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Lie algebras generated by Jordan operators

by

PENG CAO and SHANLI SUN (Beijing)

Abstract. It is proved that if J_i is a Jordan operator on a Hilbert space with the Jordan decomposition $J_i = N_i + Q_i$, where N_i is normal and Q_i is compact and quasinilpotent, $i = 1, 2$, and the Lie algebra generated by J_1, J_2 is an Engel Lie algebra, then the Banach algebra generated by J_1, J_2 is an Engel algebra. Some results for normal operators and Jordan operators on Banach spaces are given.

1. Introduction. Let $\mathcal{X}$ be a Banach space, and $\mathscr{H}$ be a Hilbert space. The Banach algebra of all bounded linear operators on $\mathcal{X}$ is denoted by $\mathbf{B}(\mathcal{X})$. It is a Lie algebra with the Lie product $[T_1, T_2] = T_1T_2 - T_2T_1$ for $T_1, T_2 \in \mathbf{B}(\mathcal{X})$. Let $M \subset \mathbf{B}(\mathcal{X})$. The Lie algebra generated by M, denoted by $\varepsilon(M)$, is the smallest Lie algebra containing M. Let $\mathcal{A}(M)$ denote the associative algebra generated by M, $\overline{M}$ the closure of M in $\mathbf{B}(\mathcal{X})$, and $\mathrm{Ker}(T)$ the kernel of an operator T.

Recall that $T \in \mathbf{B}(\mathcal{X})$ is *hermitian* if $\|\exp(itT)\| = 1$ for every $t \in \mathbb{R}$, and *hermitian-equivalent* if $\sup_{t\in\mathbb{R}} \|\exp(itT)\| < \infty$. An operator $N \in \mathbf{B}(\mathcal{X})$ is *normal* (resp., *normal-equivalent*) if $N = A + iB$, where A, B are hermitian (resp., hermitian-equivalent), and $[A, B] = 0$. Some basic properties of hermitian operators can be found in [8]; for hermitian-equivalent operators, see [4], [1].

We will also make use of the theory of decomposable operators. One can find the concepts of "spectral operator", "scalar operator", "generalized scalar operator", and "decomposable operator" in [6]. We will denote by $\mathcal{B}_{\mathrm{ad}\,N}(F)$ the maximal spectral subspace of $\mathbf{B}(\mathcal{X})$ associated with $\mathrm{ad}\,N$ and a closed subset $F \subset \mathbb{C}$. It is well known that if N is a normal-equivalent operator, then so is $\mathrm{ad}\,N$ by [4, §14, Proposition 4]. Then N and $\mathrm{ad}\,N$ are generalized scalar operators by [4, §14, Remark 6]. The following lemmas are useful.

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Lemma 1.1 ([4, §14, Theorem 6]). *If* $N \in \mathbf{B}(\mathcal{X})$ *is normal-equivalent and* $\sigma(N) = \{0\}$, *then* $N = 0$.

Lemma 1.2 ([4, §14, Corollary 4]). *If* $N \in \mathbf{B}(\mathcal{X})$ *is normal-equivalent, then*

$$\mathcal{B}_N(\{\lambda\}) = \operatorname{Ker}(N - \lambda I) = \operatorname{Ker}(A - \operatorname{Re}\lambda) = \operatorname{Ker}(B - \operatorname{Im}\lambda)$$

for each λ *in* $\sigma(N)$.

If N is normal-equivalent, then by [6, Lemma 4.4.4] we have

Lemma 1.3.

$$\mathcal{B}_{\operatorname{ad} N}(\{0\}) = \{S \in \mathbf{B}(\mathscr{H}) \mid \lim_{n\to\infty} \|(\operatorname{ad} N)^n S\|^{1/n} = 0\}.$$

We can also define an involution on normal-equivalent operators in $\mathbf{B}(\mathcal{X})$, namely for a normal-equivalent operator $N \in \mathbf{B}(\mathcal{X})$ with $N = A+iB$, define $N^* = A - iB$.

Lemma 1.4 ([2, Theorem]). *If* $N_1, N_2 \in \mathbf{B}(\mathcal{X})$ *are normal-equivalent and* $N_1 S = SN_2$ *for some* $S \in \mathbf{B}(\mathcal{X})$, *then* $N_1^* S = SN_2^*$.

For a Lie algebra $\mathcal{L} \subset \mathbf{B}(\mathcal{X})$, $\overline{\mathcal{L}}$ is the closure of $\mathcal{L}$ in $\mathbf{B}(\mathcal{X})$. For every $T \in \mathcal{L}$, $\operatorname{ad} T : \mathcal{L} \to \mathcal{L}$ is quasinilpotent (i.e., $\lim \|(\operatorname{ad} T)^n\|^{1/n} = 0$) if and only if $\operatorname{ad} T : \overline{\mathcal{L}} \to \overline{\mathcal{L}}$ is quasinilpotent if and only if $\|(\operatorname{ad} T)^n(S)\|^{1/n} \to 0$ as $n \to \infty$ for any $S \in \overline{\mathcal{L}}$. A *Volterra operator* is a compact quasinilpotent operator.

Recall that if $\mathcal{L}$ is a normed Lie algebra, then $\mathcal{L}$ is called *ad-compact* if the operator $\operatorname{ad} a$ on $\mathcal{L}$ is compact for every $a \in \mathcal{L}$; $\mathcal{L}$ is an *Engel Lie algebra* if $\operatorname{ad} a$ is quasinilpotent for every $a \in \mathcal{L}$; and $\mathcal{L}$ is called *E-solvable* if every nonzero quotient of $\mathcal{L}$ by a closed ideal has a nonzero Engel ideal. Every finite-dimensional E-solvable Lie algebra is solvable by [13, Theorem 6.19].

2. Jordan operators. Recall that $T \in \mathbf{B}(\mathcal{X})$ is a *Jordan operator* if there exists a normal-equivalent operator $S \in \mathbf{B}(\mathcal{X})$ and a quasinilpotent operator $Q \in \mathbf{B}(\mathcal{X})$ such that $[S, Q] = 0$ and $T = S + Q$. This last formula is called the *Jordan decomposition* of T. It is known that every Jordan operator is a completely regular generalized spectral operator (cf. [4, §14, Corollary 5]).

Lemma 2.1 ([4, §14, Theorem 7]). *Let* $T \in \mathbf{B}(\mathcal{X})$ *be a Jordan operator with the Jordan decomposition* $T = S + Q$. *Moreover let* $A, B \in \mathbf{B}(\mathcal{X})$ *be hermitian-equivalent operators such that* $S = A + iB$ *and* $[A, B] = 0$. *Then* A, B *and* Q *belong to the bicommutant of* T. *Moreover*, $\mathcal{B}_{\operatorname{ad} T}(F) = \mathcal{B}_{\operatorname{ad} S}(F)$ *for every closed subset* F *of* $\mathbb{C}$.

The last statement of Lemma 2.1 is in the proof of [4, §14, Theorem 7].

Recall that if $\mathcal{L}$ is a Lie algebra, then $\mathcal{A}(\mathcal{L})$ denotes the associative algebra generated by $\mathcal{L}$, $\overline{\mathcal{A}(\mathcal{L})}$ the Banach algebra generated by $\mathcal{L}$, and $\mathrm{Rad}(\overline{\mathcal{A}(\mathcal{L})})$ the Jacobson radical of $\overline{\mathcal{A}(\mathcal{L})}$. A Banach algebra $\mathcal{B}$ is called *Engel* if $\mathrm{ad}\, a : \mathcal{B} \to \mathcal{B}$ is quasinilpotent for every $a \in \mathcal{B}$. It is well known that if $\mathcal{B}$ is an Engel algebra, then $\mathcal{B}/\mathrm{Rad}\,\mathcal{B}$ is commutative (see [13, Proposition 5.21] or [3, Proposition]).

The following lemma can be found in [10].

LEMMA 2.2. *If A is a Banach algebra and $A/\mathrm{Rad}\, A$ is commutative, then $Q_A = \mathrm{Rad}(A)$, where Q_A is the set of all quasinilpotent elements in A.*

The following lemma can be found in [14].

LEMMA 2.3. *If $\mathcal{L} \subset \mathbf{B}(\mathcal{X})$ is a nilpotent (or finite-dimensional solvable) Lie algebra, then $\overline{\mathcal{A}(\mathcal{L})}/\mathrm{Rad}(\overline{\mathcal{A}(\mathcal{L})})$ is commutative.*

If $\mathcal{L}$ is an Engel Lie algebra, will $\overline{\mathcal{A}(\mathcal{L})}$ be an Engel algebra? This question was posed by Yu. V. Turovskiĭ and V. S. Shulman in [16], and a partial answer can be found in [13].

LEMMA 2.4 ([13, Theorem 5.22]). *Let $\mathcal{L} \subset \mathcal{K}^1(\mathcal{X})$ be an Engel Lie algebra, where $\mathcal{K}^1(\mathcal{X})$ is the linear space generated by the compact operators and the identity operator on $\mathcal{X}$. Then $\overline{\mathcal{A}(\mathcal{L})}$ is an Engel algebra.*

Now we begin the study of Lie algebras generated by Jordan operators.

First, note that if a Lie algebra generated by normal operators in a Hilbert space is finite-dimensional solvable, then it is commutative ([5, Theorem 2.1]). We will generalize this result to normal operators on Banach spaces. The following proposition is the pivotal step.

For two hermitian operators N_1, N_2, $i[N_1, N_2]$ is a hermitian operator. In fact, since

$$\exp([N_1, N_2]) = \lim_{n\to\infty} \left(\exp\left(\frac{1}{n}N_1\right) \exp\left(\frac{1}{n}N_2\right) \exp\left(-\frac{1}{n}N_1\right) \exp\left(-\frac{1}{n}N_2\right)\right)^{n^2},$$

we have $\|\exp(t[N_1, N_2])\| = \|\exp(-t[iN_1, iN_2])\| \le 1$ for every $t \in \mathbb{R}$. But $1 = \|\exp(t[N_1, N_2]) \exp(-t[N_1, N_2])\| \le 1$, so $\|\exp(t[N_1, N_2])\| = 1$ for every $t \in \mathbb{R}$. Hence $i[N_1, N_2]$ is hermitian (cf. [4, §14, Remark 5]).

PROPOSITION 2.1. *If $N_1, N_2 \in \mathbf{B}(\mathcal{X})$ are hermitian, and $\varepsilon(N_1, N_2)$ is an ad-compact E-solvable Lie algebra, then $N_1 N_2 = N_2 N_1$.*

Proof. Let $\mathcal{L} = \varepsilon(N_1, N_2)$. Because $\mathcal{L}$ is ad-compact E-solvable, $[\mathcal{L}, \mathcal{L}]$ is an Engel Lie algebra by [13, Theorem 6.15]. So $\mathrm{ad}(i[N_1, N_2])$ is quasinilpotent on $[\mathcal{L}, \mathcal{L}]$. Since $i[N_1, N_2]$ is hermitian, $\mathrm{ad}(i[N_1, N_2])$ is normal. By

Lemma 1.3, $[\mathcal{L},\mathcal{L}] \subset \mathcal{B}_{\mathrm{ad}(i[N_1,N_2])}(\{0\})$, so $\mathrm{ad}([N_1,N_2]) = 0$ on $[\mathcal{L},\mathcal{L}]$ by Lemma 1.2. Because $[[N_1,N_2],N_1] \in [\mathcal{L},\mathcal{L}]$, we have

$$\mathrm{ad}([N_1,N_2])([[N_1,N_2],N_1]) = \mathrm{ad}^2([N_1,N_2])(N_1) = 0.$$

Now $N_1 \in \mathcal{B}_{\mathrm{ad}([N_1,N_2])}(\{0\})$, by Lemma 1.3. So $\mathrm{ad}([N_1,N_2])(N_1) = 0$ by Lemma 1.2. That is, $\mathrm{ad}^2(N_1)(N_2) = 0$. Again $N_2 \in \mathcal{B}_{\mathrm{ad}\,N_1}(\{0\})$, so that $[N_1,N_2] = 0$. ∎

Corollary 2.1. *If $N_1,N_2 \in \mathbf{B}(\mathcal{X})$ are hermitian, and $\varepsilon(N_1,N_2)$ is a finite-dimensional solvable Lie algebra, then $N_1N_2 = N_2N_1$.*

Proof. This is evident by Proposition 2.1 and [13, Theorem 6.19]. ∎

Theorem 2.1. *If $N_1,N_2 \in \mathbf{B}(\mathcal{X})$ are normal, and $\varepsilon(N_1,N_2)$ is finite-dimensional solvable, then it is commutative, that is, $[N_1,N_2] = 0$.*

Proof. With Lemma 1.4 replacing the Fuglede–Putnam theorem in Hilbert space, the proof is similar to the proof of Theorem 2.1 in [15], so we omit it. See [5] for the details. ∎

Lemma 2.5. *For Jordan operators J_i with the Jordan decomposition $J_i = N_i + Q_i$, $i = 1,2$, if $\varepsilon(J_1,J_2)$ is an Engel Lie algebra, then $[N_1,N_2] = 0$, $[N_1,Q_2] = 0$ and $[N_2,Q_1] = 0$.*

Proof. Since $\varepsilon(J_1,J_2)$ is an Engel Lie algebra, $\mathrm{ad}\,J_1$ is a quasinilpotent operator on $\varepsilon(J_1,J_2)$. So $J_2 \in \mathcal{B}_{\mathrm{ad}\,J_1}(\{0\})$. By Lemma 2.1, $J_2 \in \mathcal{B}_{\mathrm{ad}\,N_1}(\{0\})$, and by Lemma 1.2, $[N_1,J_2] = 0$. Note that J_2 is a Jordan operator, so $[N_1,N_2] = 0$ and $[N_1,Q_2] = 0$ by Lemma 2.1. Similarly, $[N_2,Q_1] = 0$. ∎

It is known that if G is a finite-dimensional solvable Lie algebra in $\mathbf{B}(\mathcal{X})$, then the set of Jordan operators in G is an ideal of G (cf. [4, §28, Theorem 3]). Now we will give another property of Jordan operators in finite-dimensional solvable Lie algebras.

Proposition 2.2. *For Jordan operators J_i with the Jordan decomposition $J_i = N_i + Q_i$, $i = 1,2$, suppose that N_1,N_2 are normal operators, and the Lie algebra $\varepsilon(J_1,J_2)$ is finite-dimensional solvable. Then $[N_1,N_2] = 0$ and $Q_1,Q_2 \in \mathrm{Rad}(\overline{\mathcal{A}(\{N_1,N_2,Q_1,Q_2\})})$.*

Proof. Let $\mathcal{L} = \varepsilon(J_1,J_2)$ and $\{\lambda_1,\ldots,\lambda_n\} = \sigma(\mathrm{ad}_{\mathcal{L}}\,J_1)$, so $\mathrm{ad}_{\mathcal{L}}\,J_1$ has finite rank and $\mathcal{L} = \mathcal{B}_{\mathrm{ad}_{\mathcal{L}}\,J_1}(\lambda_1) \oplus \cdots \oplus \mathcal{B}_{\mathrm{ad}_{\mathcal{L}}\,J_1}(\lambda_n)$. By Lemmas 2.1 and 1.2, we have

$$\mathcal{L} = \mathrm{Ker}(\mathrm{ad}_{\mathcal{L}}\,N_1 - \lambda_1) \oplus \cdots \oplus \mathrm{Ker}(\mathrm{ad}_{\mathcal{L}}\,N_1 - \lambda_n).$$

It is easy to see that $\mathrm{span}(N_1,\mathcal{L})$ is a Lie algebra and

$$[\mathrm{span}(N_1,\mathcal{L}),\mathrm{span}(N_1,\mathcal{L})] \subset [\mathcal{L},\mathcal{L}].$$

Since $\mathcal{L}$ is finite-dimensional solvable, it follows that $[\mathcal{L},\mathcal{L}]$ is nilpotent, hence also $[\mathrm{span}(N_1,\mathcal{L}),\mathrm{span}(N_1,\mathcal{L})]$ is nilpotent, and therefore $\mathrm{span}(N_1,\mathcal{L})$ is finite-dimensional solvable. Similarly, $\mathrm{span}(N_1, N_2, \mathcal{L})$ is finite-dimensional solvable. So also is $\overline{\varepsilon(N_1, N_2)}$. Therefore, $[N_1, N_2] = 0$ by Theorem 2.1. Since $Q_1, Q_2 \in \overline{\mathrm{span}(N_1, N_2, \mathcal{L})}$, Lemmas 2.2 and 2.3 show that $Q_1, Q_2 \in \mathrm{Rad}(\overline{\mathcal{A}(\{N_1, N_2, Q_1, Q_2\})})$. ∎

Recall that for $T \in \mathcal{L}$, $\mathrm{ad}\, T : \mathcal{L} \to \mathcal{L}$ is quasinilpotent if and only if $\mathrm{ad}\, T : \overline{\mathcal{L}} \to \overline{\mathcal{L}}$ is quasinilpotent if and only if $\|(\mathrm{ad}\, T)^n(S)\|^{1/n} \to 0$ as $n \to \infty$ for any $S \in \overline{\mathcal{L}}$. A *Volterra ideal* is an ideal consisting of Volterra operators.

Recall that a Jordan operator J on a Hilbert space $\mathscr{H}$ has the Jordan decomposition $J = N + Q$, where $N \in \mathbf{B}(\mathscr{H})$ is normal, $Q \in \mathbf{B}(\mathscr{H})$ is quasinilpotent and $[N, Q] = 0$.

THEOREM 2.2. *For Jordan operators J_i on a Hilbert space with the Jordan decomposition $J_i = N_i + Q_i$, $i = 1, 2$, suppose that $\overline{\varepsilon(J_1, J_2)}$ is an Engel Lie algebra, and Q_1, Q_2 are Volterra operators. Then $\overline{\mathcal{A}(\{N_1, N_2, Q_1, Q_2\})}$ is an Engel algebra, as also is $\overline{\mathcal{A}(\varepsilon(J_1, J_2))}$.*

Proof. By Lemma 2.5, $[N_1, N_2] = 0$, $[N_1, Q_2] = 0$ and $[N_2, Q_1] = 0$. Let $\mathcal{L} = \mathrm{span}(N_1, N_2, \varepsilon(J_1, J_2))$.

CLAIM 1. *$\mathcal{L}$ is an Engel Lie algebra.*

Note that $[N_1, J_1] = 0 = [N_1, J_2]$, so $\mathrm{ad}\, N_1 = 0$ on $\mathcal{L}$. Similarly, $\mathrm{ad}\, N_2 = 0$ on $\mathcal{L}$. For every T in $\varepsilon(J_1, J_2)$, since $\overline{\varepsilon(J_1, J_2)}$ is an Engel Lie algebra, $\lim_{n\to\infty} \|(\mathrm{ad}\, T)^n(S)\|^{1/n} = 0$ for every $S \in \overline{\varepsilon(J_1, J_2)}$. As

$$\overline{\mathcal{L}} = \mathrm{span}(N_1, N_2, \overline{\varepsilon(J_1, J_2)}),$$

it follows that $\lim_{n\to\infty} \|(\mathrm{ad}\, T)^n(S)\|^{1/n} = 0$ for every $S \in \overline{\mathcal{L}}$. That is, $\mathrm{ad}\, T$ is a quasinilpotent operator on $\mathcal{L}$. For every $T' \in \mathcal{L}$ we have $T' = \lambda_1 N_1 + \lambda_2 N_2 + T$, where $T \in \varepsilon(J_1, J_2)$. So $\mathrm{ad}\, T' = \mathrm{ad}\, T$ on $\mathcal{L}$ is a quasinilpotent operator. That is, $\mathcal{L}$ is an Engel Lie algebra.

CLAIM 2. *$\overline{\mathcal{A}(\varepsilon(Q_1, Q_2))}$ consists of quasinilpotent operators.*

$\varepsilon(Q_1, Q_2)$ is an Engel Lie algebra since it is contained in $\mathcal{L}$. As Q_1, Q_2 are compact operators, so is every operator in $\overline{\varepsilon(Q_1, Q_2)}$. So by [13, Theorem 5.22], $\overline{\mathcal{A}(\varepsilon(Q_1, Q_2))}$ is an Engel algebra. Hence $\overline{\mathcal{A}(\varepsilon(Q_1, Q_2))}$ consists of quasinilpotent operators by Lemmas 2.2 and 2.3.

Let $\mathcal{A} = \mathcal{A}(N_1, N_2, Q_1, Q_2)$.

CLAIM 3. *$I := \overline{\mathrm{span}(p(N_1, N_2)q(Q_1, Q_2))}$ is a Volterra ideal of $\overline{\mathcal{A}}$, where $p(x_1, x_2)$ and $q(x_1, x_2)$ run through polynomials such that $q(0, 0) = 0$.*

First, we show that I is an ideal of $\overline{\mathcal{A}}$. Note that $[N_1, N_2] = 0 = [N_i, Q_j]$ for every $i, j = 1, 2$. Every $S \in \mathcal{A}$ has the form $\sum_{i=1}^{n} p_i(N_1, N_2) q_i(Q_1, Q_2)$, where $p_i(x_1, x_2), q_i(x_1, x_2)$ are polynomials and $p_i(0, 0) q_i(0, 0) = 0$. So it is easy to see that $SI, IS \subset I$. Note that I is closed, so $\overline{\mathcal{A}}I, I\overline{\mathcal{A}} \subset I$. That is, I is an ideal of $\overline{\mathcal{A}}$.

Next we show that I consists of Volterra operators. Note that an operator $p(N_1, N_2)q(Q_1, Q_2) \in I$ is compact and quasinilpotent, since $q(Q_1, Q_2)$ is compact and quasinilpotent, and $[p(N_1, N_2), q(Q_1, Q_2)] = 0$. For any n operators $p_i(N_1, N_2)q_i(Q_1, Q_2) \in I$, where $q_i(0, 0) = 0$, $i = 1, \dots, n$, since $\overline{\mathcal{A}(Q_1, Q_2)}$ consists of quasinilpotent operators and $[\mathcal{A}(N_1, N_2), \mathcal{A}(Q_1, Q_2)] = \{0\}$, the semigroup generated by $p_i(N_1, N_2)q_i(Q_1, Q_2)$ consists of Volterra operators. Hence so does the algebra generated by $p_i(N_1, N_2)q_i(Q_1, Q_2)$, $i = 1, \dots, n$ ([15, Theorem 4]). That is, $\operatorname{span}(p(N_1, N_2)q(Q_1, Q_2))$ consists of Volterra operators, where $p(x_1, x_2), q(x_1, x_2)$ run through polynomials such that $q(0, 0) = 0$.

Finally, since the limit of Volterra operators is a Volterra operator, I consists of Volterra operators.

By the definition of Jacobson radical, $I \subset \operatorname{Rad}(\overline{\mathcal{A}})$, so $Q_1, Q_2 \in \operatorname{Rad}(\overline{\mathcal{A}})$.

Claim 4. *For every $S \in \mathcal{A}$, $\operatorname{ad} S : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$ is quasinilpotent.*

Since $S = \sum_{i=1}^{n} p_i(N_1, N_2)q_i(Q_1, Q_2)$, where $p_i(x_1, x_2), q_i(x_1, x_2)$ are polynomials and $p_i(0, 0)q_i(0, 0) = 0$, we have $S = S_1 + S_2$, where $S_1 \in \mathcal{A}(N_1, N_2)$ and $S_2 \in I$. Note that $\operatorname{ad} N_i(\overline{\mathcal{A}}) = \{0\}$, and I consists of Volterra operators by Claim 3, so $\operatorname{ad} S = \operatorname{ad} S_2$ is a quasinilpotent operator on $\overline{\mathcal{A}}$, by Rosenblum's theorem.

Claim 5. *$\overline{\mathcal{A}} \subset C^*(N_1, N_2) + \mathcal{K}(\mathscr{H})$, where $C^*(N_1, N_2)$ is the C^*-algebra generated by N_1, N_2, and $\mathcal{K}(\mathscr{H})$ is the set of compact operators on $\mathscr{H}$.*

It is well known that the sum of a C^*-algebra and $\mathcal{K}(\mathscr{H})$ is closed in $\mathbf{B}(\mathscr{H})$ [7]. So $C^*(N_1, N_2) + \mathcal{K}(\mathscr{H})$ is closed. Note that $\mathcal{A} \subset \mathcal{A}(N_1, N_2) + I$, and I consists of Volterra operators, so $\mathcal{A} \subset C^*(N_1, N_2) + \mathcal{K}(\mathscr{H})$. As $C^*(N_1, N_2) + \mathcal{K}(\mathscr{H})$ is closed, it follows that $\overline{\mathcal{A}} \subset C^*(N_1, N_2) + \mathcal{K}(\mathscr{H})$.

Now, for every $S' \in \overline{\mathcal{A}}$, by Claim 5, there exist $N' \in C^*(N_1, N_2)$ and $Q' \in \mathcal{K}(\mathscr{H})$, such that $S' = N' + Q'$. Note that $[N_i, \overline{\mathcal{A}}] = \{0\}$ and N_i is normal, so $[N_i^*, \overline{\mathcal{A}}] = \{0\}$, $i = 1, 2$. Therefore, $\operatorname{ad} S' = \operatorname{ad} N' + \operatorname{ad} Q' = \operatorname{ad} Q'$ on $\overline{\mathcal{A}}$. But Q' is a compact operator, so $\operatorname{ad} Q' = \operatorname{ad} S'$ has countable spectrum on $\overline{\mathcal{A}}$. Since $S' \in \overline{\mathcal{A}}$, there is a sequence $\{S_n\} \subset \mathcal{A}$ such that $\lim_{n\to\infty} S_n = S'$. Hence, $\lim_{n\to\infty} \operatorname{ad} S_n = \operatorname{ad} S'$. By Claim 4, $\operatorname{ad} S_n$ is quasinilpotent on $\overline{\mathcal{A}}$; as $\operatorname{ad} S'$ has countable spectrum on $\overline{\mathcal{A}}$, it is quasinilpotent on $\overline{\mathcal{A}}$ by Newburgh's result [11]. That is, $\overline{\mathcal{A}}$ is an Engel algebra.

The last statement of the theorem is clear because $\overline{\mathcal{A}(\varepsilon(J_1, J_2))}$ is a subalgebra of $\overline{\mathcal{A}}$. ∎

Now we turn to Jordan operators on Banach spaces. We adopt the notation of the proof of Theorem 2.2.

COROLLARY 2.2. *For Jordan operators J_i with the Jordan decomposition $J_i = N_i + Q_i$, $i = 1, 2$, suppose that $\varepsilon(J_1, J_2)$ is an Engel Lie algebra, and Q_1, Q_2 are Volterra operators. Then $\overline{\mathcal{A}}/\mathrm{Rad}(\overline{\mathcal{A}})$ is commutative.*

Proof. Note that Claim 3 in the proof of Theorem 2.2 holds for Jordan operators on Banach spaces, and $\mathcal{A} \subset \mathcal{A}(N_1, N_2) + I$ with $[N_i, \overline{\mathcal{A}}] = \{0\}$, $i = 1, 2$. So $[\mathcal{A}, \mathcal{A}] \subset I \subset \mathrm{Rad}(\overline{\mathcal{A}})$ by Claim 3. As $\mathrm{Rad}(\overline{\mathcal{A}})$ is closed, also $[\overline{\mathcal{A}}, \overline{\mathcal{A}}] \subset \mathrm{Rad}(\overline{\mathcal{A}})$. ∎

COROLLARY 2.3. *For Jordan operators J_i with the Jordan decomposition $J_i = N_i + Q_i$, $i = 1, 2$, suppose that $\varepsilon(J_1, J_2)$ is an Engel Lie algebra, and Q_1, Q_2 are compact quasinilpotent operators. Then every operator in $\varepsilon(J_1, J_2)$ is a Jordan operator.*

Proof. It is easy to see that every scalar multiple of a Jordan operator is a Jordan operator. By Corollary 2.2, Q_1+Q_2 and the commutator of any two operators in $\varepsilon(J_1, J_2)$ are quasinilpotent operators. It remains to prove that $J_1 + J_2$ is a Jordan operator. But by Lemma 2.5, $[N_1, N_2] = 0$, so by [4, §14, Corollary 6], N_1+N_2 is a normal operator. Note that $[N_1+N_2, Q_1+Q_2] = 0$, and $Q_1 + Q_2$ is quasinilpotent, so $J_1 + J_2$ is a Jordan operator. ∎

COROLLARY 2.4. *For Jordan operators J_i with the Jordan decomposition $J_i = N_i + Q_i$, $i = 1, 2$, suppose that $\varepsilon(J_1, J_2)$ is an Engel Lie algebra, and Q_1, Q_2 are Volterra operators. Then $\mathcal{A}$ is reduced.*

Proof. By Claim 3 in the proof of Theorem 2.2, I is a nonzero Volterra ideal of $\overline{\mathcal{A}}$. So $\overline{\mathcal{A}}$ has a nontrivial hyperinvariant subspace by Shulman's result [14]. ∎

REMARK 2.1. Because there is a nil algebra of operators on a Hilbert space with semisimple norm closure, there is a Lie algebra $\mathcal{L}$ generated by Jordan operators which is an Engel Lie algebra, but $\overline{\mathcal{A}(\mathcal{L})}$ is not an Engel algebra (see [9]).

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Department of Mathematics
Beijing Institute of Technology
Beijing, China, 100081
E-mail: cpeng@bit.edu.cn

LMIB & Department of Mathematics
Beihang University
Beijing, China, 100083
E-mail: shlsuncn@yahoo.com.cn

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