\geq 0}$ and $(T_{2,t})_{t\geq 0}$.*

Now, we prove Theorem 1.6.

Proof of Theorem 1.6. We only prove the hyper-factorizable case. The QWEP-factorizable case is similar. We can identify $\mathrm{OH}(I)$ with ℓ^2_I completely isometrically. Fubini's theorem gives us the isometric isomorphism

$$L^2(M, \mathrm{OH}(I)) = \ell^2_I(L^2(M)).$$

Hence the Banach space $L^2(M, \mathrm{OH}(I))$ is isometric to a Hilbert space. On a Hilbert space, recall that any bounded set is R-bounded. By Theorem 3.6, we deduce that $(T_t \otimes \mathrm{Id}_{\mathrm{OH}(I)})_{t\geq 0}$ defines an R-analytic semigroup on $L^p(M, \mathrm{OH}(I))$.

Let $t_1, \ldots, t_n$ be rational numbers. We take a common denominator: we can write $t_i = s_i/d$ for some integers $d, s_1, \ldots, s_n$. The operator $T_{1/d}$ admits a Rota dilation:

$$T^k_{1/d} = Q\mathbb{E}_k\pi, \quad k \geq 1.$$

Let $x_1, \ldots, x_n \in F$. Using Proposition 4.1 and the fact that any singleton is R-bounded, we obtain

$$\begin{aligned}\Big\|\sum_{i=1}^{n} \varepsilon_i \otimes (T_{t_i} \otimes \mathrm{Id}_F)(x_i)\Big\|_{\mathrm{Rad}(L^p(M,F))} \\ &= \Big\|\sum_{i=1}^{n} \varepsilon_i \otimes (Q\mathbb{E}_{s_i}\pi \otimes \mathrm{Id}_F)(x_i)\Big\|_{\mathrm{Rad}(L^p(M,F))} \\ &\leq \Big\|\sum_{i=1}^{n} \varepsilon_i \otimes (\mathbb{E}_{s_i}\pi \otimes \mathrm{Id}_F)(x_i)\Big\|_{\mathrm{Rad}(L^p(N,F))} \\ &\leq R(\{\mathbb{E}_k \otimes \mathrm{Id}_F \colon k \geq 1\})\Big\|\sum_{i=1}^{n} \varepsilon_i \otimes (\pi \otimes \mathrm{Id}_F)x_i\Big\|_{\mathrm{Rad}(L^p(M,F))} \\ &\leq R(\{\mathbb{E}_k \otimes \mathrm{Id}_F \colon k \geq 1\})\Big\|\sum_{i=1}^{n} \varepsilon_i \otimes x_i\Big\|_{\mathrm{Rad}(L^p(M,F))}.\end{aligned}$$

By making use of the strong continuity of the semigroup, we conclude that $\{T_t \otimes \mathrm{Id}_F \colon t \geq 0\}$ is an R-bounded subset of $B(L^p(M, F))$.

Now, we have

$$L^p(M, E) = \big(L^2(M, \mathrm{OH}(I)), L^q(M, F)\big)_\alpha.$$

Furthermore, the operator space F is OUMD_p. We deduce that the Banach space $S^p(F)$ is UMD, hence K-convex. Thus the operator space F is OK-convex. By Proposition 3.5, we deduce that $L^q(M, F)$ is K-convex. Moreover the Banach space $L^2(M, \mathrm{OH}(I))$ is obviously K-convex. We conclude by using Theorem 4.2. ■

For example, we can take $E = \mathrm{OH}(I)$ in Theorem 1.6.

5. Applications to functional calculus.

We start with a little background on sectoriality and H^∞ functional calculus. We refer to [Haa], [KW], [JMX] and [L] for details and complements. A closed, densely defined linear operator $A \colon D(A) \subset X \to X$ is called *sectorial of type* ω if its spectrum $\sigma(A)$ is included in the closed sector $\overline{\Sigma_\omega}$, and for any angle $\omega < \theta < \pi$, there is a positive constant K_θ such that

$$\|(\lambda - A)^{-1}\|_{X \to X} \leq K_\theta/|\lambda|, \qquad \lambda \in \mathbb{C} - \overline{\Sigma_\theta}.$$

We recall that the sectorial operators of type $< \pi/2$ coincide with the negative generators of bounded analytic semigroups.

For any $0 < \theta < \pi$, let $H^\infty(\Sigma_\theta)$ be the algebra of all bounded analytic functions $f \colon \Sigma_\theta \to \mathbb{C}$, equipped with the supremum norm $\|f\|_{H^\infty(\Sigma_\theta)} = \sup\{|f(z)| \,:\, z \in \Sigma_\theta\}$. Let $H_0^\infty(\Sigma_\theta) \subset H^\infty(\Sigma_\theta)$ be the subalgebra of bounded analytic functions $f \colon \Sigma_\theta \to \mathbb{C}$ for which there exist $s, c > 0$ such that

$|f(z)| \leq c|z|^s(1+|z|)^{-2s}$ for any $z \in \Sigma_\theta$. Given a sectorial operator A of type $0<\omega<\pi$, a bigger angle $\omega<\theta<\pi$, and a function $f \in H_0^\infty(\Sigma_\theta)$, one can define a bounded operator $f(A)$ by means of a Cauchy integral (see e.g. [Haa, Section 2.3] or [KW, Section 9]); the resulting mapping $H_0^\infty(\Sigma_\theta) \to B(X)$ taking f to $f(A)$ is an algebra homomorphism. By definition, A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus provided that this homomorphism is bounded, that is, there exists a positive constant C such that $\|f(A)\|_{X\to X} \leq C\|f\|_{H^\infty(\Sigma_\theta)}$ for any $f \in H_0^\infty(\Sigma_\theta)$. In the case when A has a dense range, the latter boundedness condition allows a natural extension of $f \mapsto f(A)$ to the full algebra $H^\infty(\Sigma_\theta)$.

Suppose $1<p<\infty$. We say that an operator A on a vector valued noncommutative L^p-space $L^p(M,E)$ *admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus* if $\mathrm{Id}_{S^p} \otimes A$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus on the Banach space $S^p(L^p(M,E))$.

We will use the following theorem [KW, Corollary 10.9]:

Theorem 5.1. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup with generator $-A$ on a Banach space. If the semigroup has a dilation to a bounded strongly continuous group on a UMD Banach space, then A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $0<\theta<\pi$.*

We also need a particular case of [KaW, Proposition 5.1] which says that R-analyticity allows one to reduce the angle of a bounded $H^\infty(\Sigma_\theta)$ functional calculus below $\pi/2$.

Proposition 5.2. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup with generator $-A$ on a Banach space. Suppose that A has a bounded $H^\infty(\Sigma_{\theta_0})$ functional calculus for some $0<\theta_0<\pi$. If the semigroup $(T_t)_{t\geq 0}$ is R-analytic then A has indeed a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $0<\theta<\pi/2$.*

Now, we give the proof of our main result.

Proof of Theorem 1.4. The semigroup $(T_t)_{t\geq 0}$ is QWEP-dilatable. This yields the existence of a dilation of the semigroup $(T_t \otimes \mathrm{Id}_E)_{t\geq 0}$ on $L^p(M,E)$ to a strongly continuous group of isometries on $L^p(N,E)$.

Since the operator Hilbert space $\mathrm{OH}(I)$ has OUMD_2 and the operator space F has OUMD_q, we see by interpolation that the operator space E has OUMD_p. We infer that the Banach space $S^p(E)$ is UMD. Hence the Banach space $S_I^p(E)$ is also UMD for any index set I. Since N has QWEP, the Banach space $L^p(N,E)$ is a closed subspace of an ultrapower $S_I^p(E)^{\mathcal{U}}$ of the vector valued Schatten space $S_I^p(E)$ for some index set I. We deduce that the Banach space $L^p(N,E)$ is UMD.

By Theorem 5.1, we deduce that A_p has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $0<\theta<\pi$. By Theorem 1.6, the semigroup $(T_t \otimes \mathrm{Id}_E)_{t\geq 0}$

is R-analytic on $L^p(M,E)$. By Proposition 5.2, these two results imply that A_p actually admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $0<\theta<\pi/2$, as expected.

Finally, it is not hard to check that the above arguments work as well with $(\mathrm{Id}_{B(\ell^2)}\otimes T_t)_{t\geq 0}$ in place of $(T_t)_{t\geq 0}$. Thus A_p actually has a completely bounded functional calculus for some $0<\theta<\pi/2$. ■

Now, we give some natural examples to which the results of this paper can be applied.

Noncommutative Poisson semigroup. Let $n\geq 1$ be an integer. Here $\mathbb{F}_n$ denotes a free group with n generators denoted by $g_1,\dots,g_n$. Any $g\in\mathbb{F}_n$ has a unique decomposition of the form

$$g=g_{i_1}^{k_1}\cdots g_{i_l}^{k_l},$$

where $l\geq 0$ is an integer, each i_j belongs to $\{1,\dots,n\}$, each k_j is a nonzero integer, and $i_j\neq i_{j+1}$ if $1\leq j\leq l-1$. The case $l=0$ corresponds to the unit element $g=e_{\mathbb{F}_n}$. By definition, the *length* of g is defined as

$$|g|=|k_1|+\cdots+|k_l|.$$

This is the number of factors in the above decomposition of g. For any nonnegative real number $t\geq 0$, we have a normal unital completely positive selfadjoint map

$$T_t:\mathrm{VN}(\mathbb{F}_n)\to\mathrm{VN}(\mathbb{F}_n),\qquad \lambda_g\mapsto e^{-t|g|}\lambda_g.$$

These maps define a w^*-semigroup $(T_t)_{t\geq 0}$ called the *noncommutative Poisson semigroup* (see [JMX] for more information). In [A1, remark following Proposition 5.5], it is implicitly said that $(T_t)_{t\geq 0}$ is QWEP-dilatable. Moreover, using [R] and [A1, Proposition 4.8], we can show each T_t is QWEP-factorizable.

q-Ornstein–Uhlenbeck semigroup. We use the notations of [BKS]. Suppose $-1\leq q<1$. Let H be a real Hilbert space and let $(a_t)_{t\geq 0}$ be a strongly continuous semigroup of contractions on H. For any $t\geq 0$, let $T_t=\Gamma_q(a_t)$. Then $(T_t)_{\geq 0}$ is a w^*-semigroup of normal unital completely positive maps preserving the trace on the von Neumann algebra $\Gamma_q(H)$.

In the case where $a_t=e^{-t}I_H$, the semigroup $(T_t)_{\geq 0}$ is the so-called *q-Ornstein–Uhlenbeck semigroup*.

Using [R], [N] and the result [KW, Theorem 10.11] on dilation of strongly continuous semigroups of contractions on a Hilbert space it is not hard to see that we obtain examples of QWEP-dilatable semigroups of QWEP-factorizable maps.

We pass to Schur multipliers.

THEOREM 5.3. *Let $(T_t)_{t\geq 0}$ be a w^*-semigroup of self-adjoint contractive Schur multipliers on $B(\ell^2_I)$. Suppose $1 < p, q < \infty$ and $0 < \alpha < 1$. Let E be an operator space such that $E = (\mathrm{OH}(I), F)_\alpha$ for some index set I and for some OUMD$_q$-operator space F such that $1/p = (1-\alpha)/2 + \alpha/q$. Let $-A_p$ be the generator of the strongly continuous semigroup $(T_t \otimes \mathrm{Id}_E)_{t\geq 0}$ on $S^p_I(E)$. Then for some $0 < \theta < \pi/2$, the operator A_p has a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Proof. Arguing as in the proof of [A1, Corollary 4.3], we can reduce the general case to the unital and completely positive case. The proof for semigroups of unital completely positive Schur multipliers, using [A1, Proposition 5.5] and [A2], is similar to the one of Theorem 1.4. ∎

REMARK 5.4. The results of this paper lead to properties of some square functions; see [L, Section 7] for more information.

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