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Some duality results on bounded approximation properties of pairs

by

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Dedicated to the memory of Olek Pełczyński

Abstract. The main result is as follows. Let X be a Banach space and let Y be a closed subspace of X. Assume that the pair $(X^*, Y^\perp)$ has the λ-bounded approximation property. Then there exists a net (S_α) of finite-rank operators on X such that $S_\alpha(Y) \subset Y$ and $\|S_\alpha\| \le \lambda$ for all α, and (S_α) and (S_α^*) converge pointwise to the identity operators on X and X^*, respectively. This means that the pair (X, Y) has the λ-bounded duality approximation property.

1. Introduction. Let X be a Banach space and let I_X denote the identity operator on X. Let $\lambda \ge 1$. Recall that X has the λ-*bounded approximation property* if there exists a net of finite-rank operators $(S_\alpha) \subset \mathcal{F}(X)$ such that $\|S_\alpha\| \le \lambda$ for all α and $S_\alpha \to I_X$ pointwise, i.e., in the strong operator topology.

Let Y be a closed subspace of X. If the net (S_α) can be chosen with $S_\alpha(Y) \subset Y$ for all α, then the *pair* (X, Y) is said to have the λ-*bounded approximation property.* This concept was recently introduced and studied by Figiel, Johnson, and Pełczyński in the important paper [FJP] (see Theorem 4.1 below for equivalent reformulations of this concept). If $\lambda = 1$, then one speaks about the *metric approximation property* of X or of (X, Y).

Clearly, the λ-bounded approximation properties for X, (X, X), and $(X, \{0\})$ are all equivalent.

If X is *reflexive*, then the following duality conditions are equivalent:

(a) (X, Y) has the λ-bounded approximation property,

(a*) $(X^*, Y^\perp)$ has the λ-bounded approximation property.

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(This equivalence follows easily by using a convex combination argument and the fact that $S(Y) \subset Y$ if and only if $S^*(Y^\perp) \subset Y^\perp$. It may be added that, by [LisO, Corollary 5.3], the approximation property of a pair (X, Y) is always metric whenever X is reflexive.)

The implication (a)⇒(a*) does not hold in general: by a well-known Enflo–James–Lindenstrauss result (see, e.g., [LT, p. 34]), the metric approximation property of X does not imply the approximation property of X^*.

The implication (a*)⇒(a) is well known to hold for an arbitrary X in the particular case when $Y = \{0\}$ (or $Y = X$): if X^* has the λ-bounded approximation property, then also X has. This result is essentially due to Grothendieck (proved in [G, Chapter I, Proposition 40, p. 180] for the metric approximation property), but its essence resides in the following important result due to Johnson [J] (see, e.g., [C, Proposition 3.5]).

THEOREM 1.1 (Johnson). *Let X be a Banach space. If X^* has the λ-bounded approximation property, then X^* has the λ-bounded approximation property with conjugate operators.*

Recall that X^* has the *λ-bounded approximation property with conjugate operators* if there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $\|S_\alpha\| \le \lambda$ for all α and $S_\alpha^* \to I_{X^*}$ pointwise. In that case, $S_\alpha \to I_X$ in the weak operator topology, and passing to convex combinations, one may also assume that $S_\alpha \to I_X$ in the strong operator topology, i.e., X has the λ-bounded approximation property.

The principal result of this paper is as follows (see also Theorem 3.6).

THEOREM 1.2. *Let X be a Banach space and let Y be a closed subspace of X. The pair $(X^*, Y^\perp)$ has the λ-bounded approximation property if and only if there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha(Y) \subset Y$ and $\|S_\alpha\| \le \lambda$ for all α, and $S_\alpha^* \to I_{X^*}$ pointwise.*

Theorem 1.2 extends Theorem 1.1 from X^* to $(X^*, Y^\perp)$ and, by the convex combinations argument as above, it shows that the implication (a*)⇒(a) holds in full generality.

Theorem 1.1 follows easily from the principle of local reflexivity (see, e.g., [C, Proposition 3.5]), according to which finite-rank operators on a dual space are "locally conjugate" (see, e.g., [OP, Theorem 2.5]). In contrast, we cannot figure out how the principle of local reflexivity, even in its most sophisticated form (see [B] or, e.g., [OP, Theorem 2.4]), could be used to prove Theorem 1.2 (see also Remark 3.7 below).

Our proof of Theorem 1.2 will rely on the basic idea of using Grothendieck's descriptions of the dual space of the space of finite-rank operators as spaces of integral operators (Theorem 2.1). This basic idea, which comes from the alternative short proof of Theorem 1.1 in [O1, Corollary 2.3], will be considerably developed to yield a proof of Theorem 1.2.

The proof of Theorem 1.2 and its reformulation (Theorem 3.6) through the λ-*bounded duality approximation property of pairs* (see Definition 3.5) will be presented in Section 3 after establishing the Main Lemma on the structure of spaces of integral operators (Lemma 2.2) in Section 2. Section 4 contains equivalent reformulations of the λ-bounded approximation property of pairs and its duality version, with Theorem 4.2 being the main result of the section. The final Section 5 contains applications to lifting results of type (a)$\Rightarrow$(a*) in some special cases, complementing and extending results from [FJP] and [GS2].

Our notation is standard. Let X and Y be Banach spaces, both real or both complex. We denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators acting from X to Y and by $\mathcal{F}(X,Y)$ its subspace of finite-rank operators. We write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$ and, similarly, $\mathcal{F}(X)$ for $\mathcal{F}(X,X)$. The range of an operator $S : X \to Y$ is denoted by $\operatorname{ran} S := \{Sx : x \in X\}$.

A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding $j_X : X \to X^{**}$. The closed unit ball of X is denoted by B_X. The annihilator of a closed subspace Y in X is denoted by $Y^\perp := \{x^* \in X^* : x^*(y) = 0\ \forall y \in Y\}$.

2. The Main Lemma. Let X and Y be Banach spaces. The Main Lemma below (Lemma 2.2) will concern some structure of Banach spaces of integral operators considered as dual spaces of the space $\mathcal{F}(X,Y)$ of finite-rank operators.

Recall that a mapping $S : X \to Y$ belongs to $\mathcal{F}(X,Y)$ if and only if S can be represented as a finite sum of rank one operators

$$S = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $x_i^* \in X^*$ and $y_i \in Y$, and $(x_i^* \otimes y_i)(x) = x_i^*(x) y_i$ for all $x \in X$.

Also recall that an operator $T \in \mathcal{L}(X,Y)$ is an *integral operator* if there exists a probability measure space (with measure μ) and operators $a \in \mathcal{L}(X, L_\infty(\mu))$ and $b \in \mathcal{L}(L_1(\mu), Y^{**})$ such that $j_Y T = b j_1 a$, where $j_1 : L_\infty(\mu) \to L_1(\mu)$ is the identity embedding, meaning that the diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "a"'] & Y \arrow[r, "j_Y"] & Y^{**} \\ L_\infty(\mu) \arrow[rr, "j_1"] & & L_1(\mu) \arrow[u, "b"'] \end{tikzcd}$$

commutes. Let us denote by $\mathcal{I}(X,Y)$ the collection of all integral operators from X to Y. The *integral norm* $\|T\|_{\mathcal{I}}$ of an integral operator $T \in \mathcal{I}(X,Y)$

is defined by the equality

$$\|T\|_{\mathcal{I}} = \inf \|a\|\,\|b\|,$$

where the infimum is taken over all possible factorizations of T as above. It is straightforward to verify that $\mathcal{I} = (\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a Banach operator ideal (see, e.g., [DJT, Theorem 5.2]).

Integral operators were introduced by Grothendieck in [G, Chapter I, Proposition 27, pp. 124–127] with the aim of describing the dual space of an injective tensor product. Without entering into the theory of tensor products, we may reformulate this very important Grothendieck description in the following way (cf. [O3, pp. 202–203]).

Theorem 2.1 (Grothendieck). *Let X and Y be Banach spaces. Then the dual space $(\mathcal{F}(X,Y), \|\cdot\|)^*$ is linearly isometric to $\mathcal{I}(Y, X^{**})$ under the duality*

$$\Big\langle T, \sum_{i=1}^{n} x_i^* \otimes y_i \Big\rangle = \sum_{i=1}^{n} (Ty_i)(x_i^*),$$

and also to $\mathcal{I}(X^*, Y^*)$ under the duality

$$\Big\langle T, \sum_{i=1}^{n} x_i^* \otimes y_i \Big\rangle = \sum_{i=1}^{n} (Tx_i^*)(y_i).$$

We shall express Theorem 2.1 by writing $(\mathcal{F}(X,Y))^* = \mathcal{I}(Y, X^{**})$ and $(\mathcal{F}(X,Y))^* = \mathcal{I}(X^*, Y^*)$.

The link with tensor products of Banach spaces is that, in fact, $\mathcal{F}(X,Y)$ is algebraically the same as the algebraic tensor product $X^* \otimes Y$, with the rank one operator $x^* \otimes y$ corresponding to the elementary tensor $x^* \otimes y$. And the link with the *injective tensor norm* $\varepsilon = \|\cdot\|_\varepsilon$ is that, in fact, $\|T\| = \|\sum_{i=1}^n x_i^* \otimes y_i\|_\varepsilon$ for all $T \in \mathcal{F}(X,Y)$, $T = \sum_{i=1}^n x_i^* \otimes y_i$.

The following lemma (or, more precisely, its Corollary 2.3) will be needed in the proof of the necessity in Theorem 1.2. It uses the canonical identifications $(\mathcal{F}(X))^* = \mathcal{I}(X^*, X^*)$ and $(\mathcal{F}(X^*))^* = \mathcal{I}(X^*, X^{***})$ from Theorem 2.1.

Lemma 2.2. *Let X be a Banach space and let Y be a closed subspace of X. Denote $A = \{R \in \mathcal{F}(X) : R(Y) \subset Y\}$ and $B = \{S \in \mathcal{F}(X^*) : S(Y^\perp) \subset Y^\perp\}$, and consider $A^\perp$ as a subspace of $\mathcal{I}(X^*, X^*)$ and $B^\perp$ as a subspace of $\mathcal{I}(X^*, X^{***})$. If $T \in \mathcal{I}(X^*, X^*)$ is such that $T \in A^\perp$, then $j_{X^*}T \in B^\perp$.*

Proof. Let $T \in A^\perp$, i.e., $\langle T, R\rangle = 0$ for every $R \in A$. We have to show that $\langle j_{X^*}T, S\rangle = 0$ for every $S \in B$. Suppose that $S \in B$, i.e., $S \in \mathcal{F}(X^*)$ and $S(Y^\perp) \subset Y^\perp$. Consider a representation

$$S = \sum_{i=1}^{n} x_i^{**} \otimes x_i^*,$$

where $(x_i^*)_{i=1}^n \subset X^*$ and $(x_i^{**})_{i=1}^n \subset X^{**}$.

Since $Y^\perp$ is a linear subspace of X^*, $Y^\perp$ is algebraically complemented. This means that there is a linear subspace W of X^* such that $X^* = W \oplus Y^\perp$, i.e., for every $x^* \in X^*$ there is a unique representation $x^* = w + y^\perp$, where $w \in W$ and $y^\perp \in Y^\perp$. Since $x_i^* = w_i + y_i^\perp$, where $w_i \in W$ and $y_i^\perp \in Y^\perp$, we have $S = S_1 + S_2$, where

$$S_1 = \sum_{i=1}^n x_i^{**} \otimes w_i \quad \text{and} \quad S_2 = \sum_{i=1}^n x_i^{**} \otimes y_i^\perp.$$

Note that $\operatorname{ran} S_1 \subset W$. Let $(\bar{w}_i)_{i=1}^k \subset W$ be an algebraic basis of $\operatorname{ran} S_1$. Then there is a system $(\bar{x}_i^{**})_{i=1}^k \subset X^{**}$ such that

$$S_1 = \sum_{i=1}^k \bar{x}_i^{**} \otimes \bar{w}_i.$$

Let $x^* \in Y^\perp$ be arbitrary. Since $S(Y^\perp) \subset Y^\perp$,

$$Sx^* = S_1 x^* + S_2 x^* = \sum_{i=1}^k \bar{x}_i^{**}(x^*)\bar{w}_i + \sum_{i=1}^n x_i^{**}(x^*) y_i^\perp \in Y^\perp,$$

implying that

$$\sum_{i=1}^k \bar{x}_i^{**}(x^*)\bar{w}_i \in W \cap Y^\perp = \{0\}.$$

Hence, $\bar{x}_i^{**}(x^*) = 0$ for every $i \in \{1, \dots, k\}$ and for every $x^* \in Y^\perp$, i.e., $(\bar{x}_i^{**})_{i=1}^k \subset Y^{\perp\perp}$.

Let us consider the canonical isometry $I : Y^{\perp\perp} \to Y^{**}$ defined by $(Iy^{\perp\perp})(y^*) = y^{\perp\perp}(x^*)$, where $y^* \in Y^*$, $y^{\perp\perp} \in Y^{\perp\perp}$, and $x^* \in X^*$ is an arbitrary extension of y^*. Then $(I\bar{x}_i^{**})_{i=1}^k \subset Y^{**}$ and, by Theorem 2.1, we get

$$\begin{aligned}\langle j_{X^*}T, S\rangle = \langle j_{X^*}T, S_1\rangle + \langle j_{X^*}T, S_2\rangle &= \sum_{i=1}^k \bar{x}_i^{**}(T\bar{w}_i) + \sum_{i=1}^n x_i^{**}(Ty_i^\perp) \\ &= \sum_{i=1}^k (I\bar{x}_i^{**})(T\bar{w}_i|_Y) + \sum_{i=1}^n x_i^{**}(Ty_i^\perp).\end{aligned}$$

Denote $y_i^{**} := I\bar{x}_i^{**} \in Y^{**}$ and $y_i^* := T\bar{w}_i|_Y \in Y^*$, and choose elements $y_i \in Y$, $i = 1, \dots, k$, such that $y_i^{**}(y_i^*) = y_i^*(y_i)$. Also choose $x_i \in X$, $i = 1, \dots, n$, such that $x_i^{**}(Ty_i^\perp) = (Ty_i^\perp)(x_i)$. (Such elements exist. Indeed, let Z be a normed space, $z^* \in Z^*$, $z^{**} \in Z^{**}$, and denote $a := z^{**}(z^*)$. If $a = 0$, then $a = z^*(z)$ for $z = 0$. If $a \neq 0$, then there is $w \in Z$ such that $b := z^*(w) \neq 0$. Take $z = ab^{-1}w$; then $a = z^*(z)$.) Using the elements y_i

and x_i, define

$$R := \sum_{i=1}^{k} \bar{w}_i \otimes y_i + \sum_{i=1}^{n} y_i^{\perp} \otimes x_i \in \mathcal{F}(X).$$

Then $R(Y) \subset Y$, because for every $y \in Y$ we have

$$Ry = \sum_{i=1}^{k} \bar{w}_i(y) y_i + \sum_{i=1}^{n} y_i^{\perp}(y) x_i = \sum_{i=1}^{k} \bar{w}_i(y) y_i \in Y.$$

Hence, $R \in A$ and therefore $\langle T, R\rangle = 0$. On the other hand,

$$\begin{aligned}\langle T, R\rangle &= \sum_{i=1}^{k} (T\bar{w}_i)(y_i) + \sum_{i=1}^{n} (Ty_i^{\perp})(x_i) = \sum_{i=1}^{k} y_i^*(y_i) + \sum_{i=1}^{n} x_i^{**}(Ty_i^{\perp}) \\ &= \sum_{i=1}^{k} y_i^{**}(y_i^*) + \sum_{i=1}^{n} x_i^{**}(Ty_i^{\perp}) = \langle j_{X^*}T, S\rangle.\end{aligned}$$

Hence $\langle j_{X^*}T, S\rangle = 0$, as desired. ■

Thanks to the fact that

$$\|j_{X^*}T\|_{\mathcal{I}} = \|T\|_{\mathcal{I}} \quad \forall T \in \mathcal{I}(X^*, X^*)$$

(see, e.g., [DJT, Theorem 5.14]), there is a natural isometric embedding $J : \mathcal{I}(X^*, X^*) \to \mathcal{I}(X^*, X^{***})$ defined by $JT = j_{X^*}T$ for $T \in \mathcal{I}(X^*, X^*)$.

Corollary 2.3. *Let A and B be as in Lemma* 2.2, *and let*

$$\bar{J} : \mathcal{I}(X^*, X^*)/A^{\perp} \to \mathcal{I}(X^*, X^{***})/B^{\perp}$$

be defined by

$$\bar{J}(T + A^{\perp}) = JT + B^{\perp}, \quad T \in \mathcal{I}(X^*, X^*).$$

Then $\bar{J}$ is a well-defined operator, $\|\bar{J}\| \le 1$, and $\bar{J}q_1 = q_2 J$, where $q_1 : \mathcal{I}(X^*, X^*) \to \mathcal{I}(X^*, X^*)/A^{\perp}$ and $q_2 : \mathcal{I}(X^*, X^{***}) \to \mathcal{I}(X^*, X^{***})/B^{\perp}$ denote the quotient mappings.

Proof. The definition of $\bar{J}$ is correct because, by Lemma 2.2, $J(A^{\perp}) \subset B^{\perp}$. The other properties follow immediately from the definition of $\bar{J}$. ■

3. Proof of Theorem 1.2. Let X be a Banach space and let Y be a closed subspace of X. The sufficiency part of Theorem 1.2 asserts that the pair $(X^*, Y^{\perp})$ has the λ-bounded approximation property (i.e., there exists a net $(T_\alpha) \subset \mathcal{F}(X^*)$ such that $T_\alpha(Y^{\perp}) \subset Y^{\perp}$ and $\|T_\alpha\| \le \lambda$ for all α, and $T_\alpha \to I_{X^*}$ pointwise) whenever there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha(Y) \subset Y$ and $\|S_\alpha\| \le \lambda$ for all α, and $S_\alpha^* \to I_{X^*}$ pointwise. This assertion is immediate from the following easy observation.

Proposition 3.1. *Let X be a Banach space and let Y be a closed subspace of X. If $S \in \mathcal{L}(X)$, then*

$$S(Y) \subset Y \;\Leftrightarrow\; S^*(Y^\perp) \subset Y^\perp.$$

In addition to the material in Section 2, we need some more preparatory work before proceeding to the proof of the necessity part of Theorem 1.2.

Let X be a Banach space and let A be a linear subspace of $\mathcal{L}(X)$. Let $\lambda \geq 1$. Recall that X has the *λ-bounded A-approximation property* if there exists a net $(S_\alpha) \subset A$ such that $\|S_\alpha\| \leq \lambda$ for all α and $S_\alpha \to I_X$ pointwise. This concept has been studied since the early 1980s by Reinov, Grønbæk, Willis, and others (see, e.g., [BB], [Lis], [LMO] for references and recent results).

If now Y is a closed subspace of X, then $A := \{S \in \mathcal{F}(X) : S(Y) \subset Y\}$ is a linear subspace of $\mathcal{L}(X)$, and the λ-bounded approximation property of the pair (X, Y) is precisely the λ-bounded A-approximation property.

It is convenient to extend the well-known notion of the λ-duality approximation property (due to [J]; see, e.g., [C, p. 288] or [S, p. 314]) as follows.

Definition 3.2. Let X be a Banach space and let A be a linear subspace of $\mathcal{L}(X)$. Let $\lambda \geq 1$. We say that X has the *λ-bounded duality A-approximation property* if there exists a net $(S_\alpha) \subset A$ such that $\|S_\alpha\| \leq \lambda$ for all α, and $S_\alpha \to I_X$ and $S_\alpha^* \to I_{X^*}$ pointwise.

Remark 3.3. Since the weak and strong operator topologies on $\mathcal{L}(X)$ yield the same dual space (see, e.g., [DSch, Theorem VI.1.4]), by passing to convex combinations, one may always assume that $S_\alpha^* \to I_{X^*}$ and $S_\alpha \to I_X$ pointwise whenever $S_\alpha^* \to I_{X^*}$ pointwise. This means that the λ-bounded duality A-approximation property is equivalent to the well-known concept of the λ-bounded A-approximation property with conjugate operators.

Let A be a linear subspace of $\mathcal{L}(X)$. If $x^{**} \in X^{**}$ and $x^* \in X^*$, then the functional $x^* \otimes x^{**} : A \to \mathbb{K}$ is defined by the equality

$$(x^* \otimes x^{**})(T) = x^{**}(T^*x^*), \qquad T \in A.$$

Clearly $x^* \otimes x^{**} \in A^*$ and $\|x^* \otimes x^{**}\| \leq \|x^*\|\,\|x^{**}\|$.

By the proof of [O1, Theorem 2.1], the following holds (in [O1, Theorem 2.1], A was assumed to be the component of an arbitrary Banach operator ideal).

Theorem 3.4 (cf. [O1, Theorem 2.1]). *Let X be a Banach space and let A be a linear subspace of $\mathcal{L}(X)$. Let $\lambda \geq 1$. Then:*

(a) *X has the λ-bounded A-approximation property if and only if there exists $\Phi \in A^{**}$ such that $\|\Phi\| \leq \lambda$ and*

$$\Phi(x^* \otimes j_X x) = x^*(x) \quad \forall x^* \in X^*, \forall x \in X.$$

(b) *X has the λ-bounded duality A-approximation property if and only if there exists $\Phi \in A^{**}$ such that $\|\Phi\| \le \lambda$ and*

$$\Phi(x^* \otimes x^{**}) = x^{**}(x^*) \quad \forall x^* \in X^*, \forall x^{**} \in X^{**}.$$

Proof of Theorem 1.2. Necessity. Let $A = \{R \in \mathcal{F}(X) : R(Y) \subset Y\}$ and $B = \{S \in \mathcal{F}(X^*) : S(Y^\perp) \subset Y^\perp\}$ be as in Lemma 2.2.

Assume that the pair $(X^*, Y^\perp)$ has the λ-bounded approximation property. This is the same as the λ-bounded B-approximation property of X^*. According to Theorem 3.4(a), there exists $\Phi \in B^{**}$ such that $\|\Phi\| \le \lambda$ and

$$\Phi(x^{**} \otimes j_{X^*}x^*) = x^{**}(x^*) \quad \forall x^* \in X^*, \forall x^{**} \in X^{**}.$$

Since $A \subset \mathcal{F}(X)$ and $B \subset \mathcal{F}(X^*)$, A^* and B^* are isometrically isomorphic to $(\mathcal{F}(X))^*/A^\perp$ and $(\mathcal{F}(X^*))^*/B^\perp$, respectively. Hence, under canonical identifications (see Theorem 2.1), A^* and B^* are isometrically isomorphic to $\mathcal{I}(X^*, X^*)/A^\perp$ and $\mathcal{I}(X^*, X^{***})/B^\perp$, respectively. Let $t_1 : \mathcal{I}(X^*, X^*)/A^\perp \to A^*$ and $t_2 : \mathcal{I}(X^*, X^{***})/B^\perp \to B^*$ denote the corresponding isometric isomorphisms.

Let $i_1 : A \to \mathcal{F}(X)$ and $i_2 : B \to \mathcal{F}(X^*)$ be the identity embeddings and let $q_1 : \mathcal{I}(X^*, X^*) \to \mathcal{I}(X^*, X^*)/A^\perp$ and $q_2 : \mathcal{I}(X^*, X^{***}) \to \mathcal{I}(X^*, X^{***})/B^\perp$ denote the quotient mappings (as in Corollary 2.3). Then, under canonical identifications (see Theorem 2.1), we have $i_1^* = t_1 q_1$ and $i_2^* = t_2 q_2$.

Define $\Psi : A^* \to \mathbb{K}$ by $\Psi = \Phi t_2 \bar{J} t_1^{-1}$, where $\bar{J}$ is the operator from Corollary 2.3. Then the diagrams

$$\begin{tikzcd}
A^* \arrow[r, "\Psi"] \arrow[d, "t_1^{-1}"'] & \mathbb{K} & B^* \arrow[l, "\Phi"'] \\
\mathcal{I}(X^*, X^*)/A^\perp \arrow[rr, "\bar{J}"] & & \mathcal{I}(X^*, X^{***})/B^\perp \arrow[u, "t_2"'] \\
\mathcal{I}(X^*, X^*) \arrow[u, "q_1"] \arrow[rr, "J"] & & \mathcal{I}(X^*, X^{***}) \arrow[u, "q_2"']
\end{tikzcd}$$

commute, $\Psi \in A^{**}$ and $\|\Psi\| \le \|\Phi\| \le \lambda$.

Let $x^* \in X^*$ and $x^{**} \in X^{**}$. We shall show that $\Psi(x^* \otimes x^{**}) = x^{**}(x^*)$.

Denote $f := x^* \otimes x^{**} \in A^*$. Then $f(R) = x^{**}(R^*x^*)$ for all $R \in A$. Considering the rank one operator $\bar{f} := x^{**} \otimes x^* \in \mathcal{I}(X^*, X^*)$, it is easily verified that $\langle \bar{f}, R\rangle = x^{**}(R^*x^*)$ for all $R \in \mathcal{F}(X)$. Hence, $i_1^*\bar{f} = f$, i.e., $f = t_1 q_1 \bar{f}$. Now denote $g := x^{**} \otimes j_{X^*}x^* \in B^*$ and consider the rank one operator $\bar{g} := x^{**} \otimes j_{X^*}x^* \in \mathcal{I}(X^*, X^{***})$. Then, similarly, $g(S) = x^{**}(Sx^*)$ for all $S \in B$ and $\langle \bar{g}, S\rangle = x^{**}(Sx^*)$ for all $S \in \mathcal{F}(X^*)$. Hence, $g = t_2 q_2 \bar{g}$.

Therefore, since $\bar{g} = j_{X^*}\bar{f}$ and $\Phi(g) = x^{**}(x^*)$, we get

$$\begin{aligned}\Psi(x^* \otimes x^{**}) &= \Psi(f) = \Psi(t_1 q_1 \bar{f}) = \Phi(t_2 \bar{J} t_1^{-1} t_1 q_1 \bar{f}) = \Phi(t_2 \bar{J} q_1 \bar{f}) \\ &= \Phi(t_2 q_2 J \bar{f}) = \Phi(t_2 q_2 j_{X^*} \bar{f}) = \Phi(t_2 q_2 \bar{g}) = \Phi(g) = x^{**}(x^*).\end{aligned}$$

According to Theorem 3.4(b), X has the λ-bounded duality A-approximation property. This means that there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha(Y) \subset Y$ and $\|S_\alpha\| \le \lambda$ for all α, and $S_\alpha \to I_X$ and $S_\alpha^* \to I_{X^*}$ pointwise. ∎

Finally, it is natural to make the following definition.

Definition 3.5. Let X be a Banach space and let Y be a closed subspace of X. Let $\lambda \ge 1$. We say that the *pair* (X, Y) has the λ-*bounded duality approximation property* if there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha(Y) \subset Y$ and $\|S_\alpha\| \le \lambda$ for all α, and $S_\alpha \to I_X$ and $S_\alpha^* \to I_{X^*}$ pointwise.

Thus, the result we proved may be reformulated as follows.

Theorem 3.6. *Let X be a Banach space and let Y be a closed subspace of X. Let $\lambda \ge 1$. The pair $(X^*, Y^\perp)$ has the λ-bounded approximation property if and only if the pair (X, Y) has the λ-bounded duality approximation property.*

If, in the definitions of the λ-bounded and of the λ-bounded duality approximation properties of a pair (X, Y), one does not put any restriction on the norms $\|S_\alpha\|$ (i.e., one deletes "and $\|S_\alpha\| \le \lambda$"), then one obtains the notions which are naturally called the *approximation property of* (X, Y) and the *duality approximation property of* (X, Y).

Remark 3.7. The version of Theorem 3.6 stating that *the pair $(X^*, Y^\perp)$ has the approximation property if and only if the pair (X, Y) has the duality approximation property* was essentially established in [LisO, Proposition 5.11]. The proof in [LisO] uses the principle of local reflexivity. Also the special case of Theorem 3.6 when Y is of finite codimension can be proved using the principle of local reflexivity. But, as was mentioned in the Introduction, even the most sophisticated forms of the principle of local reflexivity, in our opinion, cannot be used to prove Theorem 3.6 in its full generality.

Corollary 3.8. *Let X be a Banach space and let Y be a closed subspace of X. Let X^* or X^{**} have the Radon–Nikodým property. If the pair $(X^*, Y^\perp)$ has the approximation property, then the pair (X, Y) has the metric duality approximation property.*

Proof. By [LisO, Corollary 5.12], the pair $(X^*, Y^\perp)$ has the metric approximation property. The metric duality approximation property of (X, Y) follows from Theorem 3.6. ∎

Concerning Corollary 3.8, let us recall the following famous open problem (see, e.g., [C, Problem 3.8]; for an overview see [O2, Section 3]): does the approximation property of the dual space X^* of an arbitrary Banach space X imply the metric approximation property?

4. Reformulating the bounded (duality) approximation property of pairs. Well-known reformulations of the λ-bounded approximation property of Banach spaces (see, e.g., [C, Theorem 3.3] or [S, p. 604, Theorem 18.1]) can be extended to the following conditions (a)–(d) in Theorem 4.1 below, all equivalent to the λ-bounded approximation property of pairs. The original notion of λ-bounded approximation property of (X, Y) in [FJP] was defined as property (d). In the present paper, we preferred to define the λ-bounded approximation property of (X, Y) through condition (b), since it seems to be more appropriate for expressing duality properties (cf. Definition 3.5).

THEOREM 4.1 (cf. [FJP, Lemma 1.5]). *Let X be a Banach space and let Y be a closed subspace of X. Let $\lambda \geq 1$. Then the following properties of the pair (X, Y) are equivalent:*

(a) *For every compact subset K of X and for every $\varepsilon > 0$ there exists $S \in \mathcal{F}(X)$ such that $S(Y) \subset Y$ and $\|S\| \leq \lambda$, and $\|Sx - x\| \leq \varepsilon$ for all $x \in K$.*
(b) *There exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha(Y) \subset Y$ and $\|S_\alpha\| \leq \lambda$ for all α, and $S_\alpha \to I_X$ pointwise.*
(c) *For every finite-dimensional subspace E of X and for every $\varepsilon > 0$ there exists $S \in \mathcal{F}(X)$ such that $S(Y) \subset Y$ and $\|S\| \leq \lambda$, and $\|Sx - x\| \leq \varepsilon\|x\|$ for all $x \in E$.*
(d) *For every finite-dimensional subspace E of X and for every $\varepsilon > 0$ there exists $S \in \mathcal{F}(X)$ such that $S(Y) \subset Y$ and $\|S\| \leq \lambda + \varepsilon$, and $Sx = x$ for all $x \in E$.*

Proof. The implications (a)$\Rightarrow$(b)$\Rightarrow$(c) and (d)$\Rightarrow$(a) are standard to deduce, and (c)$\Rightarrow$(d) is proved in [FJP, Lemma 1.5]. ∎

Theorem 3.6 asserts that the λ-bounded approximation property of the pair $(X^*, Y^\perp)$ is precisely the same as the λ-bounded duality approximation property of the pair (X, Y). Here the former property may be equivalently expressed using Theorem 4.1. To reformulate the latter property, one can apply Theorem 4.2 below. Note that we used condition (b) of Theorem 4.2 in Definition 3.5.

THEOREM 4.2. *Let X be a Banach space and let Y be a closed subspace of X. Let $\lambda \geq 1$. Then the following properties of the pair (X, Y) are equivalent:*

(a) *For all compact subsets K of X and L of X^*, and for every $\varepsilon > 0$, there exists $S \in \mathcal{F}(X)$ such that $S(Y) \subset Y$ and $\|S\| \leq \lambda$, and $\|Sx - x\| \leq \varepsilon$ for all $x \in K$ and $\|S^*x^* - x^*\| \leq \varepsilon$ for all $x^* \in L$.*

(b) *There exists a net* $(S_\alpha) \subset \mathcal{F}(X)$ *such that* $S_\alpha(Y) \subset Y$ *and* $\|S_\alpha\| \le \lambda$ *for all* α, *and* $S_\alpha \to I_X$ *and* $S_\alpha^* \to I_{X^*}$ *pointwise.*

(c) *For all finite-dimensional subspaces* E *of* X *and* F *of* X^*, *and for every* $\varepsilon > 0$, *there exists* $S \in \mathcal{F}(X)$ *such that* $S(Y) \subset Y$ *and* $\|S\| \le \lambda$, *and* $\|Sx - x\| \le \varepsilon\|x\|$ *for all* $x \in E$ *and* $\|S^*x^* - x^*\| \le \varepsilon\|x^*\|$ *for all* $x^* \in F$.

(d) *For all finite-dimensional subspaces* E *of* X *and* F *of* X^*, *and for every* $\varepsilon > 0$, *there exists* $S \in \mathcal{F}(X)$ *such that* $S(Y) \subset Y$ *and* $\|S\| \le \lambda + \varepsilon$, *and* $Sx = x$ *for all* $x \in E$ *and* $S^*x^* = x^*$ *for all* $x^* \in F$.

REMARK 4.3. The λ-*bounded duality approximation property* of a Banach space X is usually defined as the special case of either property (b) or (c) in Theorem 4.2 when $Y = \{0\}$ (equivalently, $Y = X$) (see, e.g., [J] or [C, p. 288] or [S, p. 314]). The equivalence of (c) and (d) in the context of the λ-bounded duality approximation property of a Banach space X has been established in [J, Lemma 3] (see also, e.g., [S, p. 315, Lemma 9.2]).

In the proof of Theorem 4.2, we shall need the following auxiliary result. Its special case when X is a Banach space was applied in the proof of [FJP, Lemma 1.5]. However, even in this special case, we have not found its proof in the literature. Therefore we include a proof for completeness.

LEMMA 4.4. *Let* X *be a locally convex Hausdorff space. Let* Y *be a closed subspace and* F *be a finite-dimensional subspace of* X. *Then there exists a continuous linear projection* P *on* X *such that* $\operatorname{ran} P = F$ *and* $P(Y) \subset Y$.

Proof. 1. Let us consider first a particular case, assuming that $F \cap Y = \{0\}$. Let $(x_i)_{i=1}^n$ be a basis of F. Denote $F_j = \operatorname{span}\{x_i : i \neq j\}$. Then $Y + F_j$ is a closed subspace of X and $x_j \notin Y + F_j$. According to a separation theorem, there exist continuous linear functionals f_j, $j = 1, \dots, n$, such that $f_j(x_j) = 1$ and $f_j|_{Y+F_j} = 0$, in particular, $f_j|_Y = 0$ and $f_i(x_j) = \delta_{ij}$, $i, j = 1, \dots, n$.

Define $P : X \to X$ by

$$Px = \sum_{i=1}^{n} f_i(x)x_i.$$

Then $\operatorname{ran} P = F$, $P(Y) = \{0\} \subset Y$, and P is a continuous linear projection on X.

2. Let us show that the general case can be reduced to the particular case above.

We start by decomposing $F = (F \cap Y) \oplus G$. Then $(F \cap Y) \cap G = \{0\}$ and $G \cap Y = \{0\}$. Hence, by the above, there exist continuous linear projections Q and R on X such that $\operatorname{ran} Q = F \cap Y$, $Q(G) \subset G$, and $\operatorname{ran} R = G$, $R(Y) \subset Y$. Define $P = Q + R - RQ$. Then $P(Y) \subset Y$. Since $QR = 0$, it is easily checked that $\operatorname{ran} P = F$ and $P^2 = P$. Hence, P is a continuous linear projection on X as desired. ∎

Proof of Theorem 4.2. The proofs of the implications (a)⇒(b)⇒(c) and (d)⇒(a) are standard.

Let us prove the implication (c)⇒(d). Let $E \subset X$ and $F \subset X^*$ be finite-dimensional subspaces and let $\varepsilon > 0$. Using Lemma 4.4, choose a projection $P \in \mathcal{L}(X)$ such that $\operatorname{ran} P = E$ and $P(Y) \subset Y$. Look at X^* endowed with its weak* topology and notice that $Y^\perp$ is weak* closed. Again using Lemma 4.4 choose a weak*-to-weak* continuous linear projection R on X^* such that $\operatorname{ran} R = F$ and $R(Y^\perp) \subset Y^\perp$. Then there exists $Q \in \mathcal{L}(X)$ such that $R = Q^*$. Hence, by Proposition 3.1, $Q(Y) \subset Y$.

Let $\delta > 0$ satisfy

$$\delta(\|P\| + \|I_X - P\|\,\|Q\|) < \varepsilon.$$

According to (c), there exists $T \in \mathcal{F}(X)$ such that $T(Y) \subset Y$ and $\|T\| \le \lambda$, and $\|Tx - x\| \le \delta\|x\|$ for all $x \in E$ and $\|T^*x^* - x^*\| \le \delta\|x^*\|$ for all $x^* \in F$.

Applying a perturbation argument inspired by [OP, proof of Lemma 1.2], we denote

$$S = I_X + (I_X - Q)(T - I_X)(I_X - P),$$

i.e.,

$$S = T + P - TP + Q - QT + QTP - QP.$$

Then, clearly, $S \in \mathcal{F}(X)$, S is the identity on $E = \operatorname{ran} P$, and S^* is the identity on $F = \operatorname{ran} Q^*$. Also, $S(Y) \subset Y$ because $Py, Ty, Qy \in Y$ for all $y \in Y$.

Finally, let us observe that

$$S = T + (I_X - T)P - Q(T - I_X)(I_X - P).$$

Let us also observe that

$$\|(I_X - T)P\| = \sup_{x \in B_X} \|TPx - Px\| \le \sup_{x \in B_X} \delta\|Px\| = \delta\|P\|$$

and

$$\|Q(T - I_X)(I_X - P)\| \le \|I_X - P\|\,\|(T^* - I_{X^*})Q^*\|$$

with

$$\|(T^* - I_{X^*})Q^*\| = \sup_{x^* \in B_{X^*}} \|T^*Q^*x^* - Q^*x^*\| \le \sup_{x^* \in B_{X^*}} \delta\|Q^*x^*\| = \delta\|Q\|.$$

Hence,

$$\begin{aligned}\|S - T\| &\le \|(I_X - T)P\| + \|Q(T - I_X)(I_X - P)\| \\ &\le \delta(\|P\| + \|I_X - P\|\,\|Q\|) \le \varepsilon\end{aligned}$$

and therefore

$$\|S\| \le \|T\| + \varepsilon \le \lambda + \varepsilon. \ \blacksquare$$

5. Lifting the bounded approximation property of pairs to dual spaces. Let X be a Banach space and let Y be a closed subspace of X. As mentioned in the Introduction, in general, the λ-bounded approximation property cannot be lifted from the pair (X, Y) to the pair $(X^*, Y^\perp)$. However, Theorems 3.4 and 3.6 appear to be good tools for studying this kind of lifting possibilities in some special cases (see Theorems 5.2 and 5.4 below).

Recall that a Banach space X is said to have the *unique extension property* if the only operator $T \in \mathcal{L}(X^{**})$ such that $\|T\| \le 1$ and $T|_X = I_X$ is the identity operator on X^{**}, i.e., $T = I_{X^{**}}$. This property was introduced and studied by Godefroy and Saphar in [GS1] (using the term "X is uniquely decomposed") and [GS2]. For instance (see [GS2]), the following Banach spaces have the unique extension property: Hahn–Banach smooth spaces, in particular, spaces which are M-ideals in their biduals (for example, closed subspaces of c_0); spaces with a Fréchet-differentiable norm; separable polyhedral Lindenstrauss spaces; spaces of compact operators $\mathcal{K}(X, Y)$ for reflexive Banach spaces X and Y.

By [GS2, Theorem 2.2], the unique extension property permits lifting the metric and metric compact approximation properties from X to X^*. An extension of this result to the metric A-approximation property was established in [O1, Corollary 2.5] for A being the component of an arbitrary Banach operator ideal. We shall need a slightly more general lifting result which follows from Theorem 3.4.

Proposition 5.1 (cf. [O1, Corollary 2.5]). *Let X be a Banach space and let A be a linear subspace of $\mathcal{L}(X)$. If X has the unique extension property and the metric A-approximation property, then X has the metric duality A-approximation property.*

Proof. The proof is essentially the same as in [O1, Corollary 2.5]. We present it for completeness. Since X has the metric A-approximation property, by Theorem 3.4(a), there exists $\Phi \in A^{**}$ such that $\|\Phi\| \le 1$ and

$$\Phi(x^* \otimes j_X x) = x^*(x) \quad \forall x^* \in X^*, \forall x \in X.$$

Define $T \in \mathcal{L}(X^{**})$ by

$$(Tx^{**})(x^*) = \Phi(x^* \otimes x^{**}), \quad x^{**} \in X^{**}, x^* \in X^*.$$

Then clearly $\|T\| \le 1$, and $T|_X = I_X$ because

$$(Tx)(x^*) = \Phi(x^* \otimes j_X x) = x^*(x) \quad \forall x \in X, \forall x^* \in X^*.$$

By the unique extension property, $T = I_{X^{**}}$. Hence,

$$\Phi(x^* \otimes x^{**}) = (I_{X^{**}} x^{**})(x^*) = x^{**}(x^*) \quad \forall x^* \in X^*, \forall x^{**} \in X^{**},$$

meaning that X has the metric duality A-approximation property (see Theorem 3.4(b)). ∎

Since the metric approximation property of the pair (X,Y) is precisely the metric A-approximation property of X, where $A=\{S\in\mathcal{F}(X):S(Y)\subset Y\}$, by Proposition 5.1, X has the metric duality A-approximation property. But the latter is precisely the metric duality approximation property of the pair (X,Y). Thus, looking also at Proposition 3.1 (or Theorem 3.6), we have obtained the following lifting result.

THEOREM 5.2. *Let X be a Banach space and let Y be a closed subspace of X. If X has the unique extension property and the pair (X,Y) has the metric approximation property, then the pair (X,Y) has the metric duality approximation property; hence, the pair $(X^*,Y^\perp)$ has the metric approximation property.*

In general, the unique extension property does not guarantee the lifting of the λ-bounded approximation property at least when $\lambda\ge 6$. Indeed, let X_{JS} be the closed subspace of c_0 constructed by Johnson and Schechtman (see [JO, Corollary JS]). Then X_{JS} has the unique extension property (all closed subspaces of c_0 do, as was already mentioned) and X_{JS} has the 6-bounded approximation property (see [Z]), but X^*_{JS} does not have the approximation property, in particular, it does not have the λ-bounded approximation property for any $\lambda\ge 1$.

Figiel, Johnson, and Pełczyński proved that *if X is a Banach space, $q:X\to Z$ is a quotient map, and* $\dim\ker q<\infty$, *then the λ-bounded approximation property of X implies the same property of Z* (see [FJP, Corollary 1.3]). Note that their proof (a straightforward one, which only uses condition (d) of Theorem 4.1) actually yields the following auxiliary result.

LEMMA 5.3 (see the proof of [FJP, Corollary 1.3]). *Let X be a Banach space and let Y be a finite-dimensional subspace of X. Let $\lambda\ge 1$. If X has the λ-bounded approximation property, then also the pair (X,Y) has the λ-bounded approximation property.*

The equivalence of conditions (a) and (c) below was established in [FJP, Proposition 1.6]. We can complement this result as follows, providing also an alternative proof for the implication (a)⇒(c).

THEOREM 5.4 (cf. [FJP, Proposition 1.6]). *Let X be a Banach space and let $\lambda\ge 1$. Then the following conditions are equivalent:*

(a) *The dual space X^* has the λ-bounded approximation property.*
(b) *The pair $(X^*,Y^\perp)$ has the λ-bounded approximation property for every finite-codimensional closed subspace Y of X.*
(c) *The pair (X,Y) has the λ-bounded approximation property for every finite-codimensional closed subspace Y of X.*

Proof. Since $Y^\perp$ is a finite-dimensional subspace of X^*, the implication (a)$\Rightarrow$(b) is immediate from Lemma 5.3. The implication (b)$\Rightarrow$(c) is clear from Theorem 3.6 because, obviously, the λ-bounded duality approximation property of the pair (X, Y) implies its λ-bounded approximation property. The implication (c)$\Rightarrow$(a) is proved in [FJP, Proposition 1.6]. ∎

Finally, let us mention that, as a by-product, we have the following slight complement to [FJP, Corollary 1.4(i)], asserting that *if the dual space X^* has the λ-bounded approximation property, then all finite-codimensional closed subspaces Y of X and their dual spaces Y^* have the λ-bounded approximation property.*

PROPOSITION 5.5. *Let X be a Banach space and let Y be a finite-codimensional closed subspace of X. Let $\lambda \geq 1$. If X^* has the λ-bounded approximation property, then all spaces $Y^\perp$, $X^*/Y^\perp$, Y^*, $(X/Y)^*$, X, Y and X/Y have the λ-bounded approximation property.*

Proof. By [FJP, Corollary 1.2], the λ-bounded approximation property of a pair (Z, W) implies that Z, W and Z/W all have the same property. According to Theorem 5.4, both pairs $(X^*, Y^\perp)$ and (X, Y) have the λ-bounded approximation property. Hence, $Y^\perp$, $X^*/Y^\perp$, X, Y, and X/Y all have the λ-bounded approximation property. Since the dual spaces Y^* and $(X/Y)^*$ are naturally isometric to $X^*/Y^\perp$ and $Y^\perp$, respectively, Y^* and $(X/Y)^*$ also have the λ-bounded approximation property. ∎

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