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Approximation of the Euclidean ball by polytopes

by

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Abstract. There is a constant c such that for every $n \in \mathbb{N}$, there is an N_n so that for every $N \geq N_n$ there is a polytope P in $\mathbb{R}^n$ with N vertices and

$$\mathrm{vol}_n(B_2^n \triangle P) \leq c\, \mathrm{vol}_n(B_2^n) N^{-\frac{2}{n-1}}$$

where B_2^n denotes the Euclidean unit ball of dimension n.

1. Main results. Let C and K be two convex bodies in $\mathbb{R}^n$. The Euclidean ball with center 0 and radius r is denoted by $B_2^n(r)$. The ball $B_2^n(1)$ is denoted by B_2^n. Let K be a convex body in $\mathbb{R}^n$ with C^2-boundary ∂K and everywhere strictly positive curvature κ. Then

$$\lim_{N\to\infty} \frac{\inf\{\mathrm{vol}_n(K \setminus P) \mid P \subseteq K \text{ and } P \text{ has at most } N \text{ vertices}\}}{N^{-\frac{2}{n-1}}} = \frac{1}{2}\, \mathrm{del}_{n-1} \Big(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}}\, d\mu_{\partial K}(x)\Big)^{\frac{n+1}{n-1}} \tag{1}$$

where $\mu_{\partial K}$ denotes the surface measure of ∂K. This theorem gives asymptotically the order of best approximation of a convex body K by polytopes contained in K with a fixed number of vertices. It was proved by McClure and Vitale [McV] in dimension 2 and by Gruber [Gr2] for general n. The constant del_{n-1} is positive and depends on the dimension n only. Its order of magnitude can be computed by considering the case $K = B_2^n$. This has been done in [GRS1] and [GRS2] by Gordon, Reisner and Schütt, namely there are numerical constants a and b such that

$$an \leq \mathrm{del}_{n-1} \leq bn.$$

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The constant del_{n-1} was determined more precisely by Mankiewicz and Schütt [MaS1], [MaS2]. They showed that

$$\frac{n-1}{n+1}\left(\mathrm{vol}_{n-1}(B_2^{n-1})\right)^{-\frac{2}{n-1}} \leq \mathrm{del}_{n-1} \leq \left(1+\frac{c\ln n}{n}\right)\frac{n-1}{n+1}\left(\mathrm{vol}_{n-1}(B_2^{n-1})\right)^{-\frac{2}{n-1}} \tag{2}$$

where c is a numerical constant. In particular,

$$\lim_{n\to\infty}\frac{\mathrm{del}_{n-1}}{n}=\frac{1}{2\pi e}.$$

What happens if we drop the condition that the polytopes have to be contained in the convex body and allow all polytopes have at most N vertices? How much better can we approximate the Euclidean ball?

In [Lud] it was shown that for all convex bodies K whose boundary is twice continuously differentiable and whose curvature is everywhere strictly positive,

$$\lim_{N\to\infty}\frac{\inf\{\mathrm{vol}_n(K\triangle P)\mid P \text{ is a polytope with at most } N \text{ vertices}\}}{N^{-\frac{2}{n-1}}} = \frac{1}{2}\,\mathrm{ldel}_{n-1}\Big(\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}\,d\mu_{\partial K}(x)\Big)^{\frac{n+1}{n-1}}.$$

The constant ldel_{n-1} is positive and depends only on n. Clearly, from the above mentioned results it follows that $\mathrm{ldel}_{n-1}\leq cn$. On the other hand, it has been shown in [Bö] that for a polytope P with at most N vertices,

$$\mathrm{vol}_n(B_2^n\triangle P)\geq\frac{1}{67e^2\pi}\,\frac{1}{n}\,\mathrm{vol}_n(B_2^n)N^{-\frac{2}{n-1}}.$$

Thus between the upper and lower estimate for ldel_{n-1} there is a gap of order n^2. In this paper we narrow this gap by showing that $\mathrm{ldel}_{n-1}\leq c$ where c is a numerical constant.

THEOREM 1. *There is a constant c such that for every $n\in\mathbb{N}$ there is an N_n so that for every $N\geq N_n$ there is a polytope P in $\mathbb{R}^n$ with N vertices such that*

$$\mathrm{vol}_n(B_2^n\triangle P)\leq c\,\mathrm{vol}_n(B_2^n)N^{-\frac{2}{n-1}}. \tag{3}$$

Gruber [Gr2] also showed

$$\lim_{N\to\infty}\frac{\inf\{\mathrm{vol}_n(K\triangle P)\mid K\subseteq P \text{ and } P \text{ is a polytope with at most } N \text{ facets}\}}{N^{-\frac{2}{n-1}}} = \frac{1}{2}\,\mathrm{div}_{n-1}\Big(\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}\,d\mu_{\partial K}(x)\Big)^{\frac{n+1}{n-1}}$$

where div_{n-1} is a positive constant that depends on n only. It is easy to show [Lud, MaS1] that there are numerical constants a and b such that $an \le \mathrm{div}_{n-1} \le bn$.

Ludwig [Lud] showed that for general polytopes

$$\lim_{N\to\infty} \frac{\inf\{\mathrm{vol}_n(K \triangle P) \mid P \text{ is a polytope with at most } N \text{ facets}\}}{N^{-\frac{2}{n-1}}} = \frac{1}{2}\,\mathrm{ldiv}_{n-1}\Big(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}}\, d\mu_{\partial K}(x)\Big)^{\frac{n+1}{n-1}}$$

where ldiv_{n-1} is a positive constant that depends on n only. Clearly, $\mathrm{ldiv}_{n-1} \le cn$ and Böröczky [Bö] showed that for polytopes P with N facets,

$$\mathrm{vol}_n(B_2^n \triangle P) \ge \frac{1}{67e^2\pi}\,\frac{1}{n}\,\mathrm{vol}_n(B_2^n) N^{-\frac{2}{n-1}}.$$

Thus again, there is a gap between the upper and lower estimates for ldiv_{n-1} of order n^2. We narrow this gap by a factor of n.

THEOREM 2. *There is a constant c such that for every $n \in \mathbb{N}$ and for every $M \ge 10^{(n-1)/2}$ and all polytopes P in $\mathbb{R}^n$ with M facets we have*

$$\mathrm{vol}_n(B_2^n \triangle P) \ge c\,\mathrm{vol}_n(B_2^n) M^{-\frac{2}{n-1}}. \tag{4}$$

For additional information on asymptotic approximation see [Gr1], [Gr3], [Sch].

2. Proof of Theorem 1.

We need the following lemmas.

LEMMA 3 (Stirling's formula). *For all $x > 0$,*

$$\sqrt{2\pi x}\, x^x e^{-x} < \Gamma(x+1) < \sqrt{2\pi x}\, x^x e^{-x} e^{1/12x}.$$

The following lemma is due to J. Müller [Mü].

LEMMA 4 ([Mü]). *Let $\mathbb{E}(\partial B_2^n, N)$ be the expected volume of a random polytope of N points that are independently chosen on the boundary of the Euclidean ball B_2^n with respect to the normalized surface measure. Then*

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(B_2^n) - \mathbb{E}(\partial B_2^n, N)}{N^{-\frac{2}{n-1}}} = \frac{(n-1)^{\frac{n+1}{n-1}} (\mathrm{vol}_{n-1}(\partial B_2^n))^{\frac{n+1}{n-1}}}{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}\,\frac{\Gamma\big(n+1+\frac{2}{n-1}\big)}{2(n+1)!}.$$

The following lemma can be found in [Mil], [SW, p. 317], and [Zä]. Let $[x_n, \dots, x_n]$ be the convex hull of $x_1, \dots, x_n$.

LEMMA 5 ([Mil]).

$$d\mu_{\partial B_2^n}(x_1)\cdots d\mu_{\partial B_2^n}(x_n) = (n-1)!\,\frac{\mathrm{vol}_{n-1}([x_1,\dots,x_n])}{(1-p^2)^{n/2}}\,d\mu_{\partial B_2^n\cap H}(x_1)\cdots d\mu_{\partial B_2^n\cap H}(x_n)\,dp\,d\mu_{\partial B_2^n}(\xi) \tag{5}$$

where ξ is the normal to the plane H through $x_1,\dots,x_n$ and p is the distance of the plane H to the origin.

LEMMA 6 ([Mil]).

$$\int_{\partial B_2^n(r)}\cdots\int_{\partial B_2^n(r)}(\mathrm{vol}_n([x_1,\dots,x_{n+1}]))^2\,d\mu_{\partial B_2^n(r)}(x_1)\cdots d\mu_{\partial B_2^n(r)}(x_{n+1}) = \frac{(n+1)r^{n^2+2n-1}}{n!n^n}\,(\mathrm{vol}_{n-1}(\partial B_2^n))^{n+1}. \tag{6}$$

For a given hyperplane H that does not contain the origin we denote by H^+ the halfspace containing the origin and by H^- the halfspace not containing the origin. A *cap* C of the Euclidean ball B_2^n is the intersection of a halfspace H^- with B_2^n. The *radius* of such a cap is the radius of the $(n-1)$-dimensional ball $B_2^n \cap H$.

LEMMA 7 ([SW]). *Let H be a hyperplane, p its distance from the origin and s the normalized surface area of $\partial B_2^n \cap H^-$, i.e.*

$$s = \frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)}.$$

Then

$$\frac{dp}{ds} = -\frac{1}{(1-p^2)^{\frac{n-3}{2}}}\,\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}. \tag{7}$$

LEMMA 8 ([SW, Lemma 3.13]). *Let C be a cap of a Euclidean ball with radius* 1. *Let u be the surface area of this cap and r its radius. Then*

$$\left(\frac{u}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} - \frac{1}{2(n+1)}\left(\frac{u}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{3}{n-1}} - c\left(\frac{u}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{5}{n-1}} \le r(u) \le \left(\frac{u}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} \tag{8}$$

where c is a numerical constant.

The right hand inequality is immediate, since $u \ge r^{n-1}\,\mathrm{vol}_{n-1}(B_2^{n-1})$.

Proof of Theorem 1. The approximating polytope is obtained in a probabilistic way. We consider a Euclidean ball that is slightly bigger than the ball with radius 1, by a carefully chosen factor. We choose N points randomly in the bigger ball and we take their convex hull. With large probability there is a random polytope that fits our requirements.

For technical reasons we choose random points in a Euclidean ball of radius 1 and we approximate a slightly smaller Euclidean ball, say with radius $1-c$ where $c=c_{n,N}$ depends on n and N only.

We now compute the expected volume difference between $B_2^n(1-c)$ and a random polytope $[x_1,\dots,x_N]$ whose vertices are chosen randomly from the boundary of B_2^n. Note that random polytopes are simplicial with probability 1. We want to estimate the expected volume difference

$$\begin{aligned}(9)\qquad &\mathbb{E}\,\mathrm{vol}_n(B_2^n(1-c)\triangle P_N)\\ &\quad=\int\limits_{\partial B_2^n}\cdots\int\limits_{\partial B_2^n}\mathrm{vol}_n(B_2^n(1-c)\triangle[x_1,\dots,x_N])\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\end{aligned}$$

where $\mathbb{P}$ denotes the uniform probability measure on ∂B_2^n. Since the volume difference between $B_2^n(1-c)$ and a polytope $P_N=[x_1,\dots,x_N]$ is

$$\begin{aligned}&\mathrm{vol}_n(B_2^n(1-c)\triangle P_N)\\ &\quad=\mathrm{vol}_n(B_2^n\setminus B_2^n(1-c))-\mathrm{vol}_n(B_2^n\setminus P_N)+2\,\mathrm{vol}_n(B_2^n(1-c)\cap P_N^{\mathrm{c}}),\end{aligned}$$

the above expression equals

$$\begin{aligned}&\mathrm{vol}_n(B_2^n\setminus B_2^n(1-c))\\ &\quad-\int\limits_{\partial B_2^n}\cdots\int\limits_{\partial B_2^n}\mathrm{vol}_n(B_2^n\setminus[x_1,\dots,x_N])\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\\ &\quad+2\int\limits_{\partial B_2^n}\cdots\int\limits_{\partial B_2^n}\mathrm{vol}_n(B_2^n(1-c)\cap[x_1,\dots,x_N]^{\mathrm{c}})\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N).\end{aligned}$$

For given N we choose c such that

$$\begin{aligned}(10)\qquad &\mathrm{vol}_n(B_2^n\setminus B_2^n(1-c))\\ &\quad=\int\limits_{\partial B_2^n}\cdots\int\limits_{\partial B_2^n}\mathrm{vol}_n(B_2^n\setminus[x_1,\dots,x_N])\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N).\end{aligned}$$

For this particular c we have

$$\begin{aligned}&\int\limits_{\partial B_2^n}\cdots\int\limits_{\partial B_2^n}\mathrm{vol}_n(B_2^n(1-c)\triangle[x_1,\dots,x_N])\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\\ &\quad=2\int\limits_{\partial B_2^n}\cdots\int\limits_{\partial B_2^n}\mathrm{vol}_n(B_2^n(1-c)\cap[x_1,\dots,x_N]^{\mathrm{c}})\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N).\end{aligned}$$

By Lemma 4 the quantity c is for large N asymptotically equal to

$$(11)\qquad N^{-\frac{2}{n-1}}(n-1)^{\frac{n+1}{n-1}}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}}\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}.$$

In particular, for large enough N,

$$c \le \left(1+\frac{1}{n^2}\right) N^{-\frac{2}{n-1}} (n-1)^{\frac{n+1}{n-1}} \times \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!} \tag{12}$$

and

$$\left(1-\frac{1}{n^2}\right) N^{-\frac{2}{n-1}} (n-1)^{\frac{n+1}{n-1}} \times \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!} \le c. \tag{13}$$

Thus there are constants a and b such that

$$aN^{-\frac{2}{n-1}} \le c \le bN^{-\frac{2}{n-1}}. \tag{14}$$

We continue the computation of the expected volume difference:

$$\begin{aligned}
&\int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n(B_2^n(1-c) \triangle [x_1,\dots,x_N])\, d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N) \\
&\quad = 2 \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n(B_2^n(1-c) \cap [x_1,\dots,x_N]^{\mathrm{c}}) \\
&\qquad \times \chi_{\{0\in[x_1,\dots,x_N]^\circ\}}\, d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N) \\
&\qquad + 2 \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n(B_2^n(1-c) \cap [x_1,\dots,x_N]^{\mathrm{c}}) \\
&\qquad \times \chi_{\{0\notin[x_1,\dots,x_N]^\circ\}}\, d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N) \\
&\quad \le 2 \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n(B_2^n(1-c) \cap [x_1,\dots,x_N]^{\mathrm{c}}) \\
&\qquad \times \chi_{\{0\in[x_1,\dots,x_N]^\circ\}}\, d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N) \\
&\qquad + \mathrm{vol}_n(B_2^n) \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \chi_{\{0\notin[x_1,\dots,x_N]^\circ\}}\, d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N).
\end{aligned}$$

By a result of [Wen] the second summand equals

$$\mathrm{vol}_n(B_2^n) 2^{-N+1} \sum_{k=0}^{n-1} \binom{N-1}{k} \le \mathrm{vol}_n(B_2^n) 2^{-N+1} n N^n,$$

so it is of much smaller order (essentially 2^{-N}) than the first summand, which, as we shall see, is of the order of $N^{-2/(n-1)}$. Therefore, in what follows we consider the first summand.

We introduce $\Phi_{j_1,\dots,j_n} : \partial B_2^n \times \cdots \times \partial B_2^n \to \mathbb{R}$ where

$$\Phi_{j_1,\dots,j_n}(x_1,\dots,x_N) = 0$$

if $[x_{j_1},\dots,x_{j_n}]$ is not an $(n-1)$-dimensional face of $[x_1,\dots,x_N]$ or if 0 is not in $[x_1,\dots,x_N]$, and

$$\begin{aligned}&\Phi_{j_1,\dots,j_n}(x_1,\dots,x_N)\\&\qquad=\mathrm{vol}_n(B_2^n(1-c)\cap[x_1,\dots,x_N]^{\mathrm{c}}\cap\mathrm{cone}(x_{j_1},\dots,x_{j_n}))\chi_{\{0\in[x_1,\dots,x_N]^\circ\}}\end{aligned}$$

if $[x_{j_1},\dots,x_{j_n}]$ is a facet of $[x_1,\dots,x_N]$ and if $0\in[x_1,\dots,x_N]$. Here

$$\mathrm{cone}(x_1,\dots,x_n)=\Big\{\sum_{i=1}^n a_ix_i\ \Big|\ \forall i: a_i\ge 0\Big\}.$$

For all random polytopes $[x_1,\dots,x_N]$ that contain 0 as an interior point,

$$\mathbb{R}^n=\bigcup_{[x_{j_1},\dots,x_{j_n}]\text{ is a facet of }[x_1,\dots,x_n]}\mathrm{cone}(x_{j_1},\dots,x_{j_n}).$$

Then

$$\begin{aligned}&\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\mathrm{vol}_n(B_2^n(1-c)\cap[x_1,\dots,x_N]^{\mathrm{c}})\chi_{\{0\in[x_1,\dots,x_N]^\circ\}}\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\\&\qquad=\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\sum_{\{j_1,\dots,j_n\}\subseteq\{1,\dots,N\}}\Phi_{j_1,\dots,j_n}(x_1,\dots,x_N)\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\end{aligned}$$

where we sum over all different subsets $\{j_1,\dots,j_n\}$. The latter expression equals

$$\binom{N}{n}\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\Phi_{1,\dots,n}(x_1,\dots,x_N)\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N).$$

Let H be the hyperplane containing the points $x_1,\dots,x_n$. The set of points where H is not well defined has measure 0 and

$$\begin{aligned}&\mathbb{P}^{N-n}(\{(x_{n+1},\dots,x_N)\mid[x_1,\dots,x_n]\text{ is a facet of }[x_1,\dots,x_N]\\&\qquad\qquad\text{and }0\in[x_1,\dots,x_N]\})=\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}.\end{aligned}$$

Therefore the above expression equals

$$\begin{aligned}&\binom{N}{n}\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\\&\times\mathrm{vol}_n(B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\dots,x_n))\chi_{\{0\in[x_1,\dots,x_N]^\circ\}}\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_n).\end{aligned}$$

Since H^- does not contain 0 this can be estimated by

$$\begin{aligned}&\binom{N}{n}\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\\&\qquad\times\mathrm{vol}_n(B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\dots,x_n))\,d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_n).\end{aligned}$$

By Lemma 5 the latter expression equals

$$\binom{N}{n}\frac{(n-1)!}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n}\int_{\partial B_2^n}\int_0^1\int_{\partial B_2^n\cap H}\cdots\int_{\partial B_2^n\cap H}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}$$
$$\times\,\mathrm{vol}_n(B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\ldots,x_n))$$
$$\times\,\frac{\mathrm{vol}_{n-1}([x_1,\ldots,x_n])}{(1-p^2)^{n/2}}\,d\mu_{\partial B_2^n\cap H}(x_1)\cdots d\mu_{\partial B_2^n\cap H}(x_n)\,dp\,d\mu_{\partial B_2^n}(\xi).$$

This in turn can be estimated by

$$\binom{N}{n}\frac{(n-1)!}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n}\tag{15}$$
$$\times\int_{\partial B_2^n}\int_{1-1/n}^1\int_{\partial B_2^n\cap H}\cdots\int_{\partial B_2^n\cap H}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}$$
$$\times\,\mathrm{vol}_n(B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\ldots,x_n))$$
$$\times\,\frac{\mathrm{vol}_{n-1}([x_1,\ldots,x_n])}{(1-p^2)^{n/2}}\,d\mu_{\partial B_2^n\cap H}(x_1)\cdots d\mu_{\partial B_2^n\cap H}(x_n)\,dp\,d\mu_{\partial B_2^n}(\xi)$$

times a factor that is less than 2 provided that N is sufficiently large. Indeed, for $p\le 1-1/n$,

$$\begin{aligned}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}&\le\exp\left(-(N-n)\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)\\&\le\exp\left(-(N-n)\left(\frac{2}{n}-\frac{1}{n^2}\right)^{\frac{n-1}{2}}\frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{n\,\mathrm{vol}_n(B_2^n)}\right)\\&\le\exp\left(-\frac{N-n}{n^{(n+1)/2}}\right)\end{aligned}$$

and the rest of the expression is bounded. We have

$$\begin{aligned}&\mathrm{vol}_n(B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\ldots,x_n))\\&\qquad\le\frac{p}{n}\max\left\{0,\left(\frac{1-c}{p}\right)^n-1\right\}\mathrm{vol}_{n-1}([x_1,\ldots,x_n]).\end{aligned}$$

This holds since $B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\ldots,x_n)$ is contained in the cone $\mathrm{cone}(x_1,\ldots,x_n)$, truncated between H and the hyperplane parallel to H at distance $1-c$ from 0. Therefore, as $p\le 1$ the above is at most

$$\binom{N}{n}\frac{(n-1)!}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n}\int\limits_{\partial B_2^n}\int\limits_{1-1/n}^{1}\int\limits_{\partial B_2^n\cap H}\cdots\int\limits_{\partial B_2^n\cap H}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}$$
$$\times\frac{1}{n}\max\left\{0,\left(\frac{1-c}{p}\right)^n-1\right\}\frac{(\mathrm{vol}_{n-1}([x_1,\dots,x_n]))^2}{(1-p^2)^{n/2}}$$
$$\times\, d\mu_{\partial B_2^n\cap H}(x_1)\cdots d\mu_{\partial B_2^n\cap H}(x_n)\,dp\,d\mu_{\partial B_2^n}(\xi).$$

By Lemma 6 this equals

$$\binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n}\frac{n}{(n-1)^{n-1}}\int\limits_{\partial B_2^n}\int\limits_{1-1/n}^{1}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}$$
$$\times\frac{1}{n}\max\left\{0,\left(\frac{1-c}{p}\right)^n-1\right\}\frac{r^{n^2-2}}{(1-p^2)^{n/2}}\,dp\,d\mu_{\partial B_2^n}(\xi)$$

where r denotes the radius of $B_2^n\cap H$. Since the integral does not depend on the direction ξ and $r^2+p^2=1$ this is

$$\binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-1}}\frac{n}{(n-1)^{n-1}}$$
$$\times\int\limits_{1-1/n}^{1}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\frac{1}{n}\max\left\{0,\left(\frac{1-c}{p}\right)^n-1\right\}r^{n^2-n-2}\,dp.$$

This equals

$$\binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-1}}\frac{n}{(n-1)^{n-1}}\tag{16}$$
$$\times\int\limits_{1-1/n}^{1-c}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\frac{1}{n}\left\{\left(\frac{1-c}{p}\right)^n-1\right\}r^{n^2-n-2}\,dp.$$

Since $p\geq 1-1/n$ and c is of the order $N^{-2/(n-1)}$, we have, for sufficiently large N,

$$\frac{1}{n}\left\{\left(\frac{1-c}{p}\right)^n-1\right\}\leq 3(1-c-p).$$

Therefore, the previous expression can be estimated by an absolute constant times

$$\binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-1}}\frac{n}{(n-1)^{n-1}}\tag{17}$$
$$\times\int\limits_{1-1/n}^{1-c}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}(1-c-p)r^{n^2-n-2}\,dp.$$

We choose

$$s = \frac{\mathrm{vol}_{n-1}(\partial B_2^n \cap H^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)}$$

as our new variable under the integral. We apply Lemma 7 in order to change the variable under the integral

$$\binom{N}{n} \frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-2}} \frac{n}{(n-1)^{n-1}} \times \int_{s(1-c)}^{1/2} (1-s)^{N-n}(1-c-p)r^{(n-1)^2}\, ds \tag{18}$$

where the normalized surface area s of the cap is a function of the distance p of the hyperplane to 0. Before we proceed we want to estimate $s(1-c)$. The radius r and the distance p satisfy $1 = p^2 + r^2$. We have

$$r^{n-1} \frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\mathrm{vol}_{n-1}(\partial B_2^n)} \le s(\sqrt{1-r^2}) \le \frac{1}{\sqrt{1-r^2}} r^{n-1} \frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\mathrm{vol}_{n-1}(\partial B_2^n)}.$$

To show this, we compare s with the surface area of the intersection $B_2^n \cap H$ of the Euclidean ball and the hyperplane H. We have

$$\frac{\mathrm{vol}_{n-1}(B_2^n \cap H)}{\mathrm{vol}_{n-1}(\partial B_2^n)} = r^{n-1} \frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\mathrm{vol}_{n-1}(\partial B_2^n)}.$$

Since the orthogonal projection onto H maps $\partial B_2^n \cap H^-$ onto $B_2^n \cap H$ the left hand inequality follows.

The right hand inequality follows again by considering the orthogonal projection onto H. The surface area of a surface element of $\partial B_2^n \cap H^-$ equals the surface area of the one it is mapped to in $B_2^n \cap H$ divided by the cosine of the angle between the normal to H and the normal to ∂B_2^n at the given point. The cosine is always greater than $\sqrt{1-r^2}$.

For $p = 1-c$ we have $r = \sqrt{2c - c^2} \le \sqrt{2c}$. Therefore by (12) we get

$$\begin{aligned} s(1-c) \le{}& \frac{e^{1/n}}{1-c} \frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\mathrm{vol}_{n-1}(\partial B_2^n)} \\ &\times \left\{ 2N^{-\frac{2}{n-1}} (n-1)^{\frac{n+1}{n-1}} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})} \right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!} \right\}^{\frac{n-1}{2}} \\ ={}& \frac{e^{1/n}}{1-c} \frac{1}{N} \left\{ \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)(n-1)}{(n+1)!} \right\}^{\frac{n-1}{2}}. \end{aligned} \tag{19}$$

The quantity c is of the order $N^{-2/(n-1)}$, therefore $1/(1-c)$ is as close to 1 as we desire for N large enough. Moreover, for large n,

$$\left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}}$$

is asymptotically equal to $1/e$. Therefore, for both n and N large enough,

$$s(1-c) \le e^{1/12}\,\frac{1}{eN}\left\{\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{n!}\right\}^{\frac{n-1}{2}}.$$

For n sufficiently large,

$$\left\{\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{n!}\right\}^{\frac{n-1}{2}} \le e^{1/12}n.$$

Indeed, by Lemma 3,

$$\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{n!} \le \left(1+\frac{2}{n(n-1)}\right)^{n+\frac{1}{2}}\left(n+\frac{2}{n-1}\right)^{\frac{2}{n-1}} e^{-\frac{2}{n-1}} e^{\frac{1}{12(n+\frac{2}{n-1})}}$$

and

$$\left(\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{n!}\right)^{\frac{n-1}{2}} \le \frac{1}{e}\left(1+\frac{2}{n(n-1)}\right)^{\frac{n-1}{2}(n+\frac{1}{2})}\left(n+\frac{2}{n-1}\right) e^{\frac{n-1}{24(n+\frac{2}{n-1})}}.$$

The right hand expression is asymptotically equal to $ne^{1/24}$. Altogether,

$$s(1-c) \le e^{1/6}\,\frac{n}{eN}. \tag{20}$$

Since $p=\sqrt{1-r^2}$ we get, for all r with $0 \le r \le 1$,

$$1-c-p = 1-c-\sqrt{1-r^2} \le \frac{1}{2}\,r^2 + r^4 - c.$$

(The estimate is equivalent to $1-\frac{1}{2}r^2-r^4 \le \sqrt{1-r^2}$. The left hand side is negative for $r \ge .9$ and thus the inequality holds for r with $.9 \le r \le 1$. For r with $0 \le r \le .9$ we square both sides.) Thus (18) is smaller than or equal to

$$\binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\mathrm{vol}_{n-1}(\partial B_2^{n}))^{n-2}}\,\frac{n}{(n-1)^{n-1}} \times \int_{s(1-c)}^{1} (1-s)^{N-n}\left(\frac{1}{2}\,r^2+r^4-c\right) r^{(n-1)^2}\,ds. \tag{21}$$

Now we evaluate the integral. We use Lemma 8. By differentiation we verify that $\left(\frac{1}{2}r^2+r^4-c\right)r^{(n-1)^2}$ is a monotone function of r. Here we use the fact that $\frac{1}{2}r^2 + r^4 - c$ is nonnegative. Hence

$$\int_{s(1-c)}^{1} (1-s)^{N-n}\left(\frac{1}{2}\,r^2 + r^4 - c\right) r^{(n-1)^2}\,ds$$
$$\leq \frac{1}{2}\int_0^1 (1-s)^{N-n}\left(s\,\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}} ds$$
$$+\int_0^1 (1-s)^{N-n}\left(s\,\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{4}{n-1}} ds$$
$$-\int_0^1 (1-s)^{N-n} c\left(s\,\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1} ds$$
$$+\int_0^{s(1-c)} (1-s)^{N-n} c\left(s\,\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1} ds.$$

By (13),

$$\int_{s(1-c)}^{1} (1-s)^{N-n}\left(\frac{1}{2}r^2 + r^4 - c\right) r^{(n-1)^2}\,ds$$
$$\leq \frac{1}{2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma\left(n+\frac{2}{n-1}\right)}{\Gamma\left(N+1+\frac{2}{n-1}\right)}$$
$$+\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{4}{n-1}} \frac{\Gamma(N-n+1)\Gamma\left(n+\frac{4}{n-1}\right)}{\Gamma\left(N+1+\frac{4}{n-1}\right)}$$
$$-\left(1-\frac{1}{n^2}\right)\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1} \frac{\Gamma(N-n+1)\Gamma(n)}{\Gamma(N+1)}$$
$$\times \frac{(n-1)^{\frac{n+1}{n-1}}(\mathrm{vol}_{n-1}(\partial B_2^n))^{\frac{2}{n-1}}}{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}\,\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}\,N^{-\frac{2}{n-1}}$$
$$+\,cs(1-c)\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1} \max_{s\in[0,s(1-c)]}(1-s)^{N-n}s^{n-1}.$$

Thus

$$\int_{s(1-c)}^{1} (1-s)^{N-n}\left(\frac{1}{2}\,r^2 + r^4 - c\right) r^{(n-1)^2}\,ds \tag{22}$$
$$\leq \frac{1}{2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma\left(n+\frac{2}{n-1}\right)}{\Gamma\left(N+1+\frac{2}{n-1}\right)}$$

$$
\begin{aligned}
&+\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{4}{n-1}} \frac{\Gamma(N-n+1)\Gamma\big(n+\frac{4}{n-1}\big)}{\Gamma\big(N+1+\frac{4}{n-1}\big)}\\
&-\frac{1}{2}\left(1-\frac{1}{n^2}\right)\frac{n-1}{(n+1)n}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}}\\
&\times\frac{\Gamma(N-n+1)\Gamma\big(n+1+\frac{2}{n-1}\big)}{\Gamma(N+1)}N^{-\frac{2}{n-1}}\\
&+cs(1-c)\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}\max_{s\in[0,s(1-c)]}(1-s)^{N-n}s^{n-1}.
\end{aligned}
$$

The second summand is asymptotically equal to

$$
\begin{aligned}
\text{(23)}\qquad &\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{4}{n-1}} \frac{(N-n)!(n-1)!n^{\frac{4}{n-1}}}{N!(N+1)^{\frac{4}{n-1}}}\\
&=\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{4}{n-1}} \frac{n^{-1+\frac{4}{n-1}}}{\binom{N}{n}(N+1)^{\frac{4}{n-1}}}.
\end{aligned}
$$

This summand is of the order $N^{-\frac{4}{n-1}}$ while the others are of the order $N^{-\frac{2}{n-1}}$.

We consider the sum of the first and third summands:

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma\big(n+\frac{2}{n-1}\big)}{\Gamma\big(N+1+\frac{2}{n-1}\big)}\\
&\qquad\times\left(1-\left(1-\frac{1}{n^2}\right)\frac{(n-1)\Gamma\big(n+1+\frac{2}{n-1}\big)\Gamma\big(N+1+\frac{2}{n-1}\big)}{n(n+1)\Gamma\big(n+\frac{2}{n-1}\big)\Gamma(N+1)N^{\frac{2}{n-1}}}\right)
\end{aligned}
$$

Since $\Gamma\big(n+1+\frac{2}{n-1}\big)=\big(n+\frac{2}{n-1}\big)\Gamma\big(n+\frac{2}{n-1}\big)$ the latter expression equals

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma\big(n+\frac{2}{n-1}\big)}{\Gamma\big(N+1+\frac{2}{n-1}\big)}\\
&\qquad\times\left(1-\left(1-\frac{1}{n^2}\right)\frac{(n-1)\big(n+\frac{2}{n-1}\big)\Gamma\big(N+1+\frac{2}{n-1}\big)}{n(n+1)\Gamma(N+1)N^{\frac{2}{n-1}}}\right).
\end{aligned}
$$

Since $\Gamma\big(N+1+\frac{2}{n-1}\big)$ is asymptotically equal to $(N+1)^{\frac{2}{n-1}}\Gamma(N+1)$ the sum of the first and third summands is for large N of the order

$$
\text{(24)}\qquad \frac{1}{n}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma\big(n+\frac{2}{n-1}\big)}{\Gamma\big(N+1+\frac{2}{n-1}\big)}
$$

which in turn is of the order

$$\frac{1}{n^2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}}\binom{N}{n}^{-1}N^{-\frac{2}{n-1}}. \tag{25}$$

We now consider the fourth summand. By (14) and (20) it is less than

$$bN^{-\frac{2}{n-1}}\frac{n}{e^{5/6}N}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}\max_{s\in[0,s(1-c)]}(1-s)^{N-n}s^{n-1}. \tag{26}$$

The maximum of the function $(1-s)^{N-n}s^{n-1}$ is attained at $(n-1)/(N-1)$ and the function is increasing on the interval $[0,(n-1)/(N-1)]$. Therefore, by (20) we have $s(1-c)<(n-1)/(N-1)$ and the maximum of this function over the interval $[0,s(1-c)]$ is attained at $s(1-c)$. By (20) we have $s(1-c)\le e^{1/6}\frac{n}{eN}$ and thus for N sufficiently large

$$\begin{aligned}\max_{s\in[0,s(1-c)]}(1-s)^{N-n}s^{n-1} &\le \left(1-\frac{n}{e^{5/6}N}\right)^{N-n}\left(e^{1/6}\,\frac{n}{eN}\right)^{n-1}\\ &\le \exp\left(\frac{n-1}{6}-\frac{n(N-n)}{e^{5/6}N}\right)\left(\frac{n}{eN}\right)^{n-1}\\ &\le \exp\left(-\frac{n}{6}\right)\left(\frac{n}{eN}\right)^{n-1}.\end{aligned}$$

Thus we get for (26) the bound, with a new constant b,

$$bN^{-\frac{2}{n-1}}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}e^{-n/6}\,\frac{n^ne^{-n}}{N^n}.$$

This is asymptotically equal to

$$bN^{-\frac{2}{n-1}}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}e^{-n/6}\,\frac{1}{\binom{N}{n}\sqrt{2\pi n}}. \tag{27}$$

Altogether, (15) for N sufficiently large can be estimated by

$$\begin{aligned}&\binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-2}}\,\frac{n}{(n-1)^{n-1}}\\ &\qquad\times\left\{\frac{1}{n^2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}}\binom{N}{n}^{-1}N^{-\frac{2}{n-1}}\right.\\ &\qquad\left.+\,bN^{-\frac{2}{n-1}}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}e^{-n/6}\,\frac{1}{\binom{N}{n}\sqrt{2\pi n}}\right\}.\end{aligned}$$

This can be estimated by a constant times

$$(\mathrm{vol}_{n-1}(\partial B_2^n))n\left\{\frac{1}{n^2}\,N^{-\frac{2}{n-1}}+bN^{-\frac{2}{n-1}}e^{-n/6}\,\frac{1}{\sqrt{2\pi n}}\right\}. \tag{28}$$

Finally, it should be noted that we have been estimating the approximation of $B_2^n(1-c)$ and not that of B_2^n. Therefore, we need to multiply (28) by $(1-c)^{-n}$. By (14),

$$(1-c)^n \geq 1 - b\,\frac{n}{N^{\frac{2}{n-1}}}$$

so that for all N with $N \geq (2bn)^{\frac{n-1}{2}}$ we have $(1-c)^{-n} \leq 2$. ∎

3. Proof of Theorem 2. We need another lemma.

LEMMA 9. *Let P_M be a polytope with M facets $F_1, \ldots, F_M$ that is best approximating for a convex body K in $\mathbb{R}^n$ with respect to the symmetric difference metric. For $k = 1, \ldots, M$, let*

$$F_k^{\mathrm{i}} = F_k \cap K, \qquad F_k^{\mathrm{a}} = F_k \cap K^{\mathrm{c}}.$$

Then, for all $j = 1, \ldots, M$,

$$\mathrm{vol}_{n-1}(F_j^{\mathrm{i}}) = \mathrm{vol}_{n-1}(F_j^{\mathrm{a}}).$$

Proof. Let H_j, $j = 1, \ldots, M$, be the hyperplane containing the face F_j. Then

$$P_M = \bigcap_{j=1}^{M} H_j^+.$$

Suppose $H_k = H(x_k, \xi_k)$, i.e. H_k is the hyperplane containing x_k and orthogonal to ξ_k. We consider

$$P_t = \bigcap_{j \neq k} H_j^+ \cap H^+\bigg(x_k + \frac{t}{\|x_k\|}\,x_k, \xi_k\bigg).$$

We have

$$\mathrm{vol}_{n-1}(P_t \mathbin{\triangle} K) = \mathrm{vol}_{n-1}(P_M \mathbin{\triangle} K) + t(\mathrm{vol}_{n-1}(F_k^{\mathrm{a}}) - \mathrm{vol}_{n-1}(F_k^{\mathrm{i}})) + \psi(t)$$

where $\psi(t)/t^2$ is a bounded function. ∎

Proof of Theorem 2. Let P_M be a best approximating polytope with M facets $F_1, \ldots, F_M$ for B_2^n with respect to the symmetric difference metric. For $k = 1, \ldots, M$, let

$$F_k^{\mathrm{i}} = F_k \cap B_2^n, \qquad F_k^{\mathrm{a}} = F_k \cap (B_2^n)^{\mathrm{c}},$$

let H_k be the hyperplane containing the facet F_k and let C_k be the cap of B_2^n with base $H_k \cap B_2^n$. (There are actually two caps, we take the one whose interior has empty intersection with P_M.) For $k = 1, \ldots, M$ we put

$$h_k = \begin{cases} \text{height of the cap } C_k & \text{if } F_k \cap (B_2^n)^\circ \neq \emptyset, \\ 0, & \text{if } F_k \cap (B_2^n)^\circ = \emptyset. \end{cases}$$

Then

$$\mathrm{vol}_{n-1}(P_M \triangle B_2^n) \geq \frac{1}{n}\sum_{k=1}^{M} h_k \,\mathrm{vol}_{n-1}(F_k^{\mathrm{i}}). \tag{29}$$

Let r_k be such that $\mathrm{vol}_{n-1}(r_k B_2^{n-1}) = \mathrm{vol}_{n-1}(F_k^{\mathrm{i}})$. Thus

$$r_k = \left(\frac{\mathrm{vol}_{n-1}(F_k^{\mathrm{i}})}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}.$$

Let $\widetilde{h}_k$ be the height of the cap of B_2^n with base $r_k B_2^{n-1}$. Then

$$\widetilde{h}_k \leq h_k \quad \text{for all } k, \tag{30}$$

and

$$\widetilde{h}_k \geq \frac{1}{2}\, r_k^2 \geq \frac{1}{2}\left(\frac{\mathrm{vol}_{n-1}(F_k^{\mathrm{i}})}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}}.$$

Thus from (29) with (30) we get

$$\begin{aligned}\mathrm{vol}_{n-1}(P_M \triangle B_2^n) &\geq \frac{1}{2n}\sum_{k=1}^{M} \frac{(\mathrm{vol}_{n-1}(F_k^{\mathrm{i}}))^{\frac{n+1}{n-1}}}{(\mathrm{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}} \\ &\geq \frac{1}{8\pi e}\sum_{k=1}^{M} (\mathrm{vol}_{n-1}(F_k^{\mathrm{i}}))^{\frac{n+1}{n-1}}.\end{aligned} \tag{31}$$

We consider two cases. The first case is

$$\sum_{k=1}^{M} \mathrm{vol}_{n-1}(F_k^{\mathrm{i}}) + \sum_{k=1}^{M} \mathrm{vol}_{n-1}(F_k^{\mathrm{a}}) \geq c\,\mathrm{vol}_{n-1}(\partial B_2^n), \tag{32}$$

where $M \geq 10^{(n-1)/2}$ and $c = 9/10$. It then follows from Lemma 9 that

$$\sum_{k=1}^{M} \mathrm{vol}_{n-1}(F_k^{\mathrm{i}}) \geq \frac{c}{2}\,\mathrm{vol}_{n-1}(\partial B_2^n). \tag{33}$$

By Hölder's inequality

$$\sum_{k=1}^{M} \mathrm{vol}_{n-1}(F_k^{\mathrm{i}}) \leq \Big(\sum_{k=1}^{M} (\mathrm{vol}_{n-1}(F_k^{\mathrm{i}}))^p\Big)^{1/p} M^{1/p'}.$$

Therefore from (31) and (33) with $p = \frac{n+1}{n-1}$ we get

$$\mathrm{vol}_{n-1}(P_M \triangle B_2^n) \geq \frac{(c/2)^{\frac{n+1}{n-1}}}{8\pi e}\,\frac{1}{M^{\frac{2}{n-1}}}\,(n\,\mathrm{vol}_n(B_2^n))^{\frac{n+1}{n-1}} \geq \frac{c^{\frac{n+1}{n-1}}}{8M^{\frac{2}{n-1}}}\,\mathrm{vol}_n(B_2^n).$$

The second case is that (32) does not hold. Thus

$$\sum_{k=1}^{M} \mathrm{vol}_{n-1}(F_k) = \sum_{k=1}^{M} \mathrm{vol}_{n-1}(F_k^{\mathrm{i}}) + \sum_{k=1}^{M} \mathrm{vol}_{n-1}(F_k^{\mathrm{a}}) < c\,\mathrm{vol}_{n-1}(\partial B_2^n).$$

Then, by the isoperimetric inequality,

$$\mathrm{vol}_n(P_M) \le \left(\frac{\sum_{k=1}^{M} \mathrm{vol}_{n-1}(F_k)}{\mathrm{vol}_{n-1}(\partial B_2^n)} \right)^{\frac{n}{n-1}} \mathrm{vol}_n(B_2^n) < c^{\frac{n}{n-1}}\,\mathrm{vol}_n(B_2^n)$$

and thus

$$\mathrm{vol}_n(P_M \triangle B_2^n) \ge (1 - c^{\frac{n}{n-1}})\,\mathrm{vol}_n(B_2^n).$$

Since $c = 9/10$, this last expression is greater than $M^{-\frac{2}{n-1}}\,\mathrm{vol}_n(B_2^n)$, provided $M \ge 10^{\frac{n-1}{2}}$, which holds by assumption. ■

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