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Almost everywhere convergence of Marcinkiewicz means of Fourier series on the group of 2-adic integers

by

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Abstract. We prove the almost everywhere convergence of the Marcinkiewicz means of integrable functions $\sigma_n f \to f$ for every $f \in L^1(I^2)$, where I is the group of 2-adic integers.

We apply the standard notions of dyadic analysis as introduced by F. Schipp, P. Simon, W. R. Wade (see e.g. [7]) and others. Set $\mathbb{N} := \{0, 1, \ldots\}$, $\mathbb{P} := \mathbb{N} \setminus \{0\}$, and $I := [0, 1)$. Denote by $\lambda(B) = |B|$ the Lebesgue measure of $B \subset I$, by $L^p(I)$ the usual Lebesgue spaces and $\|\cdot\|_p$ the corresponding norms ($1 \le p \le \infty$). Let

$$\mathcal{I} := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right) : p, n \in \mathbb{N} \right\}$$

be the set of dyadic intervals, and for given $x \in I$ let $I_n(x) \in \mathcal{I}$ denote the interval of length 2^{-n} which contains x ($n \in \mathbb{N}$). Also set $I_n := I_n(0)$ ($n \in \mathbb{N}$). Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in I$, where $x_n = 0$ or 1, and if x is a dyadic rational number ($x \in \{p/2^n : p, n \in \mathbb{N}\}$) we choose the expansion which terminates in 0's.

The 2-*adic* (or *arithmetic*) *sum* of $a, b \in I$ is $a + b := \sum_{n=0}^{\infty} r_n 2^{-(n+1)}$ where $q_n, r_n \in \{0, 1\}$ ($n \in \mathbb{N}$) are defined recursively as follows: $q_{-1} := 0$ and $a_n + b_n + q_{n-1} = 2q_n + r_n$ for $n \in \mathbb{N}$. (Since q_n, r_n take on only the values $0, 1$, these equations determine them uniquely.) The group $(I, +)$ is called the *group of* 2-*adic integers*. Set

$$\varepsilon(t) := \exp(2\pi \imath t) \qquad (t \in \mathbb{R}),$$

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where $\imath = (-1)^{1/2}$. Set

$$v_{2^n}(x) := \varepsilon\left(\frac{x_n}{2} + \cdots + \frac{x_0}{2^{n+1}}\right) \qquad (x \in I,\ n \in \mathbb{N})$$

and

$$v_n := \prod_{n=0}^{\infty} v_{2^j}^{n_j} \qquad \text{for } n = \sum_{i=0}^{\infty} n_i 2^i \ (n_i \in \{0,1\},\ i \in \mathbb{N}).$$

It is known [4] that the system $(v_n : n \in \mathbb{N})$ is the character system of $(I, +)$. Denote by

$$\widehat{f}(n) := \int_I f \overline{v}_n d\lambda, \qquad D_n := \sum_{k=0}^{n-1} v_k, \qquad K_n := \frac{1}{n} \sum_{k=0}^{n} D_k$$

the Fourier coefficients, the Dirichlet kernels and the Fejér kernels, respectively. We will also use the notation

$$K_{a,b} := \sum_{k=a}^{a+b-1} D_k.$$

It is known [5, 6, 1] that for $n \in \mathbb{N}$ and $x \in I$,

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n, \end{cases}$$

and also that

$$D_n(x) = v_n(x) \sum_{k=0}^{\infty} D_{2^k}(x) n_k (-1)^{x_k}.$$

Next we introduce some notation for two-dimensional Fourier series on the group of 2-adic integers. The normalized Haar measure is just as in the one-dimensional case.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Marcinkiewicz means and the Marcinkiewicz kernels are defined as follows:

$$\widehat{f}(n^1, n^2) := \int_{I^2} f(x^1, x^2) \overline{v}_{n^1}(x^1) \overline{v}_{n^2}(x^2)\, dx,$$

$$S_{n^1,n^2} f(x^1, x^2) := \sum_{k^1=0}^{n^1-1} \sum_{k^2=0}^{n^2-1} \widehat{f}(k^1, k^2) v_{k^1}(x^1) v_{k^2}(x^2),$$

$$\sigma_n f(x^1, x^2) := \frac{1}{n} \sum_{j=1}^{n} S_{j,j} f(x^1, x^2), \qquad K_n(x^1, x^2) = \frac{1}{n} \sum_{j=1}^{n} D_j(x^1) D_j(x^2).$$

It is well known that for $y \in I^2$,

$$\sigma_n f(y) = \int_{I^2} f(x) K_n(y-x)\, dx.$$

The next notation will prove very useful:

$$K_{a,b}(x^1, x^2) := \sum_{k=a}^{a+b-1} D_k(x^1) D_k(x^2).$$

THEOREM 1. *$\sigma_n f \to f$ for every $f \in L^1(I^2)$, where I is the group of 2-adic integers.*

This result for the trigonometric system was proved by Grünwald [2], and for some more general Nörlund means by Herriot [3]. See also the paper of Zhizhiashvili [8].

For $n, j \in \mathbb{N}$ let $n^{(j)} := \sum_{i=j}^{\infty} n_i 2^i$, for $2^B \le n < 2^{B+1}$ let $|n| := B$, and define $J_\tau := I_\tau \setminus I_{\tau+1}$.

LEMMA 2. *Let $t^1 \le t^2$. Then*

$$\sum_{t^1=0}^{m-1} \sum_{t^2=t^1}^{m-1} \int_{J_{t^1} \times J_{t^2}} \sup_{A \ge m} \sup_{|n|=A} \frac{1}{2^A} \sum_{s=t^1+1}^{A} |K_{n^{(s)},2^s}(x^1, x^2)|\, dx < c$$

where c is an absolute constant.

Proof. If $z \in J_\tau$ then

$$K_{n^{(s)},2^s}(z) = \sum_{k=n^{(s)}}^{n^{(s)}+2^s-1} v_k(z) \Big[\sum_{j=0}^{\tau-1} k_j 2^j + k_\tau 2^\tau (-1) \Big].$$

Define $\widetilde{k}_\tau := \sum_{j=0}^{\tau-1} k_j 2^j - k_\tau 2^\tau$. Then $|\widetilde{k}_\tau| \le 2^\tau$.

In the two-dimensional case, if $x^i \in J_{t^i}$ $(i = 1, 2)$ and $t^1 \le t^2$ then

$$K_{n^{(s)},2^s}(x^1, x^2) = \sum_{k=n^{(s)}}^{n^{(s)}+2^s-1} v_{k^{(t^1+1)}}(x^1) v_{k^{(t^2+1)}}(x^2) \widetilde{k}_{t^1} \widetilde{k}_{t^2}.$$

Assume $s > t^1$. Consider the following integral:

$$\begin{aligned}
\int_{J_\tau} v_k(x) \overline{v}_l(x)\, dx &= \int_{I_\tau \setminus I_{\tau+1}} \prod_{j=0}^{\tau-1} v_{2^j}^{k_j}(x) \overline{v}_{2^j}^{l_j}(x) (-1)^{k^\tau + l^\tau} \prod_{j=\tau+1}^{\infty} v_{2^j}^{k_j}(x) \overline{v}_{2^j}^{l_j}(x)\, dx \\
&= (-1)^{k^\tau + l^\tau} \int_{I_{\tau+1}(l_\tau)} v_{k^{\tau+1}}(x) \overline{v}_{l^{\tau+1}}(x)\, dx \\
&= \begin{cases} (-1)^{k_\tau + l_\tau} / 2^{\tau+1} & \text{if } k^{(\tau+1)} = l^{(\tau+1)}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

From the Cauchy–Buniakovski inequality we have

$$\int_{J_{t^1}\times J_{t^2}} |f| = \int_{I^2} 1_{J_{t^1}\times J_{t^2}} \cdot 1_{J_{t^1}\times J_{t^2}}|f| \le \|1_{J_{t^1}\times J_{t^2}}\|_2 \sqrt{\int_{I^2} 1^2_{J_{t^1}\times J_{t^2}}|f|}$$
$$= \sqrt{\frac{1}{2^{t^1+t^2}} \int_{J_{t^1}\times J_{t^2}} |f|^2}.$$

On the other hand, $\sup_n |K_{n^{(s)},2^s}(x^1,x^2)|$ depends only on $n_s, n_{s+1},\dots,n_A$ (for $|n| = A$ fixed).

Applying these facts we get

$$\int_{J_{t^1}\times J_{t^2}} \sup_n |K_{n^{(s)},2^s}(x^1,x^2)|\,dx \le \sqrt{\frac{1}{2^{t^1+t^2}} \int_{J_{t^1}\times J_{t^2}} \sup_n |K_{n^{(s)},2^s}(x^1,x^2)|^2\,dx}$$
$$\le \sqrt{\frac{1}{2^{t^1+t^2}} \int_{J_{t^1}\times J_{t^2}} \sum_{n_s,\dots,n_A\in\{0,1\}} |K_{n^{(s)},2^s}(x^1,x^2)|^2\,dx}$$
$$= \Bigg(\frac{1}{2^{t^1+t^2}} \sum_{n_s,\dots,n_A\in\{0,1\}} \sum_{k=n^{(s)}}^{n^{(s)}+2^s-1} \sum_{l=n^{(s)}}^{n^{(s)}+2^s-1} \int_{J_{t^1}\times J_{t^2}} v_{k^{(t^1+1)}}(x^1)\overline{v}_{l^{(t^1+1)}}(x^1)$$
$$\cdot v_{k^{(t^2+1)}}(x^2)\overline{v}_{l^{(t^2+1)}}(x^2)\widetilde{k}_{t^1}\widetilde{l}_{t^1}\widetilde{k}_{t^2}\widetilde{l}_{t^2}\,dx\Bigg)^{1/2}$$

These integrals are either zeros or can by estimated in absolute value by

$$\frac{1}{2^{t^1+t^2}} |\widetilde{k}_{t^1}\widetilde{l}_{t^1}\widetilde{k}_{t^2}\widetilde{l}_{t^2}| \le 2^{t^1+t^2}.$$

The latter happens if $k^{(t^1+1)} = l^{(t^1+1)}$ and $k^{(t^2+1)} = l^{(t^2+1)}$, that is, exactly in case $k^{(t^1+1)} = l^{(t^1+1)}$, because $t^1 \le t^2$.

Let us count the number of such (k,l) pairs. Since $k \in [n^{(s)}, n^{(s)} + 2^s)$ and $k^{(t^1+1)} = l^{(t^1+1)}$, for every k (their number is 2^s) the number of l's satisfying the condition above is 2^{t^1+1}. This yields altogether 2^{s+t^1+1} pairs (k,l).

So we get

$$\int_{J_{t^1}\times J_{t^2}} \sup_n |K_{n^{(s)},2^s}(x^1,x^2)|\,dx$$
$$\le \Bigg(\frac{1}{2^{t^1+t^2}} \sum_{n_s,\dots,n_A\in\{0,1\}} 2^{s+t^1+1}2^{t^1+t^2}\Bigg)^{1/2} = (2^{s+t^1+1}2^{A-s+1})^{1/2} \le c\sqrt{2^{A+t^1}}.$$

Using this inequality it follows that

$$\sum_{t^1=0}^{m-1}\sum_{t^2=t^1}^{m-1}\int_{J_{t^1}\times J_{t^2}}\sup_{A\ge m}\sup_{|n|=A}\frac{1}{2^A}\sum_{s=t^1+1}^{A}|K_{n^{(s)},2^s}(x^1,x^2)|\,dx$$
$$\le c\sum_{t^1=0}^{m-1}\sum_{t^2=t^1}^{m-1}\sum_{A=m\vee t^2}^{\infty}\sum_{s=t^1+1}^{A}\sqrt{2^{t^1-A}}\le c\sum_{t^1=0}^{m-1}\sum_{t^2=t^1}^{m-1}\sum_{A=m\vee t^2}^{\infty}(A-t^1)\sqrt{2^{t^1-A}}$$
$$\le c\sum_{t^1=0}^{m-1}\sum_{t^2=t^1}^{m-1}(m\vee t^2-t^1)\sqrt{2^{t^1-m}}\le c\sum_{t^1=0}^{m-1}(m-t^1)^2\sqrt{2^{t^1-m}}\le c.$$

Moreover,

$$\sum_{t^1=0}^{m-1}\sum_{t^2=m+1}^{\infty}\int_{J_{t^1}\times J_{t^2}}\sup_{t^2>A\ge m}\sup_{|n|=A}\frac{1}{2^A}\sum_{s=t^1+1}^{A}|K_{n^{(s)},2^s}(x^1,x^2)|\,dx$$
$$\le\sum_{t^1=0}^{m-1}\sum_{t^2=m+1}^{\infty}\frac{1}{2^{t^2}}\int_{J_{t^1}}\sup_{t^2>A\ge m}\sup_{|n|=A}\frac{1}{2^A}\sum_{s=t^1+1}^{A}|K_{n^{(s)},2^s}(x^1,0)|\,dx^1$$
$$\le c\sum_{t^1=0}^{m-1}\sum_{t^2=m+1}^{\infty}\frac{1}{2^{t^2}}\sum_{A=m}^{t^2}\frac{1}{2^A}\sum_{s=t^1}^{A}\int_{J_{t^1}}\Big|\sum_{k=n^{(s)}}^{n^{(s)}+2^s-1}v_{k^{(t^1+1)}}(x^1)\widetilde{k}_{t^1}k\,dx^1\Big|$$
$$\le c\sum_{t^1=0}^{m-1}\sum_{t^2=m+1}^{\infty}\frac{1}{2^{t^2}}\sum_{A=m}^{t^2}\frac{1}{2^A}\sum_{s=t^1}^{A}\left(\frac{1}{2^{t^1}}\right)^{1/2}\left(\frac{1}{2^{t^1}}\,2^{s+t^1+2t^1+2A}\right)^{1/2}$$
$$\le c\sum_{t^1=0}^{m-1}\sum_{t^2=m+1}^{\infty}\frac{1}{2^{t^2}}\sum_{A=m}^{t^2}\sqrt{2^{A+t^1}}\le c\sum_{t^1=0}^{m-1}\sum_{t^2=m+1}^{\infty}\sqrt{2^{t^1-t^2}}\le c.\ \blacksquare$$

Denote by $\overline{I_m^2}$ the complement of I_m^2.

LEMMA 3.

$$\int_{\overline{I_m^2}}\sup_{|n|\ge m}|K_n|\le c.$$

Proof. Using trivial estimations we get

$$|K_{n^{(s)},2^s}|\le\sum_{k=n^{(s)}}^{n^{(s)}+2^s-1}|D_k|\le\sum_{k=n^{(s)}}^{n^{(s)}+2^s-1}|\widetilde{k}_{t^1}\widetilde{k}_{t^2}|\le 2^{s+t^1+t^2\wedge|n|},$$

so

$$\sum_{s=0}^{t^1}|K_{n^{(s)},2^s}|\le c2^{2t^1+t^2\wedge|n|}.$$

From this inequality we obtain

$$\begin{aligned}
&\sum_{t^1=0}^{m-1}\sum_{t^2=t^1}^{m-1}\int_{J_{t^1}\times J_{t^2}}\sup_{A\ge m}\sup_{|n|=A}\frac{1}{2^A}\sum_{s=0}^{t^1}|K_{n^{(s)},2^s}(x^1,x^2)|\,dx\\
&\qquad\le c\sum_{t^1=0}^{m-1}\sum_{t^2=t^1}^{m-1}\int_{J_{t^1}\times J_{t^2}}\sup_{A\ge m}\frac{2^{2t^1+t^2\wedge A}}{2^A}\,dx\\
&\qquad\le c\sum_{t^1=0}^{m-1}\sum_{t^2=t^1}^{m-1}\frac{1}{2^{t^1+t^2}}\frac{2^{2t^1+t^2}}{2^m}+c\sum_{t^1=0}^{m-1}\sum_{t^2=m}^{\infty}\frac{1}{2^{t^1+t^2}}2^{2t^1}\\
&\qquad\le c\sum_{t^1=0}^{m-1}(m-t^1)2^{t^1-m}+c\sum_{t^1=0}^{m-1}\sum_{t^2=m}^{\infty}2^{t^1-t^2}\le c
\end{aligned}$$

and from this

$$\sum_{t^1=0}^{m-1}\sum_{t^2=t^1}^{m-1}\int_{J_{t^1}\times J_{t^2}}\sup_{|n|\ge m}|K_n|\le c.$$

Similarly,

$$\sum_{t^2=0}^{m-1}\sum_{t^1=t^2}^{m-1}\int_{J_{t^1}\times J_{t^2}}\sup_{|n|\ge m}|K_n|\le c.$$

Because, a.e.,

$$\overline{I_m^2}\subseteq\Big(\bigcup_{t^2=0}^{m-1}\bigcup_{t^1=t^2}^{m-1}J_{t^1}\times J_{t^2}\Big)\cup\Big(\bigcup_{t^1=0}^{m-1}\bigcup_{t^2=t^1}^{m-1}J_{t^1}\times J_{t^2}\Big)$$

the inequalities above imply

$$\int_{\overline{I_m^2}}\sup_{|n|\ge m}|K_n|\le c. \blacksquare$$

This lemma leads to the following corollary:

Corollary 4.

$$\|K_n\|_1\le c\quad \textit{for all } n\in\mathbb{N}.$$

The next step is to prove the quasi-local property of the maximal operator $\sigma^*f:=\sup_n|\sigma_nf|$.

Lemma 5. *Let* $f\in L^1(I^2)$ *with* $\int f=0$ *and* $\operatorname{supp} f\subseteq I_m(u^1)\times I_m(u^2)$ *for some* $u=(u^1,u^2)\in I^2$. *Then*

$$\int_{\overline{I_m(u^1)\times I_m(u^2)}}\sigma^*f\le c\|f\|_1.$$

Proof. From the shift invariance of the Haar measure we can suppose that $u^1 = u^2 = 0$.

In the case of $|n| < m$ the situation is simple, because

$$\sigma_n f(y) = \int_{I^2} f(\cdot) K_n(y - \cdot) = K_n(y) \int_{I_m^2} f = 0.$$

So we can suppose that $|n| \geq m$. In this case

$$\int_{I_m^2} \sigma^* f = \int_{I_m^2} \sup_{|n| \geq m} \left| \int_{I_m^2} f(x) K_n(\cdot, x)\, dx \right|$$

$$\leq \int_{I_m^2} |f(x)| \int_{I_m^2} \sup_{|n|\geq m} |K_n| \leq c\|f\|_1. \ \blacksquare$$

Proof of Theorem 1. From Corollary 4 we deduce that the maximal operator σ^* is of type (∞, ∞). Lemma 5 shows that this sublinear operator is quasi-local. Using the standard method (see e.g. [7]) it follows that it is of weak type $(1, 1)$. We get the statement of the theorem from the density of the 2-adic polynomials in $L^1(I^2)$. The proof of Theorem 1 is complete. $\blacksquare$

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