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Finite-rank perturbations of positive operators and isometries

by

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Abstract. We completely characterize the ranks of $A-B$ and $A^{1/2}-B^{1/2}$ for operators A and B on a Hilbert space satisfying $A \geq B \geq 0$. Namely, let l and m be nonnegative integers or infinity. Then $l = \operatorname{rank}(A-B)$ and $m = \operatorname{rank}(A^{1/2}-B^{1/2})$ for some operators A and B with $A \geq B \geq 0$ on a Hilbert space of dimension n ($1 \leq n \leq \infty$) if and only if $l = m = 0$ or $0 < l \leq m \leq n$. In particular, this answers in the negative the question posed by C. Benhida whether for positive operators A and B the finiteness of $\operatorname{rank}(A-B)$ implies that of $\operatorname{rank}(A^{1/2}-B^{1/2})$.

For two isometries, we give necessary and sufficient conditions in order that they be finite-rank perturbations of each other. One such condition says that, for isometries A and B, $A-B$ has finite rank if and only if $A = (I+F)B$ for some unitary operator $I+F$ with finite-rank F. Another condition is in terms of the parts in the Wold–Lebesgue decompositions of the nonunitary isometries A and B.

A bounded linear operator A on a complex separable Hilbert space H is said to be *positive*, denoted by $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all vectors x in H, where $\langle \cdot, \cdot \rangle$ denotes the inner product in H. If A is positive, then $A^{1/2}$ denotes the (unique) positive square root of A. In a recent paper by C. Benhida [1], it was asked whether for two positive operators A and B, the condition $\operatorname{rank}(A-B) < \infty$ would imply $\operatorname{rank}(A^{1/2}-B^{1/2}) < \infty$. In Section 1 below, we completely characterize the ranks of $A-B$ and $A^{1/2}-B^{1/2}$ for operators A and B satisfying $A \geq B \geq 0$. We show, in particular, that the answer to Benhida's question is "No". On the other hand, if A and B are commuting positive operators, then the rank of $A-B$ equals that of $A^{1/2}-B^{1/2}$.

We next consider, in Section 2, two isometries V_1 and V_2 and give two different necessary and sufficient conditions in order that they be finite-rank perturbations of each other. One condition converts the finite-rank perturbation of V_1 and V_2 into a multiplicative unitary relation between

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them. Another condition is in terms of the parts in the Wold–Lebesgue decompositions of the nonunitary V_1 and V_2.

For any operator A, we use $\ker A$ and $\operatorname{range} A$ to denote the kernel and range of A, respectively. The *rank* of A, $\operatorname{rank} A$, is the dimension of $\overline{\operatorname{range} A}$. A (closed) subspace K of H is said to *reduce* the operator A on H if AK and A^*K are contained in K, in which case A can be decomposed as $A_1 \oplus A_2$ on $H = K \oplus K^\perp$.

1. Positive operators. The main result of this section is the following theorem giving a characterization of the pairs of integers which are the ranks of $A - B$ and $A^{1/2} - B^{1/2}$ for operators A and B satisfying $A \geq B \geq 0$.

THEOREM 1.1. *Let l and m be nonnegative integers or infinity. Then $l = \operatorname{rank}(A - B)$ and $m = \operatorname{rank}(A^{1/2} - B^{1/2})$ for some operators A and B with $A \geq B \geq 0$ on a Hilbert space of dimension n $(1 \leq n \leq \infty)$ if and only if $l = m = 0$ or $0 < l \leq m \leq n$.*

The necessity of the condition is proved in the next lemma.

LEMMA 1.2.

(a) *If A and B are positive operators on the same Hilbert space H, then* $\operatorname{rank}(A - B) \leq 2 \operatorname{rank}(A^{1/2} - B^{1/2})$.
(b) *If A and B on H satisfy $A \geq B \geq 0$, then we have* $\operatorname{rank}(A - B) \leq \operatorname{rank}(A^{1/2} - B^{1/2})$ *and* $\ker(A^{1/2} - B^{1/2})$ *is a common reducing subspace of A and B.*

Proof. The assertion in (a) follows from the equality

$$A - B = A^{1/2}(A^{1/2} - B^{1/2}) + (A^{1/2} - B^{1/2})B^{1/2}.$$

To prove (b), let $F = A^{1/2} - B^{1/2}$ and $K = \ker F$. Assume that F and $B^{1/2}$ are represented as

$$F = F_1 \oplus 0, \qquad B^{1/2} = \begin{bmatrix} B_1 & B_2 \\ B_2^* & B_3 \end{bmatrix}$$

on $H = K^\perp \oplus K$. Then

$$\begin{aligned} A - B &= (B^{1/2} + F)^2 - B = B^{1/2}F + FB^{1/2} + F^2 \\ &= \begin{bmatrix} B_1F_1 + F_1B_1 + F_1^2 & F_1B_2 \\ B_2^*F_1 & 0 \end{bmatrix} \geq 0. \end{aligned}$$

This implies that $F_1B_2 = 0$. Since F_1 is one-to-one, we obtain $B_2 = 0$. Thus $A - B = (B_1F_1 + F_1B_1 + F_1^2) \oplus 0$, from which we derive that $\ker(A - B) \supseteq \ker(A^{1/2} - B^{1/2})$. Hence

$$\overline{\operatorname{range}(A - B)} \subseteq \overline{\operatorname{range}(A^{1/2} - B^{1/2})},$$

and so $\operatorname{rank}(A-B) \le \operatorname{rank}(A^{1/2}-B^{1/2})$. The assertion on $\ker(A^{1/2}-B^{1/2})$ follows from the arguments above. ∎

To prove the sufficiency of the condition in Theorem 1.1, we need the following lemma. This should be known to experts. We include the proof for completeness.

LEMMA 1.3. *If $A = [a_{ij}]_{i,j=1}^n$ is a matrix with $a_{ij} \neq 0$ for all i and j, and B is a diagonal matrix* $\operatorname{diag}(b_1, \ldots, b_n)$ *with distinct b_j's, then the only common reducing subspaces of A and B are the trivial ones* $\{0\}$ *and* $\mathbb{C}^n$. *The analogous assertion* (*with n replaced by infinity*) *holds for infinite matrices A and B on l^2.*

Proof. Let M be a common reducing subspace of A and B, and let P_M be the (orthogonal) projection from $\mathbb{C}^n$ onto M. Since P_M commutes with B and the b_j's are distinct, we derive that $P_M = \operatorname{diag}(p_1, \ldots, p_n)$ with $p_j = 0$ or 1 for each j. On the other hand, since P_M also commutes with A and the entries of A are all nonzero, we conclude that either $p_j = 0$ for all j or $p_j = 1$ for all j. Hence M can only be $\{0\}$ or $\mathbb{C}^n$ as asserted. ∎

Note that in the preceding lemma the requirement on the entries of A can be considerably weakened. However, for our purposes the present form suffices.

Proof of Theorem 1.1. We first assume that $0 < l \le m = n < \infty$. Let $B = \operatorname{diag}(b_1, \ldots, b_m)$ be a diagonal matrix with positive and distinct b_j's, let $x_1, \ldots, x_l$ be l linearly independent vectors in $\mathbb{C}^m$ whose components are all positive, let $C = \sum_{j=1}^l x_j x_j^*$, and let $A = B+C$. Then $A \ge B \ge 0$, $\operatorname{rank} C = l$ and the entries of A are all positive. By Lemma 1.3, the only common reducing subspaces of A and B are $\{0\}$ and $\mathbb{C}^m$. Since $\ker(A^{1/2}-B^{1/2})$ is a common reducing subspace of A and B by Lemma 1.2(b), we conclude that $\ker(A^{1/2}-B^{1/2}) = \{0\}$ or $\mathbb{C}^m$. The latter is impossible since A and B are not equal. Hence $\ker(A^{1/2}-B^{1/2}) = \{0\}$ and thus $\operatorname{rank}(A^{1/2}-B^{1/2}) = m$. For the more general case that $0 < l \le m \le n < \infty$, let A and B be the m-by-m matrices as above. Then $A \oplus 0_{n-m}$ and $B \oplus 0_{n-m}$ meet our requirements. Analogous constructions of A and B work for $n = \infty$. This completes the proof. ∎

We conclude this section with two related facts. Firstly, if A and B are commuting positive operators, then $\operatorname{rank}(A-B) = \operatorname{rank}(A^{1/2}-B^{1/2})$. This can be deduced from the spectral theory of the normal operator $A+iB$. In the next proposition, we prove this from some easily derived facts.

PROPOSITION 1.4. *If A and B are commuting positive operators*, *then*

$$\overline{\operatorname{range}(A-B)} = \overline{\operatorname{range}(A^{1/2}-B^{1/2})}.$$

Proof. The commuting of A and B implies that of $A^{1/2}$ and $B^{1/2}$. Hence $A - B = (A^{1/2} + B^{1/2})(A^{1/2} - B^{1/2})$. From this, we deduce the inclusion $\ker(A^{1/2} - B^{1/2}) \subseteq \ker(A - B)$. On the other hand, from

$$0 \le (A^{1/2} - B^{1/2})^4 = (A - B)^2 - 4A^{1/2}B^{1/2}(A^{1/2} - B^{1/2})^2 \le (A - B)^2,$$

we obtain $\ker(A-B) \subseteq \ker(A^{1/2}-B^{1/2})$. Thus $\ker(A-B) = \ker(A^{1/2}-B^{1/2})$ and our assertion follows. ■

Secondly, it is known that the compactness of $A-B$ for positive operators A and B implies that of $A^{1/2} - B^{1/2}$. Not being able to find a precise reference, we provide a proof below.

PROPOSITION 1.5. *Let A and B be positive operators on the same space. If $A - B$ is compact, then so is $A^{1/2} - B^{1/2}$.*

Proof. It is easily seen that if $A-B$ is compact, then so is $p(A)-p(B)$ for any polynomial p. Let p_n, $n = 1, 2, \ldots$, be a sequence of polynomials which converges uniformly to the square-root function $f(t) = \sqrt{t}$ on $\sigma(A) \cup \sigma(B)$. Then $p_n(A)$ and $p_n(B)$ converge in norm to $A^{1/2}$ and $B^{1/2}$, respectively. Hence $A^{1/2} - B^{1/2}$, being the norm limit of the compact operators $p_n(A) - p_n(B)$, is also compact. ■

2. Isometries. An operator A is an *isometry* if $\|Ax\| = \|x\|$ for any vector x. In this section, we obtain two different kinds of necessary and sufficient conditions for two isometries to be finite-rank perturbations of each other. The first of these is one which converts the additive finite-rank perturbation into a "left" multiplicative unitary perturbation.

THEOREM 2.1. *Let V_1, V_2 be isometries on a separable Hilbert space H. Then* $\operatorname{rank}(V_1 - V_2) < \infty$ *if and only if there is a unitary operator U of the form $I + F$ with* $\operatorname{rank} F < \infty$ *such that $V_1 = UV_2$. Moreover, in this case, F can be chosen with* $\operatorname{rank} F \le 2 \operatorname{rank}(V_1 - V_2)$.

Proof. Assume that $K \equiv \operatorname{range}(V_1^* - V_2^*)$ is finite-dimensional. Then so is $L \equiv V_1(K) + V_2(K)$. Obviously, we have $V_j(K) \subseteq L$ for $j = 1, 2$. On the other hand, since for any x in $K^\perp$ and y in K the equalities

$$\langle V_j x, V_j y\rangle = \langle x, V_j^* V_j y\rangle = \langle x, y\rangle = 0$$

hold, we obtain $V_j(K^\perp) \subseteq (V_j K)^\perp$. Together with the fact that $V_1 = V_2$ on $\ker(V_1 - V_2) = K^\perp$, this yields $V_j(K^\perp) \subseteq (V_1K)^\perp \cap (V_2K)^\perp = L^\perp$, $j = 1, 2$. Consider the 2-by-2 operator matrix representation

$$V_j = \begin{bmatrix} W_j & 0 \\ 0 & R \end{bmatrix}$$

of V_j from $H = K \oplus K^\perp$ to $H = L \oplus L^\perp$, where $W_j : K \to L$ and $R : K^\perp \to L^\perp$ are isometries. Since $W_2 W_1^* | W_1 K$ is an isometry mapping $W_1 K$

onto W_2K, it can be extended to a unitary operator U_0 on the (finite-dimensional) space L. Let $U = U_0 \oplus I$ on $H = L \oplus L^\perp$. Then U is unitary with $\operatorname{rank}(U - I) \le 2 \operatorname{rank}(V_1 - V_2)$ and satisfies

$$UV_1 = \begin{bmatrix} U_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} W_1 & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} U_0W_1 & 0 \\ 0 & R \end{bmatrix}$$

$$= \begin{bmatrix} W_2W_1^*W_1 & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} W_2 & 0 \\ 0 & R \end{bmatrix} = V_2,$$

completing the proof. ∎

It is easier to prove the corresponding "right" multiplicative unitary perturbation for isometries.

PROPOSITION 2.2. *Let V_1 and V_2 be isometries on a common Hilbert space. Then there exists a unitary operator U of the form $I + F$ with finite-rank F such that $V_1 = V_2U$ if and only if* $\operatorname{range} V_1 = \operatorname{range} V_2$ *and* $\operatorname{rank}(V_1 - V_2) < \infty$. *Moreover, in this case,* $\operatorname{rank} F$ *is equal to* $\operatorname{rank}(V_1 - V_2)$.

Proof. If $V_1 = V_2U$ as above, then obviously $\operatorname{range} V_1 = \operatorname{range} V_2$ and also

$$\operatorname{rank}(V_1 - V_2) = \operatorname{rank} V_2(U - I) = \operatorname{rank} V_2F = \operatorname{rank} F < \infty.$$

Conversely, if $\operatorname{range} V_1 = \operatorname{range} V_2$, then Douglas's factorization theorem [5, Problem 59] implies that $V_1 = V_2U$ for some invertible operator U. Since both V_1 and V_2 are isometries, so is U. Hence U is unitary. Moreover, $\operatorname{rank}(U - I) = \operatorname{rank}(V_1 - V_2) < \infty$ follows as above. This completes the proof. ∎

We now come to the second condition for the finite-rank perturbations of nonunitary isometries.

THEOREM 2.3. *Let V_1 and V_2 be nonunitary isometries on a separable Hilbert space. Then* $\operatorname{rank}(V_1 - U^*V_2U) < \infty$ *for some unitary U if and only if V_j is unitarily equivalent to $U_j \oplus W$, $j = 1, 2$, where U_1 and U_2 are singular unitary operators with finite multiplicity and W is a nonunitary isometry.*

Recall that the *multiplicity* $\mu(A)$ of an operator A on H is the minimum cardinality of a subset $\{x_\lambda\}_{\lambda\in\Lambda}$ of H for which the closed linear span of the vectors $A^n x_\lambda$, $n \ge 0$ and $\lambda \in \Lambda$, equals H. The operator A is said to be *cyclic* if $\mu(A) = 1$. By the spectral theorem, a normal operator has finite multiplicity if and only if it is the direct sum of finitely many cyclic operators (cf. [4, Section IX.10]).

For the proof of Theorem 2.3, we need the Wold–Lebesgue decomposition of isometries. This says that every isometry V can be uniquely decomposed as the direct sum of a singular unitary operator U_s, an absolutely continuous unitary operator U_a and a unilateral shift $S^{(n)}$: $V = U_s \oplus U_a \oplus S^{(n)}$. Here $S^{(n)}$ denotes the direct sum of n copies ($0 \le n \le \infty$) of the simple unilateral shift S. The proof for the sufficiency of Theorem 2.3 is based on the following lemma.

LEMMA 2.4. *Let $S^{(n)}$ ($1 \le n \le \infty$) be the direct sum of n copies of the simple unilateral shift. Then an isometry is a rank-one perturbation of $S^{(n)}$ if and only if it is unitarily equivalent to either $S^{(n)}$ or $U \oplus S^{(n)}$, where U is a cyclic singular unitary operator.*

This is proved in [6, Theorem 2 and Proposition 2].

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. For $j = 1, 2$, let $V_j = U_{js} \oplus U_{ja} \oplus S^{(n_j)}$ be the Wold–Lebesgue decomposition of V_j as above with $1 \le n_j \le \infty$.

To prove one direction, we may assume, for convenience, that V_1 and V_2 act on the same space H and $F \equiv V_1 - V_2$ has finite rank. Let $x_1, \ldots, x_k$ be vectors which span the range of F and let K be the (closed) subspace spanned by $V_1^n x_m$, $n \ge 0$ and $1 \le m \le k$. Then K is also spanned by $V_2^n x_m$, $n \ge 0$ and $1 \le m \le k$, and, in particular, K is invariant for V_1 and V_2 and hence for F. Thus we have the triangulations

$$V_1 = \begin{bmatrix} V_{11} & * \\ 0 & V_{12} \end{bmatrix}, \quad V_2 = \begin{bmatrix} V_{21} & * \\ 0 & V_{22} \end{bmatrix}, \quad F = \begin{bmatrix} F_1 & F_2 \\ 0 & 0 \end{bmatrix}$$

on $H = K \oplus K^\perp$. Let U_{ij} be the singular unitary part of V_{ij}, $i, j = 1, 2$. It was shown in [8, Lemma 4.4] that $U_{1s} = U_{11} \oplus U_{12}$ and $U_{2s} = U_{21} \oplus U_{22}$. We have $\mu(U_{i1}) \le \mu(V_{i1}) < \infty$, $i = 1, 2$, and $U_{12} = U_{22}$, the latter because $V_{12} = V_{22}$. The unitary equivalence of U_{1a} and U_{2a} follows from a result of Carey [3, Proposition], and the equality of n_1 and n_2 from the Fredholm index theory [4, Theorem XI.3.11]. This proves our necessity assertion.

To prove the sufficiency, assume that $V_j = U_j \oplus W$, $j = 1, 2$. Let $n = n_1 = n_2 \ge 1$. Since U_j is a singular unitary operator with finite multiplicity, by Lemma 2.4 there is a finite-rank operator F_j such that $U_j \oplus S^{(n)}$ and $S^{(n)} + F_j$ are unitarily equivalent. Hence $U_1 \oplus S^{(n)}$ is unitarily equivalent to a finite-rank perturbation of $U_2 \oplus S^{(n)}$. This yields our assertion that $\operatorname{rank}(V_1 - U^* V_2 U) < \infty$ for some unitary U. ■

Related results on finite-rank perturbations of more general contractions may be found in [7, 8, 2, 1].

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