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Fractional Hardy–Sobolev–Maz'ya inequality for domains

by

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Abstract. We prove a fractional version of the Hardy–Sobolev–Maz'ya inequality for arbitrary domains and L^p norms with $p \geq 2$. This inequality combines the fractional Sobolev and the fractional Hardy inequality into a single inequality, while keeping the sharp constant in the Hardy inequality.

1. Introduction. We are concerned here with the fractional Hardy inequality in an arbitrary domain $\Omega \subsetneq \mathbb{R}^N$, which states that if $1 < p < \infty$ and $0 < s < 1$ with $ps > 1$, then

$$\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \geq \mathcal{D}_{N,p,s} \int_{\Omega} \frac{|u(x)|^p}{m_{ps}(x)^{ps}}\,dx \tag{1}$$

for all $u \in W_0^{s,p}(\Omega)$, the closure of $C_c^\infty(\Omega)$ with respect to the norm defined by the left side of (1). The pseudodistance $m_{ps}(x)$ is defined in (5); its most important property for the present discussion is that for *convex* domains Ω we have $m_{ps}(x) \leq \operatorname{dist}(x, \Omega^c)$. We denote by $\mathcal{D}_{N,p,s}$ the *sharp* constant in (1), which was recently found by Loss and Sloane [13] and is explicitly given in (3) below. This constant is independent of Ω and coincides with that on the half-space which was earlier found in [3, 10].

By the (well-known) Sobolev inequality the left side of (1) dominates an L_q-norm of u. Our main result, the fractional HSM inequality, states that the left side of (1), even after subtracting the right side, is still strong enough to dominate this L_q-norm. More precisely, we shall prove

THEOREM 1.1. *Let $N \geq 2$, $2 \leq p < \infty$ and $0 < s < 1$ with $1 < ps < N$. Then there is a constant $\sigma_{N,p,s} > 0$ such that*

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$$(2)\qquad \iint\limits_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \mathcal{D}_{N,p,s}\int\limits_{\Omega} \frac{|u(x)|^p}{m_{ps}(x)^{ps}}\,dx \geq \sigma_{N,p,s}\Big(\int\limits_{\Omega}|u(x)|^q\,dx\Big)^{p/q}$$

for all open $\Omega \subsetneq \mathbb{R}^N$ *and all* $u \in W_0^{s,p}(\Omega)$, *where* $q = Np/(N-ps)$.

Inequality (2) has been conjectured in [10] in analogy to the local HSM inequalities [14, 1]. Recently, Sloane [15] found a remarkable proof of (2) for $p = 2$ and Ω being a half-space. Our result generalizes this to any $p \geq 2$ and any Ω. We emphasize that our constant $\sigma_{N,p,s}$ can be chosen independently of Ω. Therefore Theorem 1.1 is the fractional analog of the main inequality of [8], which treats the local case.

We now explain the notation in (2). The sharp constant [13] in (1) is

$$(3)\qquad \mathcal{D}_{N,p,s} = 2\pi^{(N-1)/2}\frac{\Gamma\big(\frac{1+ps}{2}\big)}{\Gamma\big(\frac{N+ps}{2}\big)}\int\limits_0^1 (1-r^{(ps-1)/p})^p \frac{dr}{(1-r)^{1+ps}}.$$

In the special case $p = 2$ we have

$$\mathcal{D}_{N,2,s} = 2\pi^{(N-1)/2}\frac{\Gamma\big(\frac{1+2s}{2}\big)}{\Gamma\big(\frac{N+2s}{2}\big)}\,\frac{B\big(\frac{1+2s}{2}, 1-s\big) - 2^{2s}}{2^{2s+1}s} = 2\kappa_{N,2s},$$

where $\kappa_{N,2s}$ is the notation used in [3, 13, 6]. We denote

$$(4)\qquad d_\omega(x) = \inf\{|t| : x + t\omega \notin \Omega\},\quad x\in\mathbb{R}^N,\ \omega\in\mathbb{S}^{N-1},$$

where $\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ is the $(N-1)$-dimensional unit sphere. Following [13] we set, for $\alpha > 0$,

$$(5)\qquad m_\alpha(x) = \bigg(\frac{2\pi^{(N-1)/2}\Gamma\big(\frac{1+\alpha}{2}\big)}{\Gamma\big(\frac{N+\alpha}{2}\big)}\bigg)^{1/\alpha}\bigg(\int\limits_{\mathbb{S}^{N-1}}\frac{d\omega}{d_\omega(x)^\alpha}\bigg)^{-1/\alpha},$$

which is analogous to the pseudodistance $m(x)$ of Davies [5, Theorem 5.3.5]. We recall that for convex domains Ω, we have $m_\alpha(x) \leq d(x)$ (see [13]).

This paper is organized as follows. In the next three sections we present three independent proofs of (2), but only the last one in full generality. In Section 2, we use the ground state representation for half-spaces as the starting point. This allows us to obtain (2) for half-spaces and any $p \geq 2$. In Section 3 we derive a fractional Hardy inequality (3.2) for balls with two additional terms, and then deduce (2) when $p = 2$ and Ω is a ball or a half-space. In the last section, we extend the method developed in [8] and use results from [11] and [13] to prove Theorem 1.1 for arbitrary domains.

2. The inequality on a half-space. In this section, we prove Theorem 1.1 in the particular case when $\Omega = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$. We note that the results of this section will not be needed to prove Theorem 1.1 in the general case. Our starting point is the inequality

$$\iint_{\mathbb{R}^N_+\times\mathbb{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \mathcal{D}_{N,p,s}\int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}}\,dx \ge c_p J[v], \tag{6}$$

where c_p is an explicit, positive constant (for $p=2$ this is an identity with $c_2 = 1$),

$$J[v] := \iint_{\mathbb{R}^N_+\times\mathbb{R}^N_+} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}(x_N y_N)^{(ps-1)/2}\,dx\,dy,$$

and $v(x) := x_N^{-(ps-1)/p}u(x)$. This inequality was derived in [10], using the 'ground state representation' method from [9]. We note that $m_{ps}(x) = x_N$ in the case of a half-space, as a quick computation shows (see also [13, (7)]).

In order to derive a lower bound on $J[v]$ we make use of the bound

$$(x_N y_N)^a \ge \min\{x_N^{2a}, y_N^{2a}\} = 2a\int_0^\infty \chi_{(t,\infty)}(x_N)\chi_{(t,\infty)}(y_N)t^{2a-1}\,dt$$

for $a > 0$. Combining this inequality with the fractional Sobolev inequality (see Lemma 2.1 below) and Minkowski's inequality, we can bound

$$\begin{aligned}
J[v] &\ge (ps-1)\int_0^\infty \iint_{\{x_N>t,\, y_N>t\}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\,t^{ps-2}\,dt \\
&\ge (ps-1)\mathcal{C}_{N,p,s}\int_0^\infty \Big(\int_{\{x_N>t\}} |v(x)|^q\,dx\Big)^{p/q} t^{ps-2}\,dt \\
&\ge (ps-1)\mathcal{C}_{N,p,s}\Big(\int_{\mathbb{R}^N_+} |v(x)|^q\Big(\int_0^{x_N} t^{ps-2}\,dt\Big)^{q/p} dx\Big)^{p/q} \\
&= \mathcal{C}_{N,p,s}\Big(\int_{\mathbb{R}^N_+} |v(x)|^q\,x_N^{q(ps-1)/p}\,dx\Big)^{p/q}.
\end{aligned}$$

Recalling the relation between u and v we arrive at (2). This completes the proof of Theorem 1.1 when $\Omega = \mathbb{R}^N_+$. ∎

In the previous proof we used the Sobolev inequality on half-spaces for functions which do not necessarily vanish on the boundary. For completeness we include a short derivation of this inequality. The precise statement involves the closure $\dot{W}^{s,p}(\mathbb{R}^N_+)$ of $C_c^\infty(\overline{\mathbb{R}^N_+})$ with respect to the left side of (1).

LEMMA 2.1. *Let $N \ge 1$, $1 \le p < \infty$ and $0 < s < 1$ with $ps < N$. Then there is a constant $\mathcal{C}_{N,p,s} > 0$ such that*

$$\iint_{\mathbb{R}^N_+\times\mathbb{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \ge \mathcal{C}_{N,p,s}\Big(\int_{\mathbb{R}^N_+} |u(x)|^q\,dx\Big)^{p/q}$$

for all $u\in \dot W^{s,p}(\mathbb{R}^N_+)$, *where* $q=Np/(N-ps)$.

Proof. If $\tilde u$ denotes the even extension of u to $\mathbb{R}^N$, then

$$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde u(x)-\tilde u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy &= 2\iint_{\mathbb{R}^N_+\times\mathbb{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\\ &\quad + 2\iint_{\mathbb{R}^N_+\times\mathbb{R}^N_+} \frac{|u(x)-u(y)|^p}{(|x'-y'|^2+(x_N+y_N)^2)^{(N+ps)/2}}\,dx\,dy\\ &\le 4\iint_{\mathbb{R}^N_+\times\mathbb{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy.\end{aligned}$$

On the other hand, by the 'standard' fractional Sobolev inequality on $\mathbb{R}^N$ (see, e.g., [9] for explicit constants) the left side is an upper bound on

$$\mathcal{S}_{N,p,s}\Big(\int_{\mathbb{R}^N} |\tilde u(x)|^q\,dx\Big)^{p/q} = 2^{p/q}\mathcal{S}_{N,p,s}\Big(\int_{\mathbb{R}^N_+} |u(x)|^q\,dx\Big)^{p/q}. \ \blacksquare$$

REMARK 2.2. The above proof of the fractional HSM inequality works analogously in the local case, that is, to show that

$$\int_{\mathbb{R}^N_+} |\nabla u|^p\,dx - \Big(\frac{p-1}{p}\Big)^p \int_{\mathbb{R}^N_+} \frac{|u|^p}{x_N^p}\,dx \ge \sigma_{N,p,1}\Big(\int_{\mathbb{R}^N_+} |u|^q\,dx\Big)^{p/q},\qquad q=\frac{Np}{N-p}, \tag{7}$$

for $u\in W^{1,p}_0(\mathbb{R}^N_+)$ when $N\ge 3$ and $2\le p<N$. Again, the starting point [9] is to bound the left side from below by an explicit constant $c_p>0$ times

$$\int_{\mathbb{R}^N_+} |\nabla v|^p x_N^{p-1}\,dx,\qquad v=x_N^{-(p-1)/p}u.$$

(For $p=2$, this is an identity with $c_2=1$.) Next, we write

$$x_N^a = a\int_0^\infty \chi_{(t,\infty)}(x_N)t^{a-1}\,dt$$

and use Sobolev's inequality on the half-space $\{x_N>t\}$ together with Minkowski's inequality. Note that the sharp constants in this half-space inequality are known explicitly (namely, given in terms of the whole-space constants via the reflection method of Lemma 2.1).

The sharp constant in (7) for $p = 2$ and $N = 3$ was found in [2]. We think it would be interesting to investigate this question for the non-local inequality (2) and we believe that [15] is a promising step in this direction.

3. The inequality on a ball. Our goal in this section is to prove a fractional Hardy–Sobolev–Maz'ya inequality on the ball $B_r \subset \mathbb{R}^N$, $N \geq 2$, of radius r centered at the origin. We again note that the results of this section will not be needed to prove Theorem 1.1 in the general case. The argument follows that from the previous section, but is more involved. More precisely, we shall prove

PROPOSITION 3.1. *Let $N \geq 2$, $p = 2$ and $1/2 < s < 1$. Then there is a constant $c = c(s, N) > 0$ such that for every $0 < r < \infty$ and $u \in W_0^{s,2}(B_r)$,*

$$\int_{B_r}\int_{B_r} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \mathcal{D}_{N,p,s}\int_{B_r} \frac{(2r)^{2s}}{(r^2-|x|^2)^{2s}}|u(x)|^2\,dx \geq c\Big(\int_{B_r}|u(x)|^q\,dx\Big)^{2/q}, \tag{8}$$

where $q = 2N/(N-2s)$.

This proves Theorem 1.1 in the special case $\Omega = B_r$ and $p = 2$ with $m_{2s}(x)$ replaced by $(r^2 - |x|^2)/2r$. We note that $(r^2 - |x|^2)/2r \leq \operatorname{dist}(x, B_r^c)$ for $x \in B_r$. (As an aside we note, however, that it is *not always* true that $(r^2 - |x|^2)/2r$ is greater than $m_{2s}(x)$. Indeed, take $x = 0$ and $N = 2$.)

We also note that Proposition 3.1 implies Theorem 1.1 for $\Omega = \mathbb{R}^N_+$ (and $p = 2$). Indeed, by translation invariance the proposition implies the inequality also on balls $B(a_r, r)$ centered at $a_r = (0, \ldots, 0, r)$. We have $\operatorname{dist}(x, B(a_r, r)^c) \leq \operatorname{dist}(x, (\mathbb{R}^N_+)^c)$, and hence the result follows by taking $r \to \infty$.

The crucial ingredient in our proof of Proposition 3.1 is

LEMMA 3.2. *Let $N \geq 2$, $1/2 < s < 1$ and for $x \in B_1 \subset \mathbb{R}^N$ define $w_N(x) = (1 - |x|^2)^{(2s-1)/2}$. Then for all $u \in W_0^{s,2}(B_1)$,*

$$\int_{B_1}\int_{B_1} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \mathcal{D}_{N,2,s}\int_{B_1} \frac{2^{2s}}{(1-|x|^2)^{2s}}|u(x)|^2\,dx \geq \tilde{J}[v] + c\int_{B_1}|v(x)|^2\,dx, \tag{9}$$

where $v = u/w_N$,

$$\tilde{J}[v] = \int_{B_1}\int_{B_1} |v(x)-v(y)|^2 \frac{w_N(x)w_N(y)}{|x-y|^{N+2s}}\,dx\,dy$$

and $c = s^{-1}(2^{2s-1} - 1)|\mathbb{S}^{N-1}| > 0$.

This inequality is somewhat analogous to (6) in the previous proof. We emphasize, however, that there are two terms on the right side of (9) and we will need both of them. Accepting this lemma for the moment, we now complete

Proof of Proposition 3.1. By scaling, we may and do assume that $r = 1$, that is, we consider only the unit ball $B_1 \subset \mathbb{R}^N$. We put $v = u/w_N$ with w_N defined in Lemma 3.2. According to that lemma, the left side of (8) is bounded from below by

$$\tilde{J}[v] + c \int_{B_1} |v(x)|^2 w_N(x)^2 \, dx. \tag{10}$$

(Here we also used that $w_N \le 1$.) For $x, y \in B_1$ we have

$$\begin{aligned} w_N(x) w_N(y) &\ge \min\{(1-|x|^2)^{2s-1}, (1-|y|^2)^{2s-1}\} \\ &= (2s-1) \int_0^1 \chi_{(t,1]}(1-|x|^2)\chi_{(t,1]}(1-|y|^2) t^{2s-2} \, dt, \end{aligned}$$

and therefore

$$\begin{aligned} &\tilde{J}[v] + c \int_{B_1} |v(x)|^2 w_N(x)^2 \, dx \\ &\ge (2s-1) \int_0^1 \Bigg(\int_{B_{\sqrt{1-t}}} \int_{B_{\sqrt{1-t}}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} \, dx\, dy + c \int_{B_{\sqrt{1-t}}} |v(x)|^2 \, dx \Bigg) t^{2s-2} \, dt. \end{aligned}$$

The fractional Sobolev inequality [4, (2.3)] and a scaling argument imply that there is a $\tilde{c} > 0$ such that for all $r > 0$,

$$r^{2s} \int_{B_r} \int_{B_r} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} \, dx\, dy + c \int_{B_r} |v(x)|^2 \, dx \ge \tilde{c} r^{2s} \Big(\int_{B_r} |v(x)|^q \, dx \Big)^{2/q}.$$

Combining the last two relations and applying Minkowski's inequality, we may bound

$$\begin{aligned} &\tilde{J}[v] + c \int_{B_1} |v(x)|^2 w_N(x)^2 \, dx \\ &\qquad \ge (2s-1)\tilde{c} \int_0^1 \Big(\int_{B_{\sqrt{1-t}}} |v(x)|^q \, dx \Big)^{2/q} (\sqrt{1-t})^{2s} t^{2s-2} \, dt \\ &\qquad \ge (2s-1)\tilde{c} \Big(\int_{B_1} |v(x)|^q \Big(\int_0^{1-|x|^2} (1-t)^s t^{2s-2} \, dt \Big)^{q/2} dx \Big)^{2/q}. \end{aligned} \tag{11}$$

We observe that

$$\int_0^{1-|x|^2} (1-t)^s t^{2s-2}\,dt \geq B(s+1,2s-1)(1-|x|^2)^{2s-1},$$

which follows from the fact that $y \mapsto \int_0^y (1-t)^s t^{2s-2}\,dt / \int_0^y t^{2s-2}\,dt$ is decreasing on $(0,1)$. This allows us to bound the right side of (11) from below by

$$(2s-1)B(s+1,2s-1)\tilde{c}\Big(\int_{B_1} |v(x)|^q (1-|x|^2)^{(s-1/2)q}\,dx\Big)^{2/q}$$
$$= (2s-1)B(s+1,2s-1)\tilde{c}\Big(\int_{B_1} |u(x)|^q\,dx\Big)^{2/q},$$

and we are done. ■

This leaves us with proving Lemma 3.2. We need to introduce some notation. The regional Laplacian (see, e.g., [12]) on an open set $\Omega \subset \mathbb{R}^N$ is, up to a multiplicative constant, given by

$$L_\Omega u(x) = \lim_{\varepsilon\to 0^+} \int_{\Omega\cap\{|y-x|>\varepsilon\}} \frac{u(y)-u(x)}{|x-y|^{N+2s}}\,dy.$$

This operator appears naturally in our context since

$$\int_\Omega \overline{u(x)}(L_\Omega u)(x)\,dx = -\frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

Our proof of Lemma 3.2 relies on a pointwise estimate for $L_{B_1} w_N$. In dimension $N=1$ this can be computed explicitly and we recall from [6, Lemma 2.1] that

$$-L_{(-1,1)} w_1(x) = \frac{(1-x^2)^{(-2s-1)/2}}{2s}(B(s+1/2,1-s) - (1-x)^{2s} + (1+x)^{2s}).$$

Hence, by [6, (2.3)],

$$-L_{(-1,1)} w_1(x) \geq c_1(1-x^2)^{(-2s-1)/2} + c_2(1-x^2)^{(-2s+1)/2}, \tag{12}$$

where

$$c_1 = \frac{B(s+1/2,1-s)-2^{2s}}{2s}, \quad c_2 = \frac{2^{2s}-2}{2s}.$$

Lemma 3.3. *Let $N\geq 2$ and let w_N be as in Lemma 3.2. Then*

$$-L_{B_1} w_N(x)$$
$$\geq \frac{c_1}{2}\int_{\mathbb{S}^{N-1}} |h_N|^{2s}\,dh \cdot (1-|x|^2)^{-(2s+1)/2} + \frac{c_2}{2}|\mathbb{S}^{N-1}|\cdot(1-|x|^2)^{-(2s-1)/2}.$$

Proof. By rotation invariance we may assume that $\mathbf{x} = (0,\dots,0,x)$. With the notation $p=(2s-1)/2$ we have

$$
\begin{aligned}
-L_{B_1} w_N(\mathbf{x}) &= \text{p.v.} \int\limits_{B_1} \frac{(1-|\mathbf{x}|^2)^p - (1-|y|^2)^p}{|\mathbf{x}-y|^{N+2s}}\, dy \\
&= \frac{1}{2} \int\limits_{\mathbb{S}^{N-1}} dh\, \text{p.v.} \int\limits_{-xh_N - \sqrt{x^2h_N^2 - x^2+1}}^{-xh_N + \sqrt{x^2h_N^2 - x^2+1}} \frac{(1-|x|^2)^p - (1-|x+ht|^2)^p}{|t|^{1+2s}}\, dt.
\end{aligned}
$$

We calculate the inner principal value integral by changing the variable $t = -xh_N + u\sqrt{x^2h_N^2 - x^2 + 1}$:

$$
\begin{aligned}
g(x,h) &:= \text{p.v.} \int\limits_{-xh_N - \sqrt{x^2h_N^2 - x^2+1}}^{-xh_N + \sqrt{x^2h_N^2 - x^2+1}} \frac{(1-|x|^2)^p - (1-|x+ht|^2)^p}{|t|^{1+2s}}\, dt \\
&= \text{p.v.} \int\limits_{-1}^{1} \frac{(1-x^2)^p - (1-u^2)^p(1-x^2+x^2h_N^2)^p}{|-xh_N + u\sqrt{x^2h_N^2 - x^2+1}|^{1+2s}} \sqrt{x^2h_N^2 - x^2 + 1}\, du \\
&= (1-x^2+x^2h_N^2)^{p-s} \text{p.v.} \int\limits_{-1}^{1} \frac{(1-\frac{x^2h_N^2}{1-x^2+x^2h_N^2})^p - (1-u^2)^p}{|u - \frac{xh_N}{\sqrt{1-x^2+x^2h_N^2}}|^{1+2s}}\, du \\
&= (1-x^2+x^2h_N^2)^{-1/2} (-L_{(-1,1)} w_1)\left(\frac{xh_N}{\sqrt{1-x^2+x^2h_N^2}}\right).
\end{aligned}
$$

Hence by (12) we have

$$
\begin{aligned}
g(x,h) \geq (1-x^2+x^2h_N^2)^{-1/2} \Bigg(c_1 &\left(1 - \frac{x^2h_N^2}{1-x^2+x^2h_N^2}\right)^{(2s-1)/2-2s} \\
&+ c_2 \left(1 - \frac{x^2h_N^2}{1-x^2+x^2h_N^2}\right)^{(2s-1)/2-2s+1} \Bigg) \\
= c_1(1-x^2+x^2h_N^2)^s &(1-x^2)^{-(2s+1)/2} \\
+ c_2(1-x^2+x^2h_N^2)^{s-1} &(1-x^2)^{-(2s-1)/2} \\
\geq c_1|h_N|^{2s}(1-x^2)^{-(2s+1)/2} &+ c_2(1-x^2)^{-(2s-1)/2}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
-L_{B_1} w_N(\mathbf{x}) &= \frac{1}{2} \int\limits_{\mathbb{S}^{N-1}} g(x,h)\, dh \\
&\geq \frac{c_1}{2} \int\limits_{\mathbb{S}^{N-1}} |h_N|^{2s}\, dh \cdot (1-x^2)^{-(2s+1)/2} \\
&\quad + \frac{c_2}{2} |\mathbb{S}^{N-1}| \cdot (1-x^2)^{-(2s-1)/2},
\end{aligned}
$$

and we are done. ∎

Finally, we are in a position to give

Proof of Lemma 3.2. We use the ground state representation formula [9] (see also [6, Lemma 2.2])

$$\int_{B_1}\int_{B_1} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + 2\int_{B_1} \frac{Lw_N(x)}{w_N(x)}|u(x)|^2\,dx = \tilde{J}[v]$$

with $u = w_N v$ and $\tilde{J}$ as defined in the lemma. The assertion now follows from Lemma 3.3, which implies that

$$-2\frac{Lw_N(x)}{w_N(x)} \geq \mathcal{D}_{N,2,s}\frac{2^{2s}}{(1-|x|^2)^{2s}} + c(1-|x|^2)^{-2s+1}$$

with $c = c_2|\mathbb{S}^{N-1}| > 0$. Indeed, here we used $2^{2s-1}\mathcal{D}_{1,2,s} = c_1$ and

$$\mathcal{D}_{N,2,s} = \mathcal{D}_{1,2,s}\cdot\frac{1}{2}\int_{\mathbb{S}^{N-1}} |h_N|^{2s}\,dh,$$

as a quick computation shows. ∎

4. The inequality in the general case. In this section we shall give a complete proof of Theorem 1.1. Our strategy is somewhat reminiscent of the proof of the Hardy–Sobolev–Maz'ya inequality in the local case in [8]. As in that paper, we use an averaging argument à la Gagliardo–Nirenberg to reduce the multi-dimensional case to the one-dimensional case. We describe this reduction in Subsection 4.1 and establish the required 1D inequality in Subsection 4.2.

4.1. Reduction to one dimension. The key ingredient in our proof of Theorem 1.1 is the following pointwise estimate of a function on an interval.

LEMMA 4.1. *Let $0 < s < 1$, $q \geq 1$ and $p \geq 2$ with $ps > 1$. Then there is a $c = c(s,q,p) < \infty$ such that for all $f \in C_c^\infty(-1,1)$,*

$$\begin{aligned} &\|f\|_\infty^{p+q(ps-1)} \\ &\quad \leq c\bigg(\int_{-1}^{1}\int_{-1}^{1} \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\,dy\,dx - \mathcal{D}_{1,p,s}\int_{-1}^{1} \frac{|f(x)|^p}{(1-|x|)^{ps}}\,dx\bigg)\|f\|_q^{q(ps-1)}. \end{aligned} \tag{13}$$

Due to the particular form of the exponents this inequality has a scale-invariant form.

COROLLARY 4.2. *Let $0 < s < 1$, $q \geq 1$ and $p \geq 2$ with $ps > 1$. Then, with the same constant $c = c(p,s,q) < \infty$ as in Lemma 4.1, we have for all* *open sets* $\Omega \subsetneq \mathbb{R}$ *and all* $f \in C_c^\infty(\Omega)$,

$$(14)\qquad \|f\|_\infty^{p+q(ps-1)} \leq c\bigg(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\,dy\,dx - \mathcal{D}_{1,p,s}\int_\Omega \frac{|f(x)|^p}{d(x)^{ps}}\,dx\bigg)\|f\|_q^{q(ps-1)}$$

where $d(x) = \operatorname{dist}(x, \Omega^c)$.

Proof. From Lemma 4.1, by translation and dilation, we obtain (14) for any interval and half-line. The extension to arbitrary open sets $\Omega \subsetneq \mathbb{R}$ is straightforward. ∎

We prove Lemma 4.1 in Subsection 4.2. Now we show how this corollary allows us to deduce our main theorem. Taking advantage of an averaging formula of Loss and Sloane [13] the argument is almost the same as in [8], but we reproduce it here to make this paper self-contained.

Proof of Theorem 1.1. Let $\omega_1, \ldots, \omega_N$ be an orthonormal basis in $\mathbb{R}^N$. We write x_j for the jth coordinate of $x \in \mathbb{R}^N$ in this basis, and $\tilde{x}_j = x - x_j\omega_j$. By skipping the jth coordinate of $\tilde{x}_j$ (which is zero), we may regard $\tilde{x}_j$ as an element of $\mathbb{R}^{N-1}$. For a given domain $\Omega \subsetneq \mathbb{R}^N$ we write

$$d_j(x) = d_{\omega_j}(x) = \inf\{|t| : x + t\omega_j \notin \Omega\}.$$

If $u \in C_c^\infty(\Omega)$, then Corollary 4.2 yields

$$|u(x)| \leq C(g_j(\tilde{x}_j)h_j(\tilde{x}_j))^{\frac{1}{p+q(ps-1)}}$$

for any $1 \leq j \leq N$, where

$$g_j(\tilde{x}_j) = \int_{\tilde{x}_j + a\omega_j \in \Omega} da \int_{\tilde{x}_j + b\omega_j \in \Omega} db\, \frac{|u(\tilde{x}_j + a\omega_j) - u(\tilde{x}_j + b\omega_j)|^p}{|a-b|^{1+ps}} - \mathcal{D}_{1,p,s}\int_{\mathbb{R}} da\, \frac{|u(\tilde{x}_j + a\omega_j)|^p}{d_j(\tilde{x}_j + a\omega_j)^{ps}}$$

and

$$h_j(\tilde{x}_j) = \Big(\int_{\mathbb{R}} da\, |u(\tilde{x}_j + a\omega_j)|^q\Big)^{ps-1}.$$

Thus

$$|u(x)|^N \leq C^N \prod_{j=1}^N (g_j(\tilde{x}_j)h_j(\tilde{x}_j))^{\frac{1}{p+q(ps-1)}}.$$

We now pick $q = \frac{pN}{N-ps}$ and rewrite the previous inequality as

$$|u(x)|^q \leq C^q \prod_{j=1}^N (g_j(\tilde{x}_j)h_j(\tilde{x}_j))^{\frac{1}{ps(N-1)}}.$$

By a standard argument based on repeated use of Hölder's inequality (see, e.g., [8, Lemma 2.4]) we deduce that

$$\int_{\mathbb{R}^N} |u(x)|^q\,dx \le C^q \prod_{j=1}^{N} \Big(\int_{\mathbb{R}^{N-1}} g_j(y)^{\frac{1}{ps}} h_j(y)^{\frac{1}{ps}}\,dy \Big)^{\frac{1}{N-1}}.$$

We note that

$$\|h_j^{\frac{1}{ps-1}}\|_{L^1(\mathbb{R}^{N-1})} = \|u\|_{L^q(\mathbb{R}^N)}^q \qquad \text{for every } j=1,\dots,N$$

and derive from the Hölder and the arithmetic-geometric mean inequality that

$$\begin{aligned}\prod_{j=1}^{N} \int_{\mathbb{R}^{N-1}} g_j(y)^{\frac{1}{ps}} h_j(y)^{\frac{1}{ps}}\,dy &\le \prod_{j=1}^{N} \|g_j\|_1^{\frac{1}{ps}} \|h_j^{\frac{1}{ps-1}}\|_1^{\frac{ps-1}{ps}} = \|u\|_q^{\frac{q(ps-1)N}{ps}} \prod_{j=1}^{N} \|g_j\|_1^{\frac{1}{ps}} \\ &\le \|u\|_q^{\frac{q(ps-1)N}{ps}} \Big(N^{-1} \sum_{j=1}^{N} \|g_j\|_1 \Big)^{\frac{N}{ps}}.\end{aligned}$$

To summarize, we have shown that

$$\|u\|_q^p \le C^{\frac{p^2 s(N-1)}{N-ps}} N^{-1} \sum_{j=1}^{N} \|g_j\|_1.$$

We now average this inequality over all choices of the coordinate system ω_j. We recall the Loss–Sloane formula [13, Lemma 2.4]

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &= \frac{1}{2} \int_{\mathbb{S}^{N-1}} d\omega \int_{\{x:\,x\cdot\omega=0\}} d\mathcal{L}_\omega(x) \int_{x+a\omega\in\Omega} da \int_{x+b\omega\in\Omega} db\, \frac{|u(x+a\omega)-u(x+b\omega)|^p}{|a-b|^{1+ps}},\end{aligned}$$

where $\mathcal{L}_\omega$ is $(N-1)$-dimensional Lebesgue measure on the hyperplane $\{x : x\cdot\omega = 0\}$. Thus we arrive at

$$\begin{aligned}\|u\|_q^p \le \frac{2\,C^{\frac{p^2 s(N-1)}{N-ps}}}{|\mathbb{S}^{N-1}|} \Big(&\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &- \mathcal{D}_{1,p,s} \frac{\pi^{(N-1)/2}\Gamma\big(\frac{1+ps}{2}\big)}{\Gamma\big(\frac{N+ps}{2}\big)} \int_\Omega \frac{|u(x)|^p}{m_{ps}(x)^{ps}}\,dx \Big).\end{aligned}$$

Recalling the definition of $\mathcal{D}_{N,p,s}$ we see that this is the inequality claimed in Theorem 1.1. ■

4.2. Proof of the key inequality. Our first step towards the proof of Proposition 4.1 is a Hardy inequality on an interval with a remainder term. Note the similarity to Lemma 3.2.

LEMMA 4.3. *Let $0 < s < 1$ and $p \geq 2$ with $ps > 1$. Then*

$$\int_0^1\int_0^1 \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\,dx\,dy - \mathcal{D}_{1,p,s}\int_0^1 \frac{|f(x)|^p}{x^{ps}}\,dx$$
$$\geq c_p \int_0^1\int_0^1 \frac{|v(x)-v(y)|^p}{|x-y|^{1+ps}}\omega(x)^{p/2}\omega(y)^{p/2}\,dx\,dy + \int_0^1 W_{p,s}(x)|v(x)|^p\omega(x)^p\,dx$$

for all f with $f(0)=0$ (and no boundary condition at $x=1$). Here $\omega(x) = x^{(ps-1)/p}$ and $f = \omega v$. The function $W_{p,s}$ is bounded away from zero and satisfies

$$W_{p,s}(x) \approx x^{-(p-1)(ps-1)/p} \qquad \textit{for } x \in (0,1/2]$$

and

$$W_{p,s}(x) \approx \begin{cases} 1 & \textit{if } p-1-ps>0, \\ |\ln(1-x)| & \textit{if } p-1-ps=0, \\ (1-x)^{-1-ps+p} & \textit{if } p-1-ps<0, \end{cases} \qquad \textit{for } x\in[1/2,1).$$

Proof. The general ground state representation [9] reads

$$\int_0^1\int_0^1 \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\,dx\,dy \geq \int_0^1 V(x)|f(x)|^p + c_p \int_0^1\int_0^1 \frac{|v(x)-v(y)|^p}{|x-y|^{1+ps}}\omega(x)^{p/2}\omega(y)^{p/2}\,dx\,dy$$

with

$$V(x) := 2\omega(x)^{-p+1}\int_0^1 (\omega(x)-\omega(y))|\omega(x)-\omega(y)|^{p-2}|x-y|^{-1-ps}\,dy$$

(understood as principal value integral). We decompose

$$\begin{aligned} V(x) &= 2\omega(x)^{-p+1}\int_0^\infty (\omega(x)-\omega(y))|\omega(x)-\omega(y)|^{p-2}|x-y|^{-1-ps}\,dy \\ &\quad - 2\omega(x)^{-p+1}\int_1^\infty (\omega(x)-\omega(y))|\omega(x)-\omega(y)|^{p-2}|x-y|^{-1-ps}\,dy \\ &= \frac{\mathcal{D}_{1,p,s}}{x^{ps}} + W_{p,s}(x). \end{aligned}$$

(The computation of the first term is in [10, Lemma 2.4].) For $x\in(0,1)$, the second term is positive, indeed,

$$W_{p,s}(x) = 2\omega(x)^{-p+1}\int_1^\infty (\omega(y)-\omega(x))^{p-1}(y-x)^{-1-ps}\,dy.$$

Note that at $x = 0$,

$$\int_1^\infty \omega(y)^{p-1} y^{-1-ps}\, dy = c_{p,s} < \infty$$

since $ps - (p-1)(ps-1)/p > 0$. Hence $W_{p,s}(x) \sim 2c_{p,s} x^{-(p-1)(ps-1)/p}$ as $x \to 0$. On the other hand, at $x = 1$, we have

$$\int_1^\infty (\omega(y) - 1)^{p-1} (y-1)^{-1-ps}\, dy = \tilde{c}_{p,s} < \infty \quad \text{if } p - 1 - ps > 0.$$

Hence $W_{p,s}(x) \to 2\tilde{c}_{p,s}$ as $x \to 1$ in that case. In the opposite case, one easily finds that for $x = 1 - \epsilon$, to leading order only y's with $y - 1$ of order ϵ contribute. Hence $W_{p,s}(x) \sim 2\tilde{c}_{p,s}(1-x)^{-1-ps+p}$ as $x \to 1$ if $p - 1 - ps < 0$ and $W_{p,s}(x) \sim 2\tilde{c}_{p,s}|\ln(1-x)|$ if $p - 1 - ps = 0$. ∎

COROLLARY 4.4. *Let $0 < s < 1$ and $p \geq 2$ with $ps > 1$. Then*

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1} \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\, dx\, dy - \mathcal{D}_{1,p,s} \int_{-1}^{1} \frac{|f(x)|^p}{(1-|x|)^{ps}}\, dx \\ \geq c_p \Big(\int_{-1}^{0}\int_{-1}^{0} + \int_0^1\int_0^1 \Big) \frac{|v(x)-v(y)|^p}{|x-y|^{1+ps}} \omega(x)^{p/2} \omega(y)^{p/2}\, dx\, dy \\ + c_{p,s} \int_{-1}^{1} |v(x)|^p \omega(x)\, dx\end{aligned}$$

for all f with $f(-1) = f(1) = 0$. Here $\omega(x) = (1-|x|)^{(ps-1)/p}$ and $f = \omega v$.

Proof. The corollary follows by applying Lemma 4.3 to the functions $f_1(x) = f(1+x)$ and $f_2(x) = f(1-x)$, where $x \in [0,1]$, and adding the resulting inequalities. ∎

The second ingredient besides Lemma 4.3 in our proof of Proposition 4.1 is the following bound due to Garsia, Rodemich and Rumsey [11].

LEMMA 4.5. *Let $p, s > 0$ with $ps > 1$. Then for any continuous function f on $[a,b]$,*

$$\int_a^b\int_a^b \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\, dy\, dx \geq c\, \frac{|f(b)-f(a)|^p}{(b-a)^{ps-1}} \tag{15}$$

with $c = (ps-1)^p (8(ps+1))^{-p}/4$.

Proof. This follows by taking $\Psi(x) = |x|^p$ and $p(x) = |x|^{s+1/p}$ in [11, Lemma 1.1]. ∎

After these preliminaries we can now turn to

Proof of Proposition 4.1. Let $\omega(x) = (1-|x|)^{(ps-1)/p}$. Substituting $v = f/\omega$ and applying Corollary 4.4, we see that it suffices to prove

$$\begin{aligned}(16)\qquad \|v\omega\|_\infty^{p+q(ps-1)} &\le c\Bigg(\int_{-1}^{1} |v(x)|^p\omega(x)\,dx \\ &+\bigg(\int_{-1}^{0}\int_{-1}^{0} + \int_0^1\int_0^1\bigg)\frac{|v(x)-v(y)|^p}{|x-y|^{1+ps}}\omega(x)^{p/2}\omega(y)^{p/2}\,dx\,dy\Bigg)\|v\omega\|_q^{q(ps-1)}.\end{aligned}$$

Without loss of generality, we may assume that v is non-negative and that for some $x_0 \in [0,1)$ we have $v(x_0)\omega(x_0) = \|v\omega\|_\infty > 0$. Let $c_1 = \omega(1/2)/(2\omega(0)) \in (0,1)$. We distinguish three cases.

Case 1: $x_0 \in [0,1/2]$ and $v\omega \ge c_1 v(x_0)\omega(x_0)$ on $[0,1/2]$. Then $\int_{-1}^1 |v|^p\omega \ge \int_0^{1/2} |v|^p\omega^p \ge (c_1^p/2)|v(x_0)\omega(x_0)|^p$ and $\int_{-1}^1 |v\omega|^q \ge (c_1^q/2)|v(x_0)\omega(x_0)|^q$, hence (16) follows.

Case 2: $x_0 \in [0,1/2]$ and there is a $z \in [0,1/2]$ such that $v(z)\omega(z) \le c_1 v(x_0)\omega(x_0)$. Let z be closest possible to x_0, so that $v(z)\omega(z) = c_1 v(x_0)\omega(x_0)$ and $v\omega \ge c_1 v(x_0)\omega(x_0)$ on the interval I with endpoints x_0 and z. We observe that

$$v(z) = c_1 v(x_0)\frac{\omega(x_0)}{\omega(z)} = \frac{v(x_0)}{2}\,\frac{\omega(x_0)}{\omega(0)}\,\frac{\omega(1/2)}{\omega(z)} \le \frac{v(x_0)}{2}.$$

By (15) we have

$$\begin{aligned}\int_0^1\int_0^1 &\frac{|v(x)-v(y)|^p}{|x-y|^{1+ps}}\omega(x)^{p/2}\omega(y)^{p/2}\,dy\,dx \\ &\ge w(1/2)^p\int_I\int_I \frac{|v(x)-v(y)|^p}{|x-y|^{1+ps}}\,dy\,dx \\ &\ge c|v(x_0)-v(z)|^p|z-x_0|^{1-ps} \ge c'|v(x_0)\omega(x_0)|^p|z-x_0|^{1-ps}.\end{aligned}$$

On the other hand,

$$\int_{-1}^{1} |v\omega|^q \ge \int_I |v\omega|^q \ge c_1^q|v(x_0)\omega(x_0)|^q|z-x_0|.$$

Hence (16) follows.

Case 3: $x_0 \in (1/2,1)$. Since the function $x \mapsto \omega(x)/\omega(x/2)$ is decreasing on $[0,1)$, we have

$$\frac{\omega(x_0)}{\omega(x_0/2)} \le \frac{\omega(1/2)}{\omega(1/4)} =: c_2.$$

Since $v(x_0/2)\omega(x_0/2) \le v(x_0)\omega(x_0)$, we get $v(x_0/2) \le c_2 v(x_0)$. Hence there exists $z \in [x_0/2, x_0)$ such that $v(z) = c_2 v(x_0)$ and $v \ge c_2 v(x_0)$ on $[z, x_0]$.

By (15) we have

$$\int_0^1\!\!\int_0^1 \frac{|v(x)-v(y)|^p}{|x-y|^{1+ps}}\,\omega(x)^{p/2}\omega(y)^{p/2}\,dy\,dx$$
$$\geq w(x_0)^p \int_z^{x_0}\int_z^{x_0} \frac{|v(x)-v(y)|^p}{|x-y|^{1+ps}}\,dy\,dx$$
$$\geq cw(x_0)^p|v(x_0)-v(z)|^p|z-x_0|^{1-ps} \geq c'|v(x_0)\omega(x_0)|^p|z-x_0|^{1-ps}.$$

Also,

$$\int_{-1}^1 |v\omega|^q \geq \omega(x_0)^q \int_z^{x_0} |v|^q \geq c_2^q|v(x_0)\omega(x_0)|^q|z-x_0|,$$

and again (16) follows. This completes the proof of Proposition 4.1. ∎

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Note added in proof. We would like to draw the reader's attention to the preprint [7] that appeared shortly after the submission of our paper and which treats related inequalities.

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