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On smooth points of boundaries of open sets

by

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Abstract. The notions of smooth points of the boundary of an open set and $\alpha(\cdot)$ intrinsically paraconvex sets are introduced. It is shown that for an $\alpha(\cdot)$ intrinsically paraconvex open set the set of smooth points is a dense G_δ-set of the boundary.

Let $(X, \|\cdot\|)$ be a Banach space. Let C be an open set in X and let $x_0 \in \overline{C}$.

The *tangent cone* $\mathcal{T}_C(x_0)$ of the set C at the point x_0 consists of $h \in X$ such that for every neighbourhood Q of h there is $t_Q > 0$ such that for $0 < t < t_Q$,

$$(x_0 + tQ) \cap C \neq \emptyset \tag{1}$$

(Dubovitskiĭ and Milyutin (1965)). It is easy to see that each tangent cone is closed.

Using the distance function we can rewrite this definition in the following form: $\mathcal{T}_C(x_0)$ consists of $h \in X$ such that for every $\varepsilon > 0$ there is a $t_0 > 0$ such that for $0 < t < t_0$,

$$\operatorname{dist}(x_0 + th, C) < \varepsilon t.$$

Here the arbitrariness of ε means that the directional derivative of the function $\operatorname{dist}(\cdot, C)$ at the point x_0 in the direction h is equal to 0, $\partial \operatorname{dist}(x, C)|_{x_0}(h) = 0$.

A point $x_0 \in \partial C$ is called a *smooth point* of ∂C if the cone $\mathcal{T}_C(x_0)$ is a halfspace, i.e.

$$\mathcal{T}_C(x_0) = \{x \in X : x^*(x) \geq 0\}$$

for some continuous linear functional x^*. The set of all smooth points of ∂C is called the *smooth set* of the ∂C and denoted by $\mathcal{S}(C)$.

It is a natural question how big part of ∂C is the smooth set $\mathcal{S}(C)$.

If $\mathcal{S}(C) = \partial C$ we say that ∂C is *smooth*.

Now we give an example of a smooth set C.

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Proposition 1. *Let* $(Y, \|\cdot\|)$ *be a Banach space. Let* Ω *be an open set in* Y. *Let* $C \subset \mathbb{R} \times \Omega$ *be the epigraph of a function* $f : \Omega \to \mathbb{R}$, $C = \{(r, y) \in \mathbb{R} \times \Omega : r \geq f(y), y \in \Omega\}$. *If* f *is Gateaux differentiable at* y_0 *then* $(f(y_0), y_0)$ *is a smooth point of* ∂C.

Proof. We take $X = \mathbb{R} \times Y$ and we put $x^*((t, y)) = t + \partial f|_{y_0}(y)$ and $H_{x^*} = \{(r, y) : r \geq f(y_0) + \partial f|_{y_0}(y)\}$. Since f is Gateaux differentiable at y_0, for every $(r, y) \in H_{x^*}$ and every neighbourhood Q of zero in X there is $s > 0$ such that for all $0 < t < s$,

$$(f(y_0), y_0) + t(r - f(y_0), y - y_0) \in C + tQ.$$

Thus $H_{x^*} \subset \mathcal{T}_C((f(y_0), y_0))$.

On the other hand, if $(r, y) \notin H_{x^*}$ then for all $s > 0$ there are $t_s > 0$ and a neighbourhood Q of zero such that $0 < t_s < s$ and $(f(y_0), y_0) + t_s(r - f(y_0), y - y_0) \notin C + t_s Q$. This implies that $(r, y) \notin \mathcal{T}_C((f(y_0), y_0))$. Therefore we have the equality

$$H_{x^*} = \mathcal{T}_C((f(y_0), y_0)). \blacksquare$$

The converse assertion does not hold:

Example 2. Let $Y = \mathbb{R}$. Let

$$f(x) = \begin{cases} |x|\bigl(-1 + \sin \frac{1}{x}\bigr) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Of course, the function $f(\cdot)$ is not differentiable at 0. On the other hand, the tangent cone of the epigraph $C = \{(t, x) : t \geq f(x)\}$ at the point $(0, 0)$ is a halfplane: $\mathcal{T}_C((0, 0)) = \{(t, x) : t \geq 0\}$.

Mazur (1933) proved that if X is separable, then every convex real-valued function defined on an open convex set $\Omega \subset X$ is Gateaux differentiable on a dense G_δ-set. Of course, such sets are residual (i.e. their complements in Ω are of the first Baire category).

Asplund (1968) found a class of Banach spaces X such that every convex real-valued function defined on an open convex set $\Omega \subset X$ is Fréchet differentiable on a dense G_δ-set. Such spaces are now called *Asplund spaces* and can be characterized in the following way. A Banach space X is an Asplund space if and only if each of its separable subspaces has a separable dual (see Phelps (1989)). As an obvious consequence of the Mazur and Asplund results we get

Proposition 3. *Let* $(X, \|\cdot\|)$ *be either a separable Banach space or an Asplund space. Let* C *be an open convex set in* X. *Then* $\mathcal{S}(C)$ *is a dense* G_δ*-set in the boundary* ∂C.

Proof. Let $x_0 \in C$ and let $f(x) = \inf\{t > 0 : (x - x_0)/t \in C\}$ be the Minkowski norm induced by the set $C - x_0$. The function $f(x)$ is convex, and thus it is Gateaux differentiable on a dense G_δ-set C_f. Since $f(x)$ is positively homogeneous, $\partial C \cap C_f$ is a dense G_δ-set in ∂C. ∎

It is a natural question to which classes of sets Proposition 3 can be extended.

Let $\alpha(\cdot) : [0, +\infty) \to [0, +\infty]$ be a nondecreasing function such that

$$\lim_{t \downarrow 0} \frac{\alpha(t)}{t} = 0. \tag{2}$$

Let $f(\cdot)$ be a real-valued continuous function defined on an open convex subset $\Omega \subset X$. We say that $f(\cdot)$ is *strongly $\alpha(\cdot)$-paraconvex* (Rolewicz (2000)) if for all $x, y \in \Omega$ and $0 \le t \le 1$ we have

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \min[t, (1-t)]\alpha(\|x - y\|).$$

We say that an open set $C \subset X$ is *$\alpha(\cdot)$-intrinsically paraconvex* (cf. Ngai–Pénot (2008)) if for all $x, y \in C$,

$$\operatorname{dist}(tx + (1-t)y, C) \le t(1-t)\alpha(\|x - y\|).$$

Proposition 4. *Let $(X, \|\cdot\|)$ be either a separable Banach space or an Asplund space. Let C be an open bounded $\alpha(\cdot)$-intrinsically paraconvex set in X. Assume that there is $x_0 \in C$ such that C is starshaped with respect to x_0. Then $\mathcal{S}(C)$ is a dense G_δ-set in ∂C.*

Proof. Let $f(x) = \inf\{t > 0 : (x - x_0)/t \in C\}$. It is easy to see that there is $c > 0$ such that $f(\cdot)$ is strongly $c\alpha(\cdot)$-paraconvex. Thus it is Gateaux differentiable on a dense G_δ-set C_f (Rolewicz (2002), (2005a), (2006)). Since $f(x)$ is positively homogeneous, $\partial C \cap C_f$ is a residual subset of ∂C. ∎

Question 5. Is Proposition 4 valid without the assumption that C is starshaped with respect to some point?

By using Proposition 4 we can show this only for locally starshaped intrinsically paraconvex sets.

We say that an open set C is a *locally starshaped intrinsically paraconvex set* if for any $x_0 \in \partial C$, there are a neighbourhood U of x_0 and $\alpha(\cdot)$ satisfying (2) such that $U \cap C$ is an α-intrinsically paraconvex set starshaped with respect to some $x_1 \in U \cap C$.

Proposition 6. *Let $(X, \|\cdot\|)$ be either a separable Banach space or an Asplund space. Let C be an open locally starshaped intrinsically paraconvex set in X. Then $\mathcal{S}(C)$ is a dense G_δ-set in ∂C.*

Proof. Fix $x_0 \in \partial C$. By our assumption there are a neighbourhood U of x_0 and $\alpha(\cdot)$ satisfying (2) such that $U \cap C$ is an intrinsically paraconvex set starshaped with respect to some $x_1 \in U \cap C$. Thus by Proposition 4, $\mathcal{S}(U \cap C)$ is a dense G_δ-set in $\partial(U \cap C)$. This implies that $\mathcal{S}(C)$ is a locally

G_δ-set in ∂C. Therefore by the Michael theorem (Michael (1954)) it is a G_δ-set in ∂C. Since x_0 was arbitrary, $\mathcal{S}(C)$ is dense in ∂C. ■

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