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Approximation properties determined by operator ideals and approximability of homogeneous polynomials and holomorphic functions

by

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Abstract. Given an operator ideal $\mathcal{I}$, a Banach space E has the $\mathcal{I}$-approximation property if the identity operator on E can be uniformly approximated on compact subsets of E by operators belonging to $\mathcal{I}$. In this paper the $\mathcal{I}$-approximation property is studied in projective tensor products, spaces of linear functionals, spaces of linear operators/homogeneous polynomials, spaces of holomorphic functions and their preduals.

1. Introduction. Given Banach spaces E and F, we denote by $\mathcal{L}(E;F)$ the Banach space of all bounded linear operators from E to F endowed with the usual operator sup norm. The subspaces of $\mathcal{L}(E;F)$ formed by all finite rank, all compact and all weakly compact operators are denoted by $\mathcal{F}(E;F)$, $\mathcal{K}(E;F)$ and $\mathcal{W}(E;F)$, respectively. For a subset S of $\mathcal{L}(E;F)$, the symbol $\overline{S}^{\tau_c}$ represents the closure of S with respect to the compact-open topology τ_c. By id_E we denote the identity operator on E.

Recall that a Banach space E has

- the *approximation property* (AP for short) if $\mathrm{id}_E \in \overline{\mathcal{F}(E;E)}^{\tau_c}$,
- the *compact approximation property* (CAP) if $\mathrm{id}_E \in \overline{\mathcal{K}(E;E)}^{\tau_c}$,
- the *weakly compact approximation property* (WCAP) if $\mathrm{id}_E \in \overline{\mathcal{W}(E;E)}^{\tau_c}$.

The AP is a classic in Banach space theory (see [13]) and is one of the main subjects of Grothendieck [29]. The CAP has been extensively studied in the last decades (see, e.g., [14, 16, 17]), but it goes back to Banach [4, p. 237]. The WCAP has been studied recently (see [17, 18]). Having in mind that $\mathcal{F}$, $\mathcal{K}$ and $\mathcal{W}$ are operator ideals, the properties above are obvious particular instances of the following general concept:

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Definition 1.1. Let $\mathcal{I}$ be an operator ideal. A Banach space E is said to have the *$\mathcal{I}$-approximation property* ($\mathcal{I}$-AP for short) if $\mathrm{id}_E \in \overline{\mathcal{I}(E;E)}^{\tau_c}$.

The $\mathcal{I}$-approximation property was studied, for instance, by Reinov [51, 52], Grønbæk and Willis [28] and Lissitsin, Mikkor and Oja [37]. Furthermore, several variants of the approximation property, including those closely related to the $\mathcal{I}$-AP, have been studied recently (see, e.g., [20, 22, 35, 36, 43, 44, 45, 53]). Even approximation properties more general than the $\mathcal{I}$-AP have already been investigated: see, for instance, Lissitsin and Oja [38].

It is clear that if E has the AP then E has the $\mathcal{I}$-AP for every operator ideal $\mathcal{I}$. In particular, the Banach spaces with a Schauder basis (e.g., ℓ_p, $1 \le p < \infty$, and c_0) have the $\mathcal{I}$-AP for every operator ideal $\mathcal{I}$.

Let us stress that different ideals may give rise to different approximation properties: (i) Willis [55] showed that there are spaces with the CAP but without the AP; (ii) Szankowski [54] proved that for $1 \le p < 2$, ℓ_p has a subspace S_p without the CAP, so $S_{3/2}$ has the WCAP but not the CAP and S_1 has the $\mathcal{CC} \cap \mathcal{C}_2$-AP but not the CAP, where $\mathcal{CC}$ and $\mathcal{C}_2$ are the ideals of completely continuous and cotype 2 operators, respectively. On the other hand, it is clear that E has the $\mathcal{I}$-AP if E has the $\overline{\mathcal{I}}$-AP ($\overline{\mathcal{I}}$ meaning the closure of $\mathcal{I}$). Thus, for example, since $\mathcal{F} \subseteq \mathcal{N}_p \subseteq \overline{\mathcal{F}} = \mathcal{A}$ [33, Proposition 19.7.3], where $\mathcal{N}_p$ and $\mathcal{A}$ are, respectively, the ideals of p-nuclear and approximable operators, we have $\mathcal{N}_p$-AP $=$ AP whereas $\mathcal{F} \neq \mathcal{N}_p \neq \overline{\mathcal{F}} = \mathcal{A}$.

The study of the approximation property and its variants—including the $\mathcal{I}$-AP—is rich and multifaceted, so to study the $\mathcal{I}$-AP, some choices have to be made. In this paper we study the $\mathcal{I}$-AP in projective tensor products (Section 3) and in spaces of mappings between Banach spaces, namely, spaces of linear functionals (Section 2), spaces of homogeneous polynomials (Section 4) and spaces of holomorphic functions and their preduals (Section 5). Proposition 4.6 fixes and generalizes a result of [18].

The results we prove in the different sections of the paper seem—at first glance—to be completely disconnected. However, several such connections are given in Section 5.

2. Preliminaries. When F is the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we shall write E' instead of $\mathcal{L}(E;\mathbb{K})$. The *compact-open topology* or the *topology of compact convergence* is the locally convex topology τ_c on $\mathcal{L}(E;F)$ which is generated by the seminorms

$$p_K(T) = \sup_{x\in K} \|T(x)\|,$$

where K ranges over all compact subsets of E.

For a given operator ideal $\mathcal{I}$, let $\overline{\mathcal{I}}$ denote the closure of $\mathcal{I}$, that is, $\overline{\mathcal{I}}(E;F) = \overline{\mathcal{I}(E;F)}$ for any Banach spaces E and F. For the theory of operator ideals we refer to [48, 19].

The results below are well known (see, e.g., [51] or [28]) or elementary. The proofs repeat verbatim their AP prototypes.

Proposition 2.1. *Let $\mathcal{I}$ be an operator ideal. The following statements are equivalent for a Banach space E:*

(a) *E has the $\mathcal{I}$-approximation property.*
(b) *For every Banach F, $\mathcal{L}(E;F) = \overline{\mathcal{I}(E;F)}^{\tau_c}$.*
(c) *For every Banach F, $\mathcal{L}(F;E) = \overline{\mathcal{I}(F;E)}^{\tau_c}$.*
(d) *$\sum_{n=1}^{\infty} x'_n(x_n) = 0$ whenever the sequences $(x_n) \subseteq E$ and $(x'_n) \subseteq E'$ are such that $\sum_{n=1}^{\infty} \|x'_n\| \, \|x_n\| < \infty$ and $\sum_{n=1}^{\infty} x'_n(T(x_n)) = 0$ for every $T \in \mathcal{I}(E;E)$.*

Just as the AP, also the $\mathcal{I}$-AP is inherited by complemented subspaces and is stable under the formation of finite cartesian products:

Proposition 2.2. *Let $\mathcal{I}$ be an operator ideal and E be a Banach space with the $\mathcal{I}$-approximation property. Then every complemented subspace of E has the $\mathcal{I}$-approximation property as well.*

Proposition 2.3. *Let $\mathcal{I}$ be an operator ideal, $k \in \mathbb{N}$ and $E_1, \dots, E_k$ be Banach spaces. Then the finite direct sum (or cartesian product) $E = \bigoplus_{n=1}^{k} E_n$ has the $\mathcal{I}$-approximation property if and only if $E_1, \dots, E_k$ have the $\mathcal{I}$-approximation property.*

Now we relate the $\mathcal{I}$-AP of E to that of its dual E'. This is a classical topic in approximation properties, and for the $\mathcal{I}$-AP it was studied, for instance, in [28, 37].

Given an operator ideal $\mathcal{I}$ and Banach spaces E and F, define

$$\mathcal{I}^{\mathrm{dual}}(E;F) = \{S \in \mathcal{L}(E;F) : S' \in \mathcal{I}(F';E')\}.$$

It is well known that $\mathcal{I}^{\mathrm{dual}}$ is an operator ideal.

Let E be a reflexive Banach space. From Proposition 2.1(a)$\Leftrightarrow$(d), it is immediate that the $\mathcal{I}$-AP of E is equivalent to the $\mathcal{I}^{\mathrm{dual}}$-AP of E'. This is used in the proof of the following result.

Theorem 2.4. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be operator ideals such that either $\mathcal{I}_2 \subseteq \mathcal{I}_1^{\mathrm{dual}}$ or $\mathcal{I}_2^{\mathrm{dual}} \subseteq \mathcal{I}_1$ and let E be a reflexive Banach space.*

(a) *If E' has the $\mathcal{I}_2$-AP then E has the $\mathcal{I}_1$-AP.*
(b) *If E has the $\mathcal{I}_2$-AP then E' has the $\mathcal{I}_1$-AP.*

Proof. Obviously (a)$\Leftrightarrow$(b) since E is reflexive. Assume that $\mathcal{I}_2 \subseteq \mathcal{I}_1^{\mathrm{dual}}$. Then (a) holds: if E' has the $\mathcal{I}_2$-AP then E' has the $\mathcal{I}_1^{\mathrm{dual}}$-AP, equivalently,

E has the $\mathcal{I}_1$-AP. Assume that $\mathcal{I}_2^{\rm dual} \subseteq \mathcal{I}_1$. Then (b) holds: if E has the $\mathcal{I}_2$-AP, equivalently, E' has the $\mathcal{I}_2^{\rm dual}$-AP, then E' has the $\mathcal{I}_1$-AP. ∎

Corollary 2.5. *Let $\mathcal{I}$ be an operator ideal such that either $\mathcal{I} \subseteq \mathcal{I}^{\rm dual}$ or $\mathcal{I}^{\rm dual} \subseteq \mathcal{I}$ and let E be a reflexive Banach space. Then E' has the $\mathcal{I}$-approximation property if and only if E has the $\mathcal{I}$-approximation property.*

Given $1 \leq p < \infty$, p^* stands for the conjugate of p, that is $1/p + 1/p^* = 1$. For the definition of the adjoint ideal $\mathcal{I}^*$ of the operator ideal $\mathcal{I}$, see, e.g., [24, p. 132].

Example 2.6. Let us see that there are plenty of ideals satisfying the conditions of Theorem 2.4 and Corollary 2.5.

(i) $\mathcal{N}_1^{\rm dual} \subseteq \mathcal{J}$ [19, Ex. 16.9], where $\mathcal{J}$ is the ideal of integral operators; $\mathcal{SS}^{\rm dual} \subseteq \mathcal{SC}$ and $\mathcal{SC}^{\rm dual} \subseteq \mathcal{SS}$ [23, 1.18], where $\mathcal{SS}$ and $\mathcal{SC}$ are, respectively the ideals of strictly singular and strictly cosingular operators; $\Gamma_p^{\rm dual} = \Gamma_{p^*}$ [24, p. 186], where Γ_p is the ideal of p-factorable operators; $\Pi_1^{\rm dual} = \Gamma_1^*$ [24, Corollary 9.5], where Π_p is the ideal of absolutely p-summing operators; $\mathcal{T}_p \subseteq \mathcal{C}_{p^*}^{\rm dual}$ and $\mathcal{C}_{p^*} \circ \mathcal{KC} \subseteq \mathcal{T}_p^{\rm dual}$ for $1 < p \leq 2$ [19, 31.2], where $\mathcal{T}_p, \mathcal{C}_p, \mathcal{KC}$ are, respectively, the ideals of type p operators, cotype p operators and K-convex operators (for the latter see [19, 31.1]); $\mathcal{N}_1^{\rm dual} \subseteq \mathcal{QN}$ [19, Ex. 9.13(b)], where $\mathcal{QN}$ is the ideal of quasinuclear operators (see [19, Ex. 9.13], [47]); $\Pi_{r,p,q}^{\rm dual} = \Pi_{r,q,p}$ [48, Theorem 17.1.5], where $\Pi_{r,p,q}$ is the ideal of absolutely (r,p,q)-summing operators; $\mathcal{L}_{p,q}^{\rm dual} = \mathcal{L}_{q,p}$ [12, p. 68], where $\mathcal{L}_{p,q}$ is the ideal of (p,q)-factorable operators; $\mathcal{K}_p = \mathcal{QN}_p^{\rm dual}$ [21], where $\mathcal{K}_p$ and $\mathcal{QN}_p^{\rm dual}$ are, respectively, the ideals of p-compact operators and quasi p-nuclear operators (for the latter see [21, 47]).

(ii) The following ideals are *completely symmetric* (that is, $\mathcal{I} = \mathcal{I}^{\rm dual}$): $\mathcal{F}; \mathcal{A}; \mathcal{K}; \mathcal{W}$ [48, Proposition 4.4.7]; $\mathcal{J}$ [19, Corollary 10.2.2]; the ideal $\mathcal{SN}$ of strongly nuclear operators [33, Theorem 19.9.3]; the ideal $\mathcal{U}_p$ of operators having approximation numbers belonging to ℓ_p, $0 < p < \infty$ [48, Theorem 14.2.5]; and $\mathcal{KC}$ [19, 31.1].

(iii) The following ideals satisfy $\mathcal{I} \subseteq \mathcal{I}^{\rm dual}$: $\mathcal{N}_1$ [19, 9.9] and the ideal $\mathcal{D}$ of dualisable operators [48, Proposition 4.4.10].

(iv) The following ideals satisfy $\mathcal{I}^{\rm dual} \subseteq \mathcal{I}$: the ideals $\mathcal{S}$ of separable operators [48, Proposition 4.4.8] and $\mathcal{DP} := \mathcal{W}^{-1} \circ \mathcal{CC}$ of Dunford–Pettis operators [23, 1.15].

Our next aim is to show that the implication E' has the $\mathcal{I}$-AP $\Rightarrow$ E has the $\mathcal{I}$-AP holds in some situations not covered by Corollary 2.5.

The *weak* topology* on $\mathcal{L}(E'; E') = (E' \hat{\otimes}_\pi E)'$ is the topology $\sigma(\mathcal{L}(E'; E'); E' \hat{\otimes}_\pi E)$. A net (T_α) in $\mathcal{L}(E'; E')$ converges to $T \in \mathcal{L}(E'; E')$ if and only if

$$\sum_{n=1}^{\infty}(T_\alpha(x'_n))x_n \to \sum_{n=1}^{\infty}(T(x'_n))x_n$$

for every $(x_n) \subseteq E$ and $(x'_n) \subseteq E'$ satisfying $\sum_{n=1}^{\infty} \|x'_n\| \, \|x_n\| < \infty$. In this case we write $T_\alpha \xrightarrow{\text{weak}^*} T$.

Given a Banach space E, we denote by w^* the weak* topology on E'. For a given operator ideal $\mathcal{I}$, we denote by $\mathcal{I}_{w^*}(E'; E')$ the set of all operators belonging to $\mathcal{I}(E'; E')$ which are w^*-to-w^* continuous. The dual space E' is said to have the *weak* density* for $\mathcal{I}$ ($\mathcal{I}$-W*D for short) if

$$\mathcal{I}(E'; E') \subseteq \overline{\mathcal{I}_{w^*}(E'; E')}^{\text{weak}^*}.$$

Every dual space with the AP has the $\mathcal{I}$-W*D for every operator ideal $\mathcal{I}$. In fact, it is well known (and an easy consequence of the principle of local reflexivity) that every dual space has the $\mathcal{F}$-W*D. Therefore if E' has the AP, then

$$\mathcal{L}(E'; E') \subseteq \overline{\mathcal{F}(E'; E')}^{\tau_c} \subseteq \overline{\mathcal{F}_{w^*}(E'; E')}^{\text{weak}^*}.$$

In particular, nonreflexive dual Banach spaces have the $\mathcal{I}$-W*D for every operator ideal $\mathcal{I}$. So, formally Corollary 2.5 does not apply to dual spaces having the $\mathcal{I}$-W*D. In this direction we have:

PROPOSITION 2.7. *Let E be a Banach space and let $\mathcal{I}$ be an operator ideal such that $\mathcal{I}^{\text{dual}} \subseteq \mathcal{I}$. If E' has the $\mathcal{I}$-AP and the $\mathcal{I}$-W*D, then E has the $\mathcal{I}$-AP.*

Proof. Let $(x_n) \subseteq E$ and $(x'_n) \subseteq E'$ be sequences with $\sum_{n=1}^{\infty} \|x'_n\| \, \|x_n\| < \infty$ and $\sum_{n=1}^{\infty} x'_n(T(x_n)) = 0$ for every $T \in \mathcal{I}(E; E)$. We know that $\mathrm{id}_{E'} \in \overline{\mathcal{I}(E'; E')}^{\tau_c}$ and $\mathcal{I}(E'; E') \subseteq \overline{\mathcal{I}_{w^*}(E'; E')}^{\text{weak}^*}$. Thus $\mathrm{id}_{E'} \in \overline{\mathcal{I}_{w^*}(E'; E')}^{\text{weak}^*}$ and so there is a net $(S_\alpha) \subseteq \mathcal{I}_{w^*}(E'; E')$ such that $S_\alpha \xrightarrow{\text{weak}^*} \mathrm{id}_{E'}$. For each α, since S_α is w^*-to-w^* continuous, there is $T_\alpha \in \mathcal{L}(E; E)$ such that $T'_\alpha = S_\alpha$. We know that $S_\alpha \in \mathcal{I}(E'; E')$, so the condition $\mathcal{I}^{\text{dual}} \subseteq \mathcal{I}$ implies that $T_\alpha \in \mathcal{I}(E; E)$ for every α. Since $T'_\alpha \xrightarrow{\text{weak}^*} (\mathrm{id}_E)'$ we get

$$\sum_{n=1}^{\infty} x'_n(T_\alpha(x_n)) \to \sum_{n=1}^{\infty} x'_n(\mathrm{id}_E(x_n)) = \sum_{n=1}^{\infty} x'_n(x_n).$$

But $\sum_{n=1}^{\infty} x'_n(T_\alpha(x_n)) = 0$ for every α because each $T_\alpha \in \mathcal{I}(E; E)$, therefore $\sum_{n=1}^{\infty} x'_n(x_n) = 0$. By Proposition 2.1 it follows that E has the $\mathcal{I}$-AP. ∎

3. Tensor stability. In this section we study the stability of the $\mathcal{I}$-AP under the formation of projective tensor products. By $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$ we mean the completed projective tensor product of the Banach spaces $E_1, \ldots, E_n$ ($\hat{\otimes}_\pi^n E$ if $E = E_1 = \cdots = E_n$), and by $\hat{\otimes}_\pi^{n,s} E$ the completed n-fold symmetric projective tensor product of the Banach space E.

Given $u_j \in \mathcal{L}(E_j; F_j)$, $j = 1, \dots, n$, we denote by $u_1 \otimes \cdots \otimes u_n$, as usual, the (unique) continuous linear operator from $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$ to $F_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi F_n$ such that

$$u_1 \otimes \cdots \otimes u_n(x_1 \otimes \cdots \otimes x_n) = u_1(x_1) \otimes \cdots \otimes u_n(x_n)$$

for every $x_1 \in E_1, \dots, x_n \in E_n$. The proof of the stability of the approximation property with respect to the formation of projective tensor products relies heavily on the fact that $u_1 \otimes \cdots \otimes u_n$ is a finite rank operator whenever $u_1, \dots, u_n$ are finite rank operators. Let us see that this does not hold for arbitrary operator ideals:

Example 3.1. The identity operator id_{ℓ_2} is weakly compact but $\mathrm{id}_{\ell_2} \otimes \mathrm{id}_{\ell_2} = \mathrm{id}_{\ell_2 \hat{\otimes}_\pi \ell_2}$ is not because $\ell_2 \hat{\otimes}_\pi \ell_2$ fails to be reflexive.

In order to settle this difficulty we need the following methods of generating ideals of multilinear mappings from operator ideals. By $\mathcal{L}(E_1, \dots, E_n; F)$ we denote the space of continuous n-linear mappings from $E_1 \times \cdots \times E_n$ to F endowed with the usual sup norm.

Definition 3.2. Let $\mathcal{I}, \mathcal{I}_1, \dots, \mathcal{I}_n$ be operator ideals.

(a) (Factorization method) A mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be *of type* $\mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n]$ if there are Banach spaces $G_1, \dots, G_n$, operators $u_j \in \mathcal{I}_j(E_j; G_j)$, $j = 1, \dots, n$, and a mapping $B \in \mathcal{L}(G_1, \dots, G_n; F)$ such that $A = B \circ (u_1, \dots, u_n)$. In this case we write $A \in \mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n](E_1, \dots, E_n; F)$. If $\mathcal{I} = \mathcal{I}_1 = \cdots = \mathcal{I}_n$ we simply write $\mathcal{L}[\mathcal{I}]$.

(b) (Composition ideals) A mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$ belongs to $\mathcal{I} \circ \mathcal{L}$ if there are a Banach space G, a mapping $B \in \mathcal{L}(E_1, \dots, E_n; G)$ and an operator $u \in \mathcal{I}(G; F)$ such that $A = u \circ B$. In this case we write $A \in \mathcal{I} \circ \mathcal{L}(E_1, \dots, E_n; F)$.

For details and examples we refer to [6, 7].

Proposition 3.3. *Given operator ideals $\mathcal{I}, \mathcal{I}_1, \dots, \mathcal{I}_n$, the following are equivalent:*

(a) $\mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n] \subseteq \mathcal{I} \circ \mathcal{L}$.

(b) *If $u_j \in \mathcal{I}_j(E_j; F_j)$, $j = 1, \dots, n$, then*

$$u_1 \otimes \cdots \otimes u_n \in \mathcal{I}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; F_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi F_n).$$

Proof. Assume (a) and let $u_j \in \mathcal{I}_j(E_j; F_j)$, $j = 1, \dots, n$, be given. Consider the canonical n-linear mapping $\sigma_n \colon E_1 \otimes \cdots \times E_n \to E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$ given by $\sigma_n(x_1, \dots, x_n) = x_1 \otimes \cdots \otimes x_n$ and observe that

$$\sigma_n \circ (u_1, \dots, u_n) \in \mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n](E_1, \dots, E_n; E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n).$$

By assumption we have

$$\sigma_n \circ (u_1, \ldots, u_n) \in \mathcal{I} \circ \mathcal{L}(E_1, \ldots, E_n; E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n).$$

Denote by T the linearization of $\sigma_n \circ (u_1, \ldots, u_n)$. Then by [7, Proposition 3.2(a)] we have $T \in \mathcal{I}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n)$. For every $x_1 \in E_1$, $\ldots, x_n \in E_n$,

$$\begin{aligned} T(x_1 \otimes \cdots \otimes x_n) &= \sigma_n \circ (u_1, \ldots, u_n)(x_1, \ldots, x_n) = \sigma_n(u_1(x_1), \ldots, u_n(x_n)) \\ &= u_1(x_1) \otimes \cdots \otimes u_n(x_n) = u_1 \otimes \cdots \otimes u_n(x_1 \otimes \cdots \otimes x_n). \end{aligned}$$

As both T and $u_1 \otimes \cdots \otimes u_n$ are linear it follows that $T = u_1 \otimes \cdots \otimes u_n$, hence $u_1 \otimes \cdots \otimes u_n \in \mathcal{I}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n)$.

Now assume (b) and let $A \in \mathcal{L}[\mathcal{I}_1, \ldots, \mathcal{I}_n](E_1, \ldots, E_n; F)$ be given. There are Banach spaces $G_1, \ldots, G_n$, operators $u_j \in \mathcal{I}_j(E_j; G_j)$, $j = 1, \ldots, n$, and $B \in \mathcal{L}(G_1, \ldots, G_n; F)$ such that $A = B \circ (u_1, \ldots, u_n)$. By assumption

$$u_1 \otimes \cdots \otimes u_n \in \mathcal{I}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; G_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi G_n),$$

so, denoting by B_L the linearization of B, the equality

$$A = B \circ (u_1, \ldots, u_n) = B_L \circ (u_1 \otimes \cdots \otimes u_n) \circ \sigma_n$$

shows that $A \in \mathcal{I} \circ \mathcal{L}(E_1, \ldots, E_n; F)$. ∎

The next result is an extension of [30, Theorem 3]:

PROPOSITION 3.4. *Let $\mathcal{I}, \mathcal{I}_1, \ldots, \mathcal{I}_n$ be operator ideals with $\mathcal{L}[\mathcal{I}_1, \ldots, \mathcal{I}_n] \subseteq \mathcal{I} \circ \mathcal{L}$. If E_j has the $\mathcal{I}_j$-AP, for $j = 1, \ldots, n$, then $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$ has the $\mathcal{I}$-AP.*

Proof. Let K be a compact subset of $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$. By [19, Corollary 3.5.1] there are compact sets $K_1 \subseteq E_1, \ldots, K_n \subseteq E_n$ such that K is contained in the closure of the absolutely convex hull of $K_1 \otimes \cdots \otimes K_n := \{x_1 \otimes \cdots \otimes x_n : x_1 \in K_1, \ldots, x_n \in K_n\}$. Since compact sets are bounded, there is $M > 0$ such that $\|x_j\| \le M$ for every $x_j \in E_j$, $j = 1, \ldots, n$. Let $\varepsilon > 0$. As E_1 has the $\mathcal{I}_1$-AP, there is an operator $u_1 \in \mathcal{I}_1(E_1; E_1)$ such that $\|u_1(x_1) - x_1\| < \varepsilon/(2nM^{n-1})$ for every $x_1 \in K_1$. As E_2 has the $\mathcal{I}_2$-AP, there is $u_2 \in \mathcal{I}_2(E_2; E_2)$ such that $\|u_2(x_2) - x_2\| < \varepsilon/(2nM^{n-1}\|u_1\|)$ for every $x_2 \in K_2$. Repeating the procedure we obtain $u_j \in \mathcal{I}_j(E_j; E_j)$ such that

$$\|u_j(x_j) - x_j\| < \frac{\varepsilon}{2nM^{n-1}\|u_1\| \cdots \|u_{j-1}\|}$$

for every $x_j \in K_j$, $j = 1, \ldots, n$. By Proposition 3.3, $u_1 \otimes \cdots \otimes u_n \in \mathcal{I}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n)$. We shall denote the projective norm of a tensor $z \in E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$ by $\|z\|$ instead of $\pi(z)$. Given $x_1 \in K_1$, $\ldots, x_n \in K_n$,

$$
\begin{aligned}
&\|u_1 \otimes \cdots \otimes u_n(x_1 \otimes \cdots \otimes x_n) - x_1 \otimes \cdots \otimes x_n\| \\
&\quad = \|u_1(x_1) \otimes \cdots \otimes u_n(x_n) - x_1 \otimes \cdots \otimes x_n\| \\
&\quad = \bigg\| u_1(x_1) \otimes \cdots \otimes u_n(x_n) - \sum_{j=1}^{n-1} u_1(x_1) \otimes \cdots \otimes u_j(x_j) \otimes x_{j+1} \otimes \cdots \otimes x_n \\
&\qquad + \sum_{j=1}^{n-1} u_1(x_1) \otimes \cdots \otimes u_j(x_j) \otimes x_{j+1} \otimes \cdots \otimes x_n - x_1 \otimes \cdots \otimes x_n \bigg\| \\
&\quad = \bigg\| \sum_{j=1}^{n} u_1(x_1) \otimes \cdots \otimes u_{j-1}(x_{j-1}) \otimes (u_j(x_j) - x_j) \otimes x_{j+1} \otimes \cdots \otimes x_n \bigg\| \\
&\quad \le \sum_{j=1}^{n} \|u_1(x_1) \otimes \cdots \otimes u_{j-1}(x_{j-1}) \otimes (u_j(x_j) - x_j) \otimes x_{j+1} \otimes \cdots \otimes x_n\| \\
&\quad \le \sum_{j=1}^{n} \|u_1\|\,\|x_1\| \cdots \|u_{j-1}\|\,\|x_{j-1}\|\,\|u_j(x_j) - x_j\|\,\|x_{j+1}\| \cdots \|x_n\| \\
&\quad < \sum_{j=1}^{n} \|u_1\| \cdots \|u_{j-1}\| M^{n-1} \frac{\varepsilon}{2nM^{n-1}\|u_1\| \cdots \|u_{j-1}\|} = \frac{\varepsilon}{2}.
\end{aligned}
$$

In summary,

$$\|u_1 \otimes \cdots \otimes u_n(x_1 \otimes \cdots \otimes x_n) - x_1 \otimes \cdots \otimes x_n\| < \varepsilon/2 \tag{3.1}$$

for every $x_1 \in K_1, \ldots, x_n \in K_n$. Take z in the absolutely convex hull of $K_1 \otimes \cdots \otimes K_n$. Then $z = \sum_{j=1}^{k} \lambda_j x_j^1 \otimes \cdots \otimes x_j^n$, where $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k$ are scalars with $|\lambda_1| + \cdots + |\lambda_k| \le 1$, and $x_j^m \in K_m$ for $j = 1, \ldots, k, m = 1, \ldots, n$. Using (3.1), a routine computation shows that $\|u_1 \otimes \cdots \otimes u_n(z) - z\| < \varepsilon/2$. By continuity we have

$$\|u_1 \otimes \cdots \otimes u_n(z) - z\| \le \varepsilon/2 < \varepsilon$$

for every z in the closure of the absolutely convex hull of $K_1 \otimes \cdots \otimes K_n$, hence for every $z \in K$. ∎

As to ideals satisfying the conditions above we have:

EXAMPLE 3.5.

(a) It is plain that $\mathcal{L}[\mathcal{F}] \subseteq \mathcal{F} \circ \mathcal{L}$ and $\mathcal{L}[\mathcal{S}] \subseteq \mathcal{S} \circ \mathcal{L}$.
(b) $\mathcal{L}[\mathcal{N}_1] \subseteq \mathcal{N}_1 \circ \mathcal{L}$ [31, Theorem 3.7] (see also [33, Proposition 17.3.9]).
(c) $\mathcal{L}[\mathcal{J}] \subseteq \mathcal{J} \circ \mathcal{L}$ [32, Theorem 2].
(d) Let $\mathcal{L}_{\mathcal{K}}$ denote the ideal of compact multilinear mappings (bounded sets are sent to relatively compact sets). Pełczyński [46] proved that $\mathcal{K} \circ \mathcal{L} = \mathcal{L}_{\mathcal{K}}$. Now it follows easily that $\mathcal{L}[\mathcal{K}] \subseteq \mathcal{K} \circ \mathcal{L}$. So the projective tensor product of spaces with the CAP has the CAP too.

(e) $\mathcal{L}[\mathcal{L}_{\infty,q,\gamma}] \subseteq \mathcal{L}_{\infty,q,\gamma} \circ \mathcal{L}$ for $0 < q \le 1$ and $-1/q < \gamma < \infty$ [15, Theorem 3.1], where $\mathcal{L}_{\infty,q,\gamma}$ is the ideal of Lorentz–Zygmund operators.
(f) $\mathcal{L}[\mathcal{L}_{1,q}] \subseteq \mathcal{L}_{1,q} \circ \mathcal{L}$ for $q > 1$ and $\mathcal{L}[\mathcal{K}_{1,p}] \subseteq \mathcal{K}_{1,p} \circ \mathcal{L}$ for $p \ge 1$ [12, Theorem 2.1], where $\mathcal{K}_{1,p}$ is the ideal of $(1,p)$-compact operators.
(g) It is unknown if the projective tensor product of Schur spaces is a Schur space (see, e.g., [8]), so it is unknown if $\mathcal{L}[\mathcal{CC}] \subseteq \mathcal{CC} \circ \mathcal{L}$.

Here are other concrete situations to which Proposition 3.4 applies:

Example 3.6. Let $n \in \mathbb{N}$.

(a) If $1 \le p_1, \ldots, p_n < \infty$, then $\mathcal{L}[\mathcal{W}, \mathcal{I}_1, \ldots, \mathcal{I}_n] \subseteq \mathcal{W} \circ \mathcal{L}$ where $\mathcal{I}_j$ is either $\mathcal{K}$ or Π_{p_j}, $j = 1, \ldots, n$ (Racher [50]).
(b) $\mathcal{L}[\Pi_1, \mathcal{J}, \overset{(n)}{\ldots}, \mathcal{J}] \subseteq \Pi_1 \circ \mathcal{L}$ (Holub [32]).
(c) $\mathcal{L}[\mathcal{QN}, \mathcal{N}_1, \overset{(n)}{\ldots}, \mathcal{N}_1] \subseteq \mathcal{QN} \circ \mathcal{L}$ (Holub [32]).
(d) If $p_1 > p_j$ for $j = 2, \ldots, n$, then $\mathcal{L}[\mathcal{U}_{p_1}, \mathcal{U}_{p_2}, \ldots, \mathcal{U}_{p_n}] \subseteq \mathcal{U}_{p_1} \circ \mathcal{L}$ (König [34, p. 79], Pietsch [49]).

Combining Proposition 3.4 and Example 3.6 we get:

Proposition 3.7.

(a) *Let $E_1, \ldots, E_n$ be Banach spaces, one of which has the WCAP and the others E_j have either the CAP or the Π_{p_j}-AP for some $1 \le p_j < \infty$. Then $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$ has the WCAP.*
(b) *Let $E_1, \ldots, E_n$ be Banach spaces, one of which has the Π_1-AP and the others have the $\mathcal{J}$-AP. Then $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$ has the Π_1-AP.*
(c) *Let $E_1, \ldots, E_n$ be Banach spaces, one of which has the $\mathcal{QN}$-AP and the others have the AP. Then $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$ has the AP.*
(d) *Let $p_1, \ldots, p_n > 0$. If $E_1, \ldots, E_n$ are Banach spaces, each E_j with the $\mathcal{U}_{p_j}$-AP, then $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$ has the $\mathcal{U}_{p_k}$-AP if $p_k > p_j$ for every $j \ne k$.*

Corollary 3.8. *Let $\mathcal{I}$ be an operator ideal such that $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$. The following are equivalent for a Banach space E:*

(a) *E has the $\mathcal{I}$-AP.*
(b) *$\hat{\otimes}_\pi^n E$ has the $\mathcal{I}$-AP for every $n \in \mathbb{N}$.*
(c) *$\hat{\otimes}_\pi^n E$ has the $\mathcal{I}$-AP for some $n \in \mathbb{N}$.*
(d) *$\hat{\otimes}_\pi^{n,s} E$ has the $\mathcal{I}$-AP for every $n \in \mathbb{N}$.*
(e) *$\hat{\otimes}_\pi^{n,s} E$ has the $\mathcal{I}$-AP for some $n \in \mathbb{N}$.*

Proof. (a)⇒(b) follows from Proposition 3.4; (b)⇒(c) is obvious; (c)⇒(a) follows from Proposition 2.2 because E is obviously a complemented subspace of $\hat{\otimes}_\pi^n E$; (b)⇒(d) follows from Proposition 2.2 because $\hat{\otimes}_\pi^{n,s} E$ is a complemented subspace of $\hat{\otimes}_\pi^n E$ via the symmetrization operator; (d)⇒(e) is obvious; (e)$\Rightarrow$(a) follows from Proposition 2.2 because E is a complemented subspace of $\hat{\otimes}_\pi^{n,s} E$ (see [5, Corollary 4]). ∎

4. Polynomial ideals and the $\mathcal{I}$-AP. The symbol $\mathcal{P}(^nE;F)$ stands for the space of continuous n-homogeneous polynomials from E to F. A *polynomial ideal* is a subclass $\mathcal{Q}$ of the class of all continuous homogeneous polynomials between Banach spaces such that, for every $n \in \mathbb{N}$ and all Banach spaces E and F, the component $\mathcal{Q}(^nE;F) := \mathcal{P}(^nE;F) \cap \mathcal{Q}$ satisfies:

(a) $\mathcal{Q}(^nE;F)$ is a linear subspace of $\mathcal{P}(^nE;F)$ which contains the n-homogeneous polynomials of finite type.
(b) If $T \in \mathcal{L}(G;E)$, $P \in \mathcal{Q}(^nE;F)$ and $S \in \mathcal{L}(F;H)$, then $S \circ P \circ T \in \mathcal{Q}(^nG;H)$.

There are different ways to construct a polynomial ideal from a given operator ideal $\mathcal{I}$. Let us see three of such methods (see [6, 7]):

Definition 4.1. Let $\mathcal{I}$ be an operator ideal.

(a) (Factorization method) A polynomial $P \in \mathcal{P}(^nE;F)$ is said to be *of type* $\mathcal{P}_{\mathcal{L}[\mathcal{I}]}$ if there are a Banach space G, an operator $u \in \mathcal{I}(E;G)$ and a polynomial $Q \in \mathcal{P}(^nG;F)$ such that $P = Q \circ u$. In this case we write $P \in \mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^nE;F)$.
(b) (Composition ideals) A polynomial $P \in \mathcal{P}(^nE;F)$ belongs to $\mathcal{I} \circ \mathcal{P}$ if there are a Banach space G, a polynomial $Q \in \mathcal{P}(^nG;F)$ and an operator $u \in \mathcal{I}(E;G)$ such that $P = u \circ Q$. In this case we write $P \in \mathcal{I} \circ \mathcal{P}(^nE;F)$.
(c) (Linearization method) A polynomial $P \in \mathcal{P}(^nE;F)$ is said to be *of type* $\mathcal{P}_{[\mathcal{I}]}$ if the linear operator
$$\bar{P} : E \to \mathcal{P}(^{n-1}E;F), \qquad \bar{P}(x)(y) = \check{P}(x, y, \ldots, y),$$
belongs to $\mathcal{I}$. In this case we write $P \in \mathcal{P}_{[\mathcal{I}]}(^nE;F)$.

It is well known that $\mathcal{P}_{\mathcal{L}[\mathcal{I}]}$, $\mathcal{I} \circ \mathcal{P}$ and $\mathcal{P}_{[\mathcal{I}]}$ are polynomial ideals.

Given a polynomial $P \in \mathcal{P}(^nE;F)$, we denote by $\check{P}$ the (unique) continuous symmetric n-linear mapping from E^n to F such that $P(x) = \check{P}(x, \ldots, x)$ for every $x \in E$.

Theorem 4.2. *Let $\mathcal{I}$ be an operator ideal. The following are equivalent for a Banach space E:*

(a) *E has the $\mathcal{I}$-approximation property.*
(b) $\mathcal{P}(^nE;F) = \overline{\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^nE;F)}^{\tau_c}$ *for every $n \in \mathbb{N}$ and every Banach space F.*
(c) $\mathcal{P}(^nE;F) = \overline{\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^nE;F)}^{\tau_c}$ *for some $n \in \mathbb{N}$ and every Banach space F.*
(d) $\mathcal{P}(^nF;E) = \overline{\mathcal{I} \circ \mathcal{P}(^nF;E)}^{\tau_c}$ *for every $n \in \mathbb{N}$ and every Banach space F.*
(e) $\mathcal{P}(^nF;E) = \overline{\mathcal{I} \circ \mathcal{P}(^nF;E)}^{\tau_c}$ *for some $n \in \mathbb{N}$ and every Banach space F.*

Furthermore, if $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$, then the conditions above are also equivalent to:

(f) $\mathcal{P}(^nE; F) = \overline{\mathcal{I} \circ \mathcal{P}(^nE; F)}^{\tau_c}$ *for every* $n \in \mathbb{N}$ *and every Banach space* F.
(g) $\mathcal{P}(^nE; F) = \overline{\mathcal{I} \circ \mathcal{P}(^nE; F)}^{\tau_c}$ *for some* $n \in \mathbb{N}$ *and every Banach space* F.

Proof. (a)⇒(b). Let $P \in \mathcal{P}(^nE; F)$, K be a compact subset of E and $\varepsilon > 0$. Since P is uniformly continuous on K, there is $\delta > 0$ such that $\|P(y) - P(x)\| < \varepsilon$ whenever $\|y - x\| < \delta$, $x \in K$ and $y \in E$. By the $\mathcal{I}$-AP of E there is an operator $T \in \mathcal{I}(E; E)$ such that $\|T(x) - x\| < \delta$ for every $x \in K$. It follows that $\|P(T(x)) - P(x)\| < \varepsilon$ for every $x \in K$. But $P \circ T \in \mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^nE; F)$, so $P \in \overline{\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^nE; F)}^{\tau_c}$.

(c)⇒(a). Let $u \in \mathcal{L}(E; F)$, K be a compact subset of E and $\varepsilon > 0$. Let $\varphi \in E'$, $\varphi \neq 0$, and $a \in K$ be such that $\varphi(a) = 1$. Define $P \in \mathcal{P}(^nE; F)$ by $P(x) = \varphi(x)^{n-1}u(x)$. Since $K_1 := \bigcup_{\varepsilon_i = \pm 1}(\varepsilon_1 K + \cdots + \varepsilon_n K)$ is a compact subset of E, by assumption there is a polynomial $Q \in \mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^nE; F)$ such that $\|Q(x) - P(x)\| < n!\varepsilon/n$ for every $x \in K_1$. By the polarization formula, for every $(x_1, \dots, x_n) \in K \times \cdots \times K$ we have

$$\|\check{Q}(x_1, \dots, x_n) - \check{P}(x_1, \dots, x_n)\|$$
$$\left\| \frac{1}{n!2^n} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_n \Big[Q\Big(\sum_{i=1}^n \varepsilon_i x_i \Big) - P\Big(\sum_{i=1}^n \varepsilon_i x_i \Big) \Big] \right\| < \frac{\varepsilon}{n}.$$

From
$$\check{P}(x, a, \dots, a) = \frac{1}{n} u(x) + \frac{n-1}{n} \varphi(x) u(a),$$
it follows that
$$\|n\check{Q}(x, a, \dots, a) - u(x) - (n-1)\varphi(x)u(a)\|$$
$$= n \left\| \check{Q}(x, a, \dots, a) - \left(\frac{1}{n} u(x) + \frac{n-1}{n} \varphi(x) u(a) \right) \right\| < \varepsilon$$
for every $x \in K$. Considering $S = n\check{Q}(\cdot, a, \dots, a) - (n-1)\varphi(\cdot)u(a) \in \mathcal{L}(E; F)$, we have $\|S(x) - u(x)\| < \varepsilon$ for every $x \in K$. Let us check that $S \in \mathcal{I}(E; F)$. Indeed, as $Q \in \mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^nE; F)$, there are a Banach space G, an operator $v \in \mathcal{I}(E; G)$ and a polynomial $R \in \mathcal{P}(^nG; F)$ such that $Q = R \circ v$. Define $T \colon G \to F$ by $T(y) = \check{R}(y, v(a), \dots, v(a))$. Then $T \circ v \in \mathcal{I}(E; F)$ and
$$T \circ v(x) = T(v(x)) = \check{R}(v(x), v(a), \dots, v(a)) = \check{Q}(x, a, \dots, a)$$
for every $x \in E$, proving that $\check{Q}(\cdot, a, \dots, a) \in \mathcal{I}(E; F)$. On the other hand, $\varphi(\cdot)u(a) \in \mathcal{I}(E; F)$, being a finite rank operator. Thus $S \in \mathcal{I}(E; F)$ and $\mathcal{L}(E; F) = \overline{\mathcal{I}(E; F)}^{\tau_c}$. Proposition 2.1 shows that E has the $\mathcal{I}$-AP.

(a)⇒(d). Let $P \in \mathcal{P}(^nF; E)$, K be a compact subset of E and $\varepsilon > 0$. Since $P(K)$ is a compact subset of E and E has the $\mathcal{I}$-approximation property, there is an operator $T \in \mathcal{I}(E; E)$ such that $\|T(z) - z\| < \varepsilon$ for every $z \in P(K)$. Hence $\|T(P(x)) - P(x)\| < \varepsilon$ for every $x \in K$. Since $T \circ P \in \mathcal{I} \circ \mathcal{P}(^nF; E)$ we have $P \in \overline{\mathcal{I} \circ \mathcal{P}(^nF; E)}^{\tau_c}$.

(e)⇒(a). The same argument of (c)⇒(a), *mutatis mutandis*, works in this case. We just sketch the proof: given $u \in \mathcal{L}(F; E)$, a compact set $K \subseteq F$ and $\varepsilon > 0$, take $\varphi \in F'$, $\varphi \neq 0$, and $a \in K$ such that $\varphi(a) = 1$. Defining $P = \varphi(\cdot)^{n-1}u(\cdot) \in \mathcal{P}(^nF; E)$ and a compact subset K_1 of F as before, by assumption there is $Q \in \mathcal{I} \circ \mathcal{P}(^nF; E)$ such that $\|Q(x) - P(x)\| < n!\varepsilon/n$ for every $x \in K_1$. Define $S = n\check{Q}(\cdot, a, \ldots, a) - (n-1)\varphi(\cdot)u(a) \in \mathcal{L}(F; E)$ and proceed exactly as above to get $\|S(x) - u(x)\| < \varepsilon$ for every $x \in K$. Write $Q = v \circ R$ with $v \in \mathcal{I}(G; E)$ and $R \in \mathcal{P}(^nF; G)$ and define $T \in \mathcal{L}(F; G)$ by $T(y) = \check{R}(y, a, \ldots, a)$. Thus $v \circ T = \check{Q}(\cdot, a, \ldots, a) \in \mathcal{I}(F; E)$ and this implies that $S \in \mathcal{I}(F; E)$.

Since (b)⇒(c) and (d)⇒(e) are obvious, the first part of the proof is complete.

Assume now that $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$.

(a)⇒(f). By assumption, E has the $\mathcal{I}$-AP. Let $n \in \mathbb{N}$, $P \in \mathcal{P}(^nE; F)$, K be a compact subset of E and $\varepsilon > 0$. Note that $P = P_L \circ \sigma_n$ where $\sigma_n \in \mathcal{P}(^nE; \hat{\otimes}_\pi^{n,s} E)$ is the canonical n-homogeneous polynomial defined by $\sigma_n(x) = x \otimes \cdots \otimes x$ and $P_L \in \mathcal{L}(\hat{\otimes}_\pi^{n,s} E; F)$ is the linearization of P, that is, $P_L(x \otimes \cdots \otimes x) = P(x)$. By Corollary 3.8, $\hat{\otimes}_\pi^{n,s} E$ has the $\mathcal{I}$-AP, hence $\mathcal{L}(\hat{\otimes}_\pi^{n,s} E; F) = \overline{\mathcal{I}(\hat{\otimes}_\pi^{n,s} E; F)}^{\tau_c}$ by Proposition 2.1. So for the compact subset $\sigma(K)$ of $\hat{\otimes}_\pi^{n,s} E$ there is an operator $u \in \mathcal{I}(\hat{\otimes}_\pi^{n,s} E; F)$ such that

$$\|u \circ \sigma_n(x) - P(x)\| = \|u(\sigma_n(x)) - P_L(\sigma_n(x))\| < \varepsilon$$

for every $x \in K$. Since $Q = u \circ \sigma_n \in \mathcal{I} \circ \mathcal{P}(^nE; F)$, it follows that $P \in \overline{\mathcal{I} \circ \mathcal{P}(^nE; F)}^{\tau_c}$.

The implication (f)⇒(g) is obvious and (g)⇒(a) follows from a repetition of the arguments for (c)⇒(a) and (e)⇒(a), therefore the proof is complete. ∎

The spaces $\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^nE; E)$ and $\mathcal{I} \circ \mathcal{P}(^nE; E)$ are often different. We have obtained situations where, however, their τ_c-closures coincide:

COROLLARY 4.3. *Let $\mathcal{I}$ be an operator ideal.*

(a) *If Banach spaces E and F have the $\mathcal{I}$-approximation property, then $\overline{\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^nE; F)}^{\tau_c} = \mathcal{P}(^nE; F) = \overline{\mathcal{I} \circ \mathcal{P}(^nF; E)}^{\tau_c}$ for every $n \in \mathbb{N}$.*
(b) *A Banach space E has the $\mathcal{I}$-approximation property if and only if $\overline{\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^nE; E)}^{\tau_c} = \mathcal{P}(^nE; E) = \overline{\mathcal{I} \circ \mathcal{P}(^nE; E)}^{\tau_c}$ for every $n \in \mathbb{N}$.*

EXAMPLE 4.4. It is not difficult to check that neither $\mathcal{P}_{\mathcal{L}[\mathcal{W}]}(^2\ell_1; \ell_1) \subseteq \mathcal{W} \circ \mathcal{P}(^2\ell_1; \ell_1)$ nor $\mathcal{W} \circ \mathcal{P}(^2\ell_1; \ell_1) \subseteq \mathcal{P}_{\mathcal{L}[\mathcal{W}]}(^2\ell_1; \ell_1)$ (see [6, Examples 27 and 28]). Nevertheless, by Corollary 4.3(b) both subspaces are τ_c-dense in $\mathcal{P}(^2\ell_1; \ell_1)$ because ℓ_1 has the approximation property (hence the weakly compact approximation property).

The following result appears in Çaliskan [18]:

THEOREM 4.5 ([18, Theorem 11]). *The following are equivalent for a Banach space E:*

(a) *E has the weakly compact approximation property.*
(b) $\mathcal{P}(^nE;F)=\overline{\mathcal{P}_{[\mathcal{W}]}(^nE;F)}^{\tau_c}$ *for every* $n\in\mathbb{N}$ *and every Banach space* F.
(c) $\mathcal{P}(^nE;F)=\overline{\mathcal{P}_{[\mathcal{W}]}(^nE;F)}^{\tau_c}$ *for some* $n\in\mathbb{N}$ *and every Banach space* F.

Unfortunately there is a gap in the proof of this theorem (see the MathSciNet review by Boyd [9] and the Erratum of [18]). In this direction we have:

PROPOSITION 4.6. *Let $\mathcal{I}$ be a closed injective operator ideal. The following are equivalent for a Banach space E:*

(a) *E has the $\mathcal{I}$-approximation property.*
(b) $\mathcal{P}(^nE;F)=\overline{\mathcal{P}_{[\mathcal{I}]}(^nE;F)}^{\tau_c}$ *for every* $n\in\mathbb{N}$ *and every Banach space* F.
(c) $\mathcal{P}(^nE;F)=\overline{\mathcal{P}_{[\mathcal{I}]}(^nE;F)}^{\tau_c}$ *for some* $n\in\mathbb{N}$ *and every Banach space* F.

Proof. Just combine Theorem 4.2 with the fact that $\mathcal{P}_{[\mathcal{I}]}=\mathcal{P}_{\mathcal{L}[\mathcal{I}]}$ whenever the operator ideal $\mathcal{I}$ is closed and injective (see [11]). ∎

Recalling that $\mathcal{W}$ is closed and injective, Proposition 4.6 fixes Theorem 4.5 and generalizes it to arbitrary closed injective operator ideals.

5. Spaces of holomorphic functions. The approximation property and its variants in spaces of holomorphic functions and their preduals have been extensively investigated (see, e.g., [3, 10, 16, 17, 26, 27, 42]). In this section we study the $\mathcal{I}$-approximation property in spaces of holomorphic functions of bounded type, spaces of weakly uniformly continuous holomorphic functions, spaces of bounded holomorphic functions and/or their preduals. For background on infinite-dimensional holomorphy we refer to [25, 40]. An important issue of this section is the combination of results from different sections of the paper.

All spaces in this section are supposed to be complex.

Spaces of holomorphic functions, spaces of bounded holomorphic functions and spaces of weakly uniformly continuous holomorphic functions, as well as their respective preduals, are locally convex spaces, so we have to say a few words about the definition of the $\mathcal{I}$-approximation property in the setting of locally convex spaces. The definition of operator ideals (on Banach spaces) can be naturally generalized to locally convex spaces (details can be found in [48, Chapter 29]). We say that an operator ideal $\mathcal{U}$ on locally convex spaces is an *extension* of an operator ideal $\mathcal{I}$ on Banach spaces if $\mathcal{U}(E;F)=\mathcal{I}(E;F)$ for all Banach spaces E and F. There are several ways to extend an operator ideal on Banach spaces to one on locally convex spaces (see [48, Section 29.5]). In this paper we shall work with the smallest of such natural extensions, which we describe next. Given an operator ideal $\mathcal{I}$ on Banach spaces, an operator $S \in \mathcal{L}(U;V)$ between locally convex spaces belongs to the *inferior extension of* $\mathcal{I}$ if there exist Banach spaces E and F and operators $A \in \mathcal{L}(U,E)$, $T \in \mathcal{I}(E,F)$ and $Y \in \mathcal{L}(F,V)$ such that $S = Y \circ T \circ A$. In this case, for simplicity, we still write $S \in \mathcal{I}(U;V)$. Of course we can consider the compact-open topology on $\mathcal{L}(U;U)$ for a locally convex space U, so Definition 1.1 makes sense for an operator ideal $\mathcal{I}$ on Banach spaces and a locally convex space U, hence the $\mathcal{I}$-approximation property is well defined for locally convex spaces.

Unless explicitly stated otherwise, an operator ideal means an operator ideal on Banach spaces and a statement like $\mathcal{I}_1 \subseteq \mathcal{I}_2$ means that $\mathcal{I}_1(E;F) \subseteq \mathcal{I}_2(E;F)$ for all Banach spaces E and F.

REMARK 5.1. It is easy to see that the basic facts, including Propositions 2.2 and 2.3, hold true in the realm of locally convex spaces. Of course, whenever necessary, $\|T(x) - x\|$ should be replaced by $p(T(x) - x)$ where p is an arbitrary continuous seminorm.

DEFINITION 5.2. A sequence $\{E_n\}_{n=1}^{\infty}$ of subspaces of a locally convex space E is said to be a *decomposition* of E if any $x \in E$ can be written in a unique way as $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in E_n$ for every n and the projection $\sum_{n=1}^{\infty} x_n \mapsto \sum_{n=1}^{m} x_n$ is continuous for every $m \in \mathbb{N}$.

Let $\mathcal{S} = \{(\alpha_n)_{n=1}^{\infty} : \alpha_n \in \mathbb{C} \text{ and } \limsup_{n\to\infty} |\alpha_n|^{1/n} \le 1\}$. A decomposition $\{E_n\}_{n=1}^{\infty}$ of E is $\mathcal{S}$*-absolute* if:

(1) $\sum_{n=1}^{\infty} x_n \in E$, $x_n \in E_n$ for all n and $(\alpha_n)_{n=1}^{\infty} \in \mathcal{S}$ implies $\sum_{n=1}^{\infty} \alpha_n x_n \in E$.
(2) If p is a continuous seminorm on E and $(\alpha_n)_{n=1}^{\infty} \in \mathcal{S}$ then

$$p_\alpha\Big(\sum_{n=1}^{\infty} x_n\Big) := \sum_{n=1}^{\infty} |\alpha_n| p(x_n)$$

defines a continuous seminorm on E.

Further details can be found in [25, Section 3.3].

An obvious modification in the proof of [10, Proposition 1] provides the following lemma.

LEMMA 5.3. *Let $\mathcal{I}$ be an operator ideal. If $\{E_n\}_{n=1}^{\infty}$ is an $\mathcal{S}$-absolute decomposition of the locally convex space E, then E has the $\mathcal{I}$-approximation property if and only if each E_n has the $\mathcal{I}$-approximation property.*

Let E be a Banach space. We denote by $\mathcal{P}_w(^nE)$ the closed subspace of $\mathcal{P}(^nE)$ of all continuous n-homogeneous polynomials that are weakly continuous on bounded sets. Let U be an open subset of a Banach space E.

A bounded subset A of U is *U-bounded* if there is a 0-neighborhood V such that $A+V \subseteq U$. We denote by $\mathcal{H}_b(U;F)$ the space of holomorphic functions $f\colon U \to F$, where F is a Banach space, of *bounded type*, that is, f is bounded on U-bounded sets. If $F = \mathbb{C}$ we simply write $\mathcal{H}_b(U)$. The symbol $\mathcal{H}_{wu}(U)$ stands for the space of all holomorphic functions $f\colon U \to \mathbb{C}$ that are weakly uniformly continuous on U-bounded sets. When endowed with the topology of uniform convergence on U-bounded sets, both $\mathcal{H}_b(U;F)$ and $\mathcal{H}_{wu}(U)$ are locally convex spaces.

PROPOSITION 5.4. *Let $\mathcal{I}$ be an operator ideal, U be a balanced open subset of a Banach space E, and F be a Banach space.*

(a) *$\mathcal{H}_b(U;F)$ has the $\mathcal{I}$-AP if and only if $\mathcal{P}(^nE;F)$ has the $\mathcal{I}$-AP for every $n \in \mathbb{N}$.*
(b) *$\mathcal{H}_{wu}(U)$ has the $\mathcal{I}$-AP if and only if $\mathcal{P}_w(^nE)$ has the $\mathcal{I}$-AP for every $n \in \mathbb{N}$.*

Proof. Just combine Lemma 5.3 with the facts that $\{\mathcal{P}(^nE;F)\}_{n=1}^{\infty}$ is an $\mathcal{S}$-absolute decomposition of $\mathcal{H}_b(U;F)$ (this follows from an adaptation of the proof of [25, Proposition 3.36]) and that $\{\mathcal{P}_w(^nE)\}_{n=1}^{\infty}$ is an $\mathcal{S}$-absolute decomposition of $\mathcal{H}_{wu}(U)$ (see the proof of [10, Theorem 9]). ∎

In the following some of our apparently disconnected results will be combined together. A Banach space E is said to be *polynomially reflexive* if $\mathcal{P}(^nE)$ is reflexive for every $n \in \mathbb{N}$. For example, Tsirelson's original space T^* is polynomially reflexive [1].

PROPOSITION 5.5. *Let $\mathcal{I}$ be an operator ideal such that $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$ and either $\mathcal{I} \subseteq \mathcal{I}^{\mathrm{dual}}$ or $\mathcal{I}^{\mathrm{dual}} \subseteq \mathcal{I}$. The following are equivalent for a polynomially reflexive Banach space E and a balanced open subset U of E:*

(a) *E has the $\mathcal{I}$-AP.*
(b) *$\mathcal{P}(^nE)$ has the $\mathcal{I}$-AP for every $n \in \mathbb{N}$.*
(c) *$\mathcal{P}(^nE)$ has the $\mathcal{I}$-AP for some $n \in \mathbb{N}$.*
(d) *$\mathcal{H}_b(U)$ has the $\mathcal{I}$-AP.*

Proof. (a)⇒(b). Let $n \in \mathbb{N}$. By Corollary 3.8 we know that $\hat{\otimes}_{\pi}^{n,s}E$ has the $\mathcal{I}$-AP. Since $\mathcal{P}(^nE)$ is isomorphic to $(\hat{\otimes}_{\pi}^{n,s}E)'$ and these spaces are reflexive, Corollary 2.5 shows that $\mathcal{P}(^nE)$ has the $\mathcal{I}$-AP.

(b)⇒(c). This implication is obvious.

(c)⇒(a). Use the same argument as for (a)⇒(b).

(d)⇔(b). This equivalence follows from Proposition 5.4(a). ∎

To get another connection of results from different sections we consider the predual of the space of holomorphic functions. Given an open subset U of a Banach space, Mazet [39] proved the existence of a complete locally convex space $G(U)$ and of a canonical holomorphic function $\delta_U\colon U \to G(U)$

such that for every Banach space F and every holomorphic function f from U to F there is a unique continuous linear operator T_f from $G(U)$ to F such that $f = T_f \circ \delta_U$.

The following result follows directly from [25, Proposition 3.38], Lemma 5.3 and Corollary 3.8.

Proposition 5.6. *Let U be a balanced open subset of the Banach space E and $\mathcal{I}$ be an operator ideal such that $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$. Then E has the $\mathcal{I}$-AP if and only if $G(U)$ has the $\mathcal{I}$-AP.*

The results from Section 4 have not been combined with results from other sections yet. For results of Section 4 to come into play we investigate the $\mathcal{I}$-approximation property in the predual of the space $\mathcal{H}^\infty(U;F)$ of bounded holomorphic functions from an open subset U of a Banach space E to a Banach space F; $\mathcal{H}^\infty(U;F)$ is a Banach space with the sup norm. Let U be an open subset of a Banach space E. Mujica [41] proved the existence of a Banach space $G^\infty(U)$ and of a canonical bounded holomorphic mapping $\delta_U \in \mathcal{H}^\infty(U;G^\infty(U))$ with the following universal property: to every $f \in \mathcal{H}^\infty(U;F)$ corresponds a unique linear operator $T_f \in \mathcal{L}(G^\infty(U);F)$ such that $f = T_f \circ \delta_U$. He also introduced a very useful locally convex topology on $\mathcal{H}^\infty(U;F)$:

Theorem 5.7 ([41, Theorem 4.8]). *Let E and F be Banach spaces, and let U be an open subset of E. Let τ_γ denote the locally convex topology on $\mathcal{H}^\infty(U;F)$ generated by the seminorms*

$$p(f) = \sup_j \alpha_j \|f(x_j)\|,$$

where (x_j) varies over all sequences in U and (α_j) varies over all sequences of positive real numbers tending to zero. Then the mapping

$$f \in (\mathcal{H}^\infty(U;F), \tau_\gamma) \mapsto T_f \in (\mathcal{L}(G^\infty(U);F), \tau_c)$$

is a topological isomorphism.

We denote by $\mathcal{I} \circ \mathcal{H}^\infty(U;F)$ the collection of all $f \in \mathcal{H}^\infty(U;F)$ such that $f = u \circ g$, where G is a Banach space, $g \in \mathcal{H}^\infty(U;G)$ and $u \in \mathcal{I}(G;F)$. The next result extends [17, Theorem 5].

Theorem 5.8. *Let $\mathcal{I}$ be an operator ideal such that $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$. The following conditions are equivalent for a Banach space E and a bounded open subset U of E:*

(a) *E has the $\mathcal{I}$-AP.*
(b) *$\mathcal{H}^\infty(U;F) = \overline{\mathcal{I} \circ \mathcal{H}^\infty(U;F)}^{\tau_\gamma}$ for every Banach space F.*
(c) *$G^\infty(U)$ has the $\mathcal{I}$-AP.*

Proof. (a)⇒(b). Let $f \in \mathcal{H}^\infty(U;F)$. Let p be a continuous seminorm on $(\mathcal{H}^\infty(U;F), \tau_\gamma)$. By [41, Proposition 5.2] there are homogeneous polynomials $P_j \in P(^jE;F)$, $j = 0,1,\dots,n$, such that $p(P - f) < \varepsilon/2$ where $P = P_0 + P_1 + \dots + P_n$. Since E has the $\mathcal{I}$-AP and $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$, it follows from Theorem 4.2 that $\mathcal{P}(^jE;F) = \overline{\mathcal{I} \circ \mathcal{P}(^jE;F)}^{\tau_c}$ for every $j \in \mathbb{N}$. On the other hand, by [41, Proposition 4.9], $\tau_\gamma = \tau_c$ on $P \in P(^jE;F)$ for every $j \in \mathbb{N}$. So there are homogeneous polynomials $Q_j \in \mathcal{I} \circ \mathcal{P}(^jE;F)$ such that

$$p(Q_j - P_j) < \frac{\varepsilon}{2(n+1)}$$

for every $j = 0,1,\dots,n$. Putting $Q = Q_0 + Q_1 + \dots + Q_n$ and mimicking the argument used in the proof of [2, Theorem 2.4] one can easily prove that $Q = u \circ R$ where $u \in \mathcal{I}(G;F)$, G is a Banach space and R is a finite sum of homogeneous polynomials from E to G. Then the restriction of Q to U, still denoted by Q, is a bounded holomorphic function, so $Q \in \mathcal{I} \circ \mathcal{H}^\infty(U;F)$. Since

$$p(Q-P) = p\Big(\sum_{j=0}^n Q_j - \sum_{j=0}^n P_j\Big) \le \sum_{j=0}^n p(Q_j - P_j) < \frac{\varepsilon}{2},$$

it follows that

$$p(Q-f) \le p(Q-P) + p(P-f) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which proves (b).

(b)⇒(c). By [41, Theorem 2.1], $\delta_U \in \mathcal{H}^\infty(U;G^\infty(U))$. Taking $F = G^\infty(U)$ in (b), we find that $\delta_U \in \overline{\mathcal{I} \circ \mathcal{H}^\infty(U;G^\infty(U))}^{\tau_\gamma}$. Hence there is a net $(f_\alpha) \subseteq \mathcal{I} \circ \mathcal{H}^\infty(U;G^\infty(U))$ such that $f_\alpha \xrightarrow{\tau_\gamma} \delta_U$. For the corresponding net (T_{f_α}) of linear operators, by Theorem 5.7 we get

$$T_{f_\alpha} \xrightarrow{\tau_c} T_{\delta_U} = \mathrm{id}_{G^\infty(U)}.$$

But [2, Theorem 3.2] implies that $(T_{f_\alpha}) \subseteq \mathcal{I}(G^\infty(U);G^\infty(U))$. Therefore we obtain $\mathrm{id}_{G^\infty(U)} \in \overline{\mathcal{I}(G^\infty(U);G^\infty(U))}^{\tau_c}$, and Proposition 2.1 shows that $G^\infty(U)$ has the $\mathcal{I}$-AP.

(c)⇒(a). By [41, Proposition 2.3], E is topologically isomorphic to a complemented subspace of $G^\infty(U)$, which has the $\mathcal{I}$-AP by assumption. It follows from Proposition 2.2 that E has the $\mathcal{I}$-AP. ∎

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