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Optimal domains for kernel operators on $[0,\infty)\times[0,\infty)$

by

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Abstract. Let T be a kernel operator with values in a rearrangement invariant Banach function space X on $[0,\infty)$ and defined over simple functions on $[0,\infty)$ of bounded support. We identify the optimal domain for T (still with values in X) in terms of interpolation spaces, under appropriate conditions on the kernel and the space X. The techniques used are based on the relation between linear operators and vector measures.

Introduction. Let $T\colon E\to X$ be a continuous linear operator between function spaces. The problem of determining the *optimal domain* for T within a class $\mathcal{F}$ of spaces consists in identifying the "largest" space $Y\in\mathcal{F}$ with $E\subset Y$ to which T can be extended as a continuous operator, still with values in X. The space Y is the largest in the sense that if T can be extended to $F\in\mathcal{F}$ then F is continuously embedded into Y. A procedure for solving this problem is to associate to T a vector measure ν, defined by $\nu(A)=T(\chi_A)$, and to study the space $L^1(\nu)$ of integrable functions with respect to ν. In the case when E and X are Banach function spaces over a finite measure space, optimal domains for classical operators have been studied in [7], [8], [9] and [19]. In this setting, the space $L^1(\nu)$ turns out to be the optimal domain for T within the class of Banach function spaces with absolutely continuous norm provided T satisfies an appropriate "monotone weak convergence" property. This identification allows one to deduce properties of the optimal domain for T from those of T (and so of ν) and X; see [4], [5], [6], [10], [13], [18], [20], [21].

For operators T defined on Banach function spaces over an infinite measure space, the associated vector measure ν may not be defined for measurable sets of infinite measure (e.g. if T is the Hilbert transform on the real

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line). Thus, we are led to consider ν defined on a structure weaker than a σ-algebra (e.g. bounded measurable subsets of the real line).

In this paper, we consider operators T defined over simple functions with respect to a δ-ring $\mathcal{R}$. We consider the associated vector measure ν on $\mathcal{R}$. Under appropriate conditions, $L^1(\nu)$ is the optimal domain for T within a certain class of Banach function spaces containing the simple functions with respect to $\mathcal{R}$ (Theorem 2.5). In this case, the integration operator $f \mapsto \int f\,d\nu$ extends T to $L^1(\nu)$.

An important class of operators between function spaces are kernel operators, defined as $T(f) = \int_0^\infty f(y)K(\cdot,y)\,dy$, where K is a measurable function on $[0,\infty)\times[0,\infty)$. If the kernel K is nonnegative and satisfies some integrability and monotonicity conditions, then we give a precise description, as in [8], of the optimal domain for T as an interpolation space of weighted L^1-spaces (Theorems 3.5, 3.8 and 3.9).

1. Preliminaries. Given a σ-finite measure space (Ω,Σ,μ), a *Banach function space* (abbreviated B.f.s.) X is a Banach space of (classes of) measurable functions which are integrable over sets of finite measure, such that X contains the simple functions supported on sets of finite measure and satisfies the condition that $g \in X$ with $\|g\|_X \le \|f\|_X$ whenever $f \in X$ and $|g| \le |f|$ μ-a.e. [15, Definition 1.b.17]. Note that a B.f.s. is a Banach lattice for the μ-a.e. order. A Banach lattice has *absolutely continuous* (abbreviated a.c.) norm if order bounded, increasing sequences are norm convergent. A B.f.s. X has the *Fatou property* if for every sequence $(f_n) \subset X$ of nonnegative functions with $\sup_n \|f_n\|_X < \infty$ that increases μ-a.e. to f, we have $f \in X$ and $\|f_n\|_X \to \|f\|_X$.

Consider $[0,\infty)$ with Lebesgue measure m. A *rearrangement invariant* (r.i.) space X on $[0,\infty)$ is a B.f.s. on $[0,\infty)$ which has the Fatou property and $f \in X$ implies that its decreasing rearrangement f^* also belongs to X with $\|f^*\|_X = \|f\|_X$. The *decreasing rearrangement* f^* of a function f is the left continuous inverse of its distribution function $m_f(\lambda) := m(\{t \in \Omega : |f(t)| > \lambda\})$. Relevant r.i. spaces on $[0,\infty)$ are $L^1 \cap L^\infty$, with norm $\|f\|_{L^1\cap L^\infty} = \max\{\|f\|_1, \|f\|_\infty\}$, and the space $L^1 + L^\infty$ of all functions f such that $f = g + h$ for some $g \in L^1$ and $h \in L^\infty$, endowed with the norm $\|f\|_{L^1+L^\infty} = \int_0^1 f^*(s)\,ds$. Moreover, if X is a r.i. space on $[0,\infty)$, then $L^1 \cap L^\infty \subset X \subset L^1 + L^\infty$ continuously. For issues related to r.i. spaces, see [1, Chp. 2].

We briefly recall the integration theory of real functions with respect to vector measures defined on δ-rings, due to Lewis [14] and Masani and Niemi [16], [17]. Let $\mathcal{R}$ be a *δ-ring* of subsets of a set Ω, that is, a ring closed under countable intersections, and $\mathcal{R}^{\mathrm{loc}}$ the σ-algebra of all subsets A of Ω such that $A \cap B \in \mathcal{R}$ for all $B \in \mathcal{R}$. We denote by $\mathcal{M}$ the space of measurable real functions on $(\Omega, \mathcal{R}^{\mathrm{loc}})$ and by $\mathcal{S}(\mathcal{R})$ the space of $\mathcal{R}$-*simple functions*, that is, simple functions supported in $\mathcal{R}$. Let $\lambda\colon \mathcal{R} \to \mathbb{R}$ be a countably additive measure, that is, $\sum \lambda(A_n)$ converges to $\lambda(\bigcup A_n)$ for (A_n) a family of disjoint sets in $\mathcal{R}$ with $\bigcup A_n \in \mathcal{R}$. The *variation* of λ is the nonnegative countably additive measure on $\mathcal{R}^{\mathrm{loc}}$ given by

$$|\lambda|(A) = \sup\Big\{ \sum |\lambda(A_i)| : (A_i) \text{ finite disjoint sequence in } \mathcal{R} \cap 2^A \Big\}.$$

A function $f \in \mathcal{M}$ is *integrable with respect to* λ if $|f|_{1,\lambda} = \int |f|\, d|\lambda| < \infty$. If we identify functions which are equal $|\lambda|$-a.e., the space $L^1(\lambda)$ of integrable functions with respect to λ is a Banach space with norm $|\cdot|_{1,\lambda}$, in which $\mathcal{S}(\mathcal{R})$ is dense. The integral of an $\mathcal{R}$-simple function $f = \sum_{i=1}^n a_i \chi_{A_i}$ is defined by $\int f\, d\lambda := \sum_{i=1}^n a_i \lambda(A_i)$. For $f \in L^1(\lambda)$ the integral is defined by $\int f\, d\lambda := \lim \int f_n\, d\lambda$, where (f_n) is a sequence in $\mathcal{S}(\mathcal{R})$ converging to f in $L^1(\lambda)$.

Let X be a real Banach space and $\nu\colon \mathcal{R} \to X$ a *vector measure*, that is, ν has the property that $\sum \nu(A_n)$ converges to $\nu(\bigcup A_n)$ in X, for (A_n) disjoint sets in $\mathcal{R}$ with $\bigcup A_n \in \mathcal{R}$. A set $A \in \mathcal{R}^{\mathrm{loc}}$ is ν-*null* if $\nu(B) = 0$ for all $B \in \mathcal{R} \cap 2^A$. A property holds ν-*almost everywhere* (ν-a.e.) if it holds except on a ν-null set. Let X^* be the dual space of X, and $|x^*\nu|$ the variation of the measure $x^*\nu\colon \mathcal{R} \to \mathbb{R}$. A function $f \in \mathcal{M}$ is *integrable with respect to* ν if it is integrable with respect to $|x^*\nu|$ for all $x^* \in X^*$ and for each $A \in \mathcal{R}^{\mathrm{loc}}$ there is a vector, denoted by $\int_A f d\nu \in X$, such that

$$x^*\Big(\int_A f\, d\nu \Big) = \int_A f\, dx^*\nu \qquad \text{for all } x^* \in X^*.$$

We denote by $L^1(\nu)$ the space of integrable functions with respect to ν, where functions which are equal ν-a.e. are identified. An $\mathcal{R}$-simple function $f = \sum_{i=1}^n a_i \chi_{A_i}$ is in $L^1(\nu)$ with $\int_A f\, d\nu = \sum_{i=1}^n a_i \nu(A_i \cap A)$ for $A \in \mathcal{R}^{\mathrm{loc}}$. The space $L^1(\nu)$ is a Banach lattice for the ν-a.e. order and the a.c. norm given by

$$\|f\|_\nu = \sup\Big\{ \int |f|\, d|x^*\nu| : x^* \in X^*,\ \|x^*\| \le 1 \Big\}.$$

The $\mathcal{R}$-simple functions are dense in $L^1(\nu)$. Moreover, $L^1(\nu)$ is an ideal of measurable functions, that is, $g \in L^1(\nu)$ whenever $|g| \le |f|$ ν-a.e. for some $f \in L^1(\nu)$. The integration operator defined by $f \in L^1(\nu) \mapsto \int f\, d\nu \in X$ is continuous with $\|\int f\, d\nu\|_X \le \|f\|_\nu$. Also, each $f \in L^1(\nu)$ satisfies

$$\frac{1}{2}\|f\|_\nu \le \sup\Big\{ \Big\| \int_A f d\nu \Big\|_X : A \in \mathcal{R} \Big\} \le \|f\|_\nu. \tag{1}$$

For results concerning the space $L^1(\nu)$ when ν is defined on a δ-ring, see [11].

2. Optimal domain for operators on a B.f.s. on a δ-ring. Let $\mathcal{R}$ be a δ-ring of sets and X a Banach space. Given a linear operator $T\colon \mathcal{S}(\mathcal{R}) \to X$ we consider the finitely additive set function $\nu\colon \mathcal{R} \to X$ defined by $\nu(A) = T(\chi_A)$. In this section we show that if ν is a vector measure (i.e. is countably additive in X over $\mathcal{R}$) then T can be extended to $L^1(\nu)$ as a continuous operator with values still in X. Indeed, $L^1(\nu)$ is the optimal domain for T within a certain class of spaces (see below).

The next definition extends that of B.f.s. by considering a δ-ring $\mathcal{R}$ in the role played by sets of finite measure in the classical setting.

Definition 2.1. Let $\mathcal{R}$ be a δ-ring of sets in Ω and $\mu\colon \mathcal{R} \to [0,\infty]$ a countably additive measure. A *Banach function space* over $(\Omega,\mathcal{R},\mu)$ is a Banach space E of (classes of) functions in $\mathcal{M}$ (i.e. $\mathcal{R}^{\mathrm{loc}}$-measurable) satisfying:

(i) If $g \in \mathcal{M}$, $f \in E$ and $|g| \le |f|$ μ-a.e., then $g \in E$ and $\|g\|_E \le \|f\|_E$.
(ii) $\chi_A \in E$ for every $A \in \mathcal{R}$.

Note that a B.f.s. over $(\Omega,\mathcal{R},\mu)$ is a Banach lattice with the μ-a.e. order, in which the convergence in norm of a sequence implies the convergence μ-a.e. for some subsequence.

Example 2.2.

(a) A B.f.s. with respect to a σ-finite measure space (Ω,Σ,μ) is a B.f.s. over $(\Omega,\mathcal{R},\mu)$, where $\mathcal{R}$ is the δ-ring of sets in Σ with finite measure. In this case, $\mathcal{R}^{\mathrm{loc}} = \Sigma$.
(b) Let $\nu\colon \mathcal{R} \to X$ be a vector measure and $\lambda\colon \mathcal{R} \to [0,\infty]$ a *local control measure* for ν, that is, a countably additive measure which has the same null sets as ν. For the existence of such a measure, see [2, Theorem 3.2] and [17, Proposition 3.6]. As noted before, the space $L^1(\nu)$ is an ideal of measurable functions containing the $\mathcal{R}$-simple functions. Since the ν-a.e. order is equivalent to the λ-a.e. order, $L^1(\nu)$ is a B.f.s. over $(\Omega,\mathcal{R},\lambda)$; see [11].

The next proposition extends Theorem 3.1 of [8] to the setting of δ-rings.

Proposition 2.3. *Let E be a B.f.s. over $(\Omega,\mathcal{R},\mu)$, X a Banach space and $T : E \to X$ a linear operator satisfying*:

(i) *If $f_n, f \in E$ with $0 \le f_n \uparrow f$ μ-a.e., then Tf_n converges weakly in X to Tf.*
(ii) *If $A \in \mathcal{R}^{\mathrm{loc}}$ with $\chi_A \in E$ and $x^* \in X^*$, we have*

$$\sup_{B\in\mathcal{R}\cap 2^A} |x^*T(\chi_B)| = 0 \;\Rightarrow\; x^*T(\chi_A) = 0.$$

Then the set function $\nu : \mathcal{R} \to X$ given by $\nu(A) = T(\chi_A)$ is a vector measure and for each $f \in E$ we have $f \in L^1(\nu)$ with $\int f\,d\nu = Tf$. Even more, if μ and

ν are equivalent, that is, they have the same null sets, then E is continuously embedded into $L^1(\nu)$ and the integration operator extends T.

Proof. Let (A_n) be disjoint sets in $\mathcal{R}$ with $\bigcup A_n \in \mathcal{R}$. For any subsequence (A_{n_j}), from (i) we see that $T(\chi_{\bigcup_{j=1}^N A_{n_j}}) = \sum_{j=1}^N \nu(A_{n_j})$ converges weakly in X to $T(\chi_{\bigcup A_{n_j}}) = \nu(\bigcup A_{n_j})$. From the Orlicz–Pettis theorem, it follows that $\sum \nu(A_n)$ is unconditionally convergent to $\nu(\bigcup A_n)$; see [12, Corollary I.4.4]. Thus, ν is a vector measure on $\mathcal{R}$.

Suppose that for simple functions $\psi \in E$ we have $\psi \in L^1(\nu)$ and $\int \psi\, d\nu = T(\psi)$. Consider $0 \le f \in E$ and let (ψ_n) be a sequence of simple functions increasing to f. Then $\psi_n \in E$ and so $\psi_n \in L^1(\nu)$. Since $\mathcal{R}$-simple functions are dense in $L^1(\nu)$, we can take $\varphi_n \in \mathcal{S}(\mathcal{R})$ such that $\|\psi_n - \varphi_n\|_\nu \to 0$. Then there is a subsequence such that $\varphi_{n_k} \to f$ ν-a.e. Let $x^* \in X^*$. If we prove that $(\int_A \varphi_{n_k} dx^*\nu)$ converges for every $A \in \mathcal{R}^{\mathrm{loc}}$, applying [11, Proposition 2.3] to the measure $x^*\nu\colon \mathcal{R} \to \mathbb{R}$, we will deduce that f is integrable with respect to $x^*\nu$ and $\int_A f\, dx^*\nu = \lim_{k\to\infty} \int_A \varphi_{n_k}\, dx^*\nu$ for $A \in \mathcal{R}^{\mathrm{loc}}$.

Note that

$$\Big|\int_A \varphi_{n_k}\, dx^*\nu - \int_A \psi_{n_k}\, dx^*\nu\Big| \le \int_A |\varphi_{n_k} - \psi_{n_k}|\, d|x^*\nu| \le \|x^*\|\, \|\varphi_{n_k} - \psi_{n_k}\|_\nu \to 0.$$

Then from condition (i) it follows that

$$x^*T(f\chi_A) = \lim_{k\to\infty} x^*T(\psi_{n_k}\chi_A) = \lim_{k\to\infty} \int_A \psi_{n_k}\, dx^*\nu = \lim_{k\to\infty} \int_A \varphi_{n_k}\, dx^*\nu.$$

So, f is integrable with respect to $x^*\nu$ and

$$\int_A f dx^*\nu = \lim_{k\to\infty} \int_A \varphi_{n_k}\, dx^*\nu = x^*T(f\chi_A).$$

Hence, $f \in L^1(\nu)$ and $\int f\, d\nu = T(f)$. Since a μ-null set is ν-null, the map that takes the class of E represented by f to the class of $L^1(\nu)$ represented by f is well defined. In the case μ and ν are equivalent, this map is one-to-one, that is, it is the identity map. In this case, E is embedded in $L^1(\nu)$ and the embedding is continuous since it is a positive linear operator between Banach lattices; see [15, p. 2].

Therefore, we only have to prove that if $\chi_A \in E$ then $\chi_A \in L^1(\nu)$ with $\int \chi_A\, d\nu = T(\chi_A)$. If $A \in \mathcal{R}^{\mathrm{loc}}$ with $\chi_A \in E$, then $|x^*\nu|(A) < \infty$ for every $x^* \in X^*$. Suppose not; then for some $x^* \in X^*$ we can find, via a standard procedure, an increasing sequence of sets $B_n \in \mathcal{R} \cap 2^A$ such that $|x^*\nu(B_n)| > n$. But from (i), $\nu(B_n) = T(\chi_{B_n})$ is weakly convergent to $T(\chi_{\bigcup B_n}) \in X$. The contradiction establishes the claim.

Let $B \in \mathcal{R}^{\mathrm{loc}}$ and $x^* \in X^*$. Then $|x^*\nu|(B \cap A) < \infty$. Since $|x^*\nu|(B \cap A) = \sup\{|x^*\nu|(H) : H \in \mathcal{R} \cap 2^{B\cap A}\}$, there exists an increasing sequence

(H_n) of sets in $\mathcal{R}\cap 2^{B\cap A}$ with $|x^*\nu|((B\cap A)\backslash H_n)$ converging to zero. Thus, $\chi_{H_n}\uparrow\chi_{A\cap B}$ $x^*\nu$-a.e. and so

$$\int_B \chi_A\,dx^*\nu=\lim_{n\to\infty}\int\chi_{H_n}\,dx^*\nu=\lim_{n\to\infty}x^*\nu(H_n)$$
$$=\lim_{n\to\infty}x^*T(\chi_{H_n})=x^*T(\chi_{B\cap A}).$$

The last equality is obtained from condition (ii), since $(B\cap A)\backslash\bigcup H_n$ is a $x^*\nu$-null set and so $x^*T(\chi_{(B\cap A)\backslash\bigcup H_n})=0$. Hence, $\chi_A\in L^1(\nu)$ and $\int_B\chi_A\,d\nu=T(\chi_{A\cap B})$ for every $B\in\mathcal{R}^{\mathrm{loc}}$. ∎

An important consequence for applications follows.

Corollary 2.4. *Let E be a B.f.s. over $(\Omega,\mathcal{R},\mu)$ with a.c. norm, X a Banach space and $T:E\to X$ a continuous linear operator such that B is μ-null whenever $B\in\mathcal{R}^{\mathrm{loc}}$ with $T(\chi_A)=0$ for every set $A\in\mathcal{R}\cap 2^B$. Then E is continuously embedded into $L^1(\nu)$ and the integration operator extends T, where $\nu:\mathcal{R}\to X$ is the vector measure given by $\nu(A)=T(\chi_A)$.*

Proof. Since the condition satisfied by T means that μ and ν are equivalent, we only have to show that conditions (i) and (ii) in Proposition 2.3 hold. From the absolute continuity of the norm of E and the continuity of T, (i) follows. For (ii), since T is continuous it suffices to show $\mathcal{S}(\mathcal{R})$ is dense in E. Let $\{A_\alpha:\alpha\in\Delta\}$ be a maximal family of sets $A_\alpha\in\mathcal{R}$ with $\mu(A_\alpha)>0$ and $\mu(A_\alpha\cap A_\beta)=0$ for $\alpha\neq\beta$. Observe that the maximality of this family implies that if $B\in\mathcal{R}^{\mathrm{loc}}$ satisfies $\mu(B\cap A_\alpha)=0$ for every $\alpha\in\Delta$, then B is a μ-null set. Given $f\in E$, for any sequence $(\alpha_j)\subset\Delta$ we have

$$\sum_{j=1}^n|f|\chi_{A_{\alpha_j}}=|f|\chi_{\bigcup_{j=1}^n A_{\alpha_j}}\uparrow|f|\chi_{\bigcup A_{\alpha_j}}\quad\mu\text{-a.e.}$$

Since E has a.c. norm, $\sum_{j\geq1}|f|\chi_{A_{\alpha_j}}$ converges in E. Thus, $\sum_{\alpha\in\Delta}|f|\chi_{A_\alpha}$ satisfies the Cauchy condition. So, $f\chi_{A\alpha}=0$ μ-a.e. except for a countable set $\{\alpha_j\}$. Hence, $f=f\chi_{\bigcup A_{\alpha_j}}$ μ-a.e. Suppose $f\geq0$ and let (ψ_n) be a sequence of simple functions with $0\leq\psi_n\uparrow f$. Then $\varphi_n=\psi_n\chi_{\bigcup_{j=1}^n A_{\alpha_j}}$ are $\mathcal{R}$-simple functions and $0\leq\varphi_n\uparrow f$ μ-a.e. Hence, (φ_n) converges to f in E. ∎

The next result identifies, under very mild conditions, the optimal domain for an operator.

Theorem 2.5. *Let $\mathcal{R}$ be a δ-ring, X a Banach space and $T:\mathcal{S}(\mathcal{R})\to X$ a linear operator such that $T(\chi_{A_n})$ is weakly convergent to $T(\chi_A)$ in X for every increasing sequence $(A_n)\subset\mathcal{R}$ with $A=\bigcup A_n\in\mathcal{R}$. Then the set function $\nu:\mathcal{R}\to X$ given by $\nu(A)=T(\chi_A)$ is a vector measure and $L^1(\nu)$ is the optimal domain for T within the class of B.f.s.'s over $(\Omega,\mathcal{R},\mu)$ with a.c. norm and μ equivalent to ν.*

Proof. Note that the required condition on T is necessary and sufficient for ν being a vector measure. The space $L^1(\nu)$ is a B.f.s. over $(\Omega, \mathcal{R}, \lambda)$ with a.c. norm and λ a local control measure for ν, that is, λ is equivalent to ν. Moreover, the integration operator extends T. Suppose now that $\widetilde{T}\colon F \to X$ is a continuous linear extension of T, where F is a B.f.s. over $(\Omega, \mathcal{R}, \mu)$ with a.c. norm and μ equivalent to ν. Let $B \in \mathcal{R}^{\mathrm{loc}}$ be such that $\widetilde{T}(\chi_A) = 0$ for all $A \in \mathcal{R} \cap 2^B$. Since $\widetilde{T}(\chi_A) = T(\chi_A) = \nu(A)$, it follows that B is ν-null, equivalently B is μ-null. Hence, the hypotheses of Corollary 2.4 are satisfied by $\widetilde{T}$ and so F is continuously embedded in $L^1(\widetilde{\nu})$, where $\widetilde{\nu}\colon \mathcal{R} \to X$ is the vector measure given by $\widetilde{\nu}(A) = \widetilde{T}(\chi_A)$. Since $\widetilde{\nu}$ is equal to ν, we have $L^1(\widetilde{\nu}) = L^1(\nu)$. ■

Again, an important consequence for applications follows.

Corollary 2.6. *Let E be a B.f.s. over $(\Omega, \mathcal{R}, \mu)$ with a.c. norm, X a Banach space and $T\colon E \to X$ a continuous linear operator such that B is μ-null whenever $B \in \mathcal{R}^{\mathrm{loc}}$ with $T(\chi_A) = 0$ for all $A \in \mathcal{R} \cap 2^B$. Then the optimal domain for T within the class of B.f.s.'s on $(\Omega, \mathcal{R}, \lambda)$ with a.c. norm and λ equivalent to ν is the space $L^1(\nu)$ with $\nu\colon \mathcal{R} \to X$ the vector measure given by $\nu(A) = T(\chi_A)$.*

The required condition on T in Corollary 2.6 is necessary for the operator integration being an extension of T to $L^1(\nu)$, that is, for E being embedded injectively in $L^1(\nu)$.

Example 2.7. Let $\mathcal{R}$ be the δ-ring of all bounded Borel subsets of $\mathbb{R}$ and m the Lebesgue measure. Note that $\mathcal{R}^{\mathrm{loc}}$ is the Borel σ-algebra of $\mathbb{R}$. For $1 \le p < \infty$, the space $L^p(\mathbb{R})$ is a B.f.s. over $(\mathbb{R}, \mathcal{R}, m)$ with a.c. norm. An isomorphism $T\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ satisfies the hypothesis of Corollary 2.6, in particular m-null and ν-null sets coincide. Then, for the vector measure $\nu\colon \mathcal{R} \to L^p(\mathbb{R})$ given by $\nu(A) = T(\chi_A)$, it follows that $L^1(\nu)$ is the optimal domain for T within the B.f.s.'s over $(\Omega, \mathcal{R}, \mu)$ with a.c. norm and μ equivalent to ν, in particular $\mu = m$. Also, it is known that $L^1(\nu)$ is order isomorphic to $L^p(\mathbb{R})$; see [11, Example 4.1].

3. Optimal domains for kernel operators. Throughout this section, $\mathcal{B}$ is the σ-algebra of the Borel subsets of $[0,\infty)$ and $\mathcal{B}_{\mathrm{b}}$ the δ-ring of bounded sets of $\mathcal{B}$. Note that $\mathcal{B}_{\mathrm{b}}^{\mathrm{loc}} = \mathcal{B}$. Lebesgue measure on $[0,\infty)$ is denoted by m.

Let $K\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ be a measurable function such that, for every $x \in [0,\infty)$, the function K_x defined by $K_x(y) = K(x,y)$ is integrable over sets in $\mathcal{B}_{\mathrm{b}}$. We say that K is an *admissible kernel*. Associated to K we have the finitely additive set function ν defined over $\mathcal{B}_{\mathrm{b}}$ by

$$\nu(A) = \int_A K(\cdot\,, y)\, dy,$$

and the linear operator T given by

$$T(f)=\int_0^\infty f(y)K(\cdot\,,y)\,dy,$$

provided the integral exists.

Proposition 3.1. *Let X be a B.f.s. on $[0,\infty)$, K an admissible kernel, T the operator associated to K and ν the associated set function. Suppose that ν takes values in X, that is, $\nu\colon \mathcal{B}_{\mathrm{b}}\to X$ is well defined.*

(a) *If X has a.c. norm, then $\nu\colon \mathcal{B}_{\mathrm{b}}\to X$ is countably additive.*
(b) *If $\nu\colon \mathcal{B}_{\mathrm{b}}\to X$ is countably additive, then $f\in L^1(\nu)$ implies $T(f)\in X$ and $T(f)=\int f\,d\nu$.*
(c) *If $\nu\colon \mathcal{B}_{\mathrm{b}}\to X$ is countably additive, then $L^1(\nu)$ is the optimal domain for $T\colon \mathcal{S}(\mathcal{B}_{\mathrm{b}})\to X$ within the class of B.f.s.'s over $([0,\infty),\mathcal{B}_{\mathrm{b}},\mu)$ with a.c. norm and μ equivalent to ν.*

Proof. (a) Let $(A_n)\subset\mathcal{B}_{\mathrm{b}}$ be such that $\bigcup A_n\in\mathcal{B}_{\mathrm{b}}$. As K is nonnegative, $\nu(\bigcup_{j=1}^n A_j)\uparrow\nu(\bigcup A_j)\in X$. Since X has a.c. norm, $\sum_{j=1}^n\nu(A_j)=\nu(\bigcup_{j=1}^n A_j)$ converges in X to $\nu(\bigcup A_j)$.

(b) Suppose $\nu\colon\mathcal{B}_{\mathrm{b}}\to X$ is countably additive. Let $0\le f\in L^1(\nu)$ and (ψ_n) be a sequence of simple functions with $0\le\psi_n\uparrow f$. The functions $\varphi_n=\psi_n\chi_{[0,n]}$ are $\mathcal{B}_{\mathrm{b}}$-simple and $0\le\varphi_n\uparrow f$. Since $L^1(\nu)$ has a.c. norm, φ_n converges to f in $L^1(\nu)$, so $\int\varphi_n\,d\nu\to\int f\,d\nu$ in X. Consider a subsequence $\int\varphi_{n_k}\,d\nu=T(\varphi_{n_k})$ converging a.e. to $\int f\,d\nu$. Since K is nonnegative, $0\le T\varphi_{n_k}=\int_0^\infty\varphi_{n_k}(y)K(\cdot\,,y)\,dy$ increases to $\int_0^\infty f(y)K(\cdot\,,y)\,dy$. Hence, $T(f)=\int f\,d\nu\in X$.

(c) If $\nu\colon\mathcal{B}_{\mathrm{b}}\to X$ is countably additive, then $T\colon\mathcal{S}(\mathcal{B}_{\mathrm{b}})\to X$ satisfies the hypothesis of Theorem 2.5. ∎

In view of Proposition 3.1, we will focus our attention on the problem of determining conditions on the admissible kernel K and the B.f.s. X so that $\nu\colon\mathcal{B}_{\mathrm{b}}\to X$ is a vector measure. Let X be a r.i. space on $[0,\infty)$. Since $L^1\cap L^\infty$ is continuously embedded in X, a vector measure with values in $L^1\cap L^\infty$ is also a vector measure with values in X. Thus, we look for conditions on K guaranteeing the associated set function $\nu\colon\mathcal{B}_{\mathrm{b}}\to L^1\cap L^\infty$ is a vector measure.

Proposition 3.2. *Given an admissible kernel K, the set function $\nu\colon\mathcal{B}_{\mathrm{b}}\to L^1\cap L^\infty$ associated to K is a vector measure if and only if K satisfies*:

(i) *The maps K_y defined by $K_y(x)=K(x,y)$ are integrable for m-a.e. $y\in[0,\infty)$.*
(ii) *The map $y\mapsto\int_0^\infty K_y(x)\,dx$ is integrable over sets in $\mathcal{B}_{\mathrm{b}}$.*
(iii) *For every $A\in\mathcal{B}_{\mathrm{b}}$, we have $\operatorname{ess\,sup}_{x\ge0}\int_A K(x,y)\,dy<\infty$.*
(iv) *For every $A\in\mathcal{B}_{\mathrm{b}}$, we have $\lim_{m(B)\to0}\operatorname{ess\,sup}_{x\ge0}\int_{A\cap B}K(x,y)\,dy=0$.*

Proof. Clearly $\nu(A) \in L^\infty$ for every $A \in \mathcal{B}_\mathrm{b}$ if and only if K satisfies (iii). The condition that $\nu(A) \in L^1$ is precisely

$$\int_0^\infty \int_A K(x,y)\,dy\,dx = \int_A \int_0^\infty K(x,y)\,dx\,dy < \infty.$$

This holds for every $A \in \mathcal{B}_\mathrm{b}$ if and only if K satisfies (i) and (ii). Thus, ν is well defined if and only if K satisfies (i)–(iii). Since L^1 has a.c. norm, from Proposition 3.1(a) it follows that $\nu\colon \mathcal{B}_\mathrm{b} \to L^1 \cap L^\infty$ is countably additive if and only if $\nu\colon \mathcal{B}_\mathrm{b} \to L^\infty$ is countably additive.

Suppose K satisfies (iv). Let (A_n) be disjoint sets in $\mathcal{B}_\mathrm{b}$, with $A = \bigcup A_n \in \mathcal{B}_\mathrm{b}$. Since $m(A) < \infty$, we have $m(\bigcup_{j>n} A_j) \to 0$. From (iv), it follows that $\|\nu(\bigcup_{j>n} A_j)\|_\infty = \operatorname{ess\,sup}_{x\ge0} \int_{\bigcup_{j\ge n} A_j} K(x,y)\,dy$ converges to zero. So, $\nu\colon \mathcal{B}_\mathrm{b} \to L^\infty$ is countably additive.

Conversely, suppose $\nu\colon \mathcal{B}_\mathrm{b} \to L^\infty$ is countably additive. If (iv) does not hold for some $A \in \mathcal{B}_\mathrm{b}$, then there exists $\delta > 0$ and sets (B_n) with $m(B_n) \le 1/2^n$ such that

$$\delta < \operatorname*{ess\,sup}_{x\ge0} \int_{B_n\cap A} K_x(y)\,dy = \|\nu(B_n \cap A)\|_\infty \le \|\chi_{B_n\cap A}\|_\nu \le \|\chi_{H_n}\|_\nu,$$

where $H_n = \bigcup_{j\ge n} B_j \cap A$. Since χ_{H_n} are $\mathcal{B}_\mathrm{b}$-simple functions decreasing to zero m-a.e., so ν-a.e., absolute continuity of the norm in $L^1(\nu)$ implies $\|\chi_{H_n}\|_\nu \to 0$. We have arrived at a contradiction. ∎

Example 3.3. Let $\phi\colon [0,\infty) \to [0,\infty)$ be a measurable function and define K on $[0,\infty) \times [0,\infty)$ by $K(x,y) = \phi(x-y)\chi_{[0,x]}(y)$. The function K is an admissible kernel satisfying (i)–(iv) in Proposition 3.2 if and only if ϕ is integrable. In this case, $\nu\colon \mathcal{B}_\mathrm{b} \to L^1 \cap L^\infty$ is a vector measure. These kernels were considered in [7] for the interval $[0,1]$.

We also focus our attention on the problem of identifying the optimal domain for T, the associated operator to an admissible kernel K, which under the conditions of Proposition 3.1(c) corresponds to identifying the space $L^1(\nu)$, where ν is the vector measure associated to K. For this, we consider *decreasing* kernels K, that is, satisfying $K_{x_1}(y) \ge K_{x_2}(y)$, for every $y \in [0,\infty)$, whenever $x_1 \le x_2$. Decreasing admissible kernels K satisfy (iii) and (iv) in Proposition 3.2. Thus, for these kernels, $\nu\colon \mathcal{B}_\mathrm{b} \to L^1 \cap L^\infty$ is a vector measure if and only if K satisfies (i) and (ii) in Proposition 3.2. Observe that *increasing* kernels ($K_{x_1}(y) \le K_{x_2}(y)$, for every $y \in [0,\infty)$, whenever $x_1 \le x_2$) do not satisfy (i) in Proposition 3.2.

Example 3.4. The kernel $K(x,y) = \exp(-\lambda(y-x))\chi_{[x,\infty)}(y)$ with $\lambda \in \mathbb{R}$ is admissible and satisfies conditions (i)–(iv) in Proposition 3.2. Thus, the associated set function $\nu\colon \mathcal{B}_{\mathrm{b}} \to L^1 \cap L^\infty$ is a vector measure. Moreover, K is decreasing whenever $\lambda \le 0$.

For each decreasing admissible kernel K satisfying (i) and (ii) in Proposition 3.2, we consider the functions ω and ξ given by

$$\omega(y) = \int_0^\infty K_y(x)\,dx, \quad \xi(y) = K(0^+, y) = \lim_{x\to 0^+} K(x,y), \qquad \text{for } y \in [0,\infty).$$

Note that ω and ξ are integrable over sets in $\mathcal{B}_{\mathrm{b}}$. Denote by L^1_ω the space of integrable functions with respect to Lebesgue measure with density ω and by $\|\cdot\|_\omega$ its norm; similarly for L^1_ξ.

We first consider the smallest r.i. space $L^1 \cap L^\infty$.

Theorem 3.5. *Let K be a decreasing admissible kernel satisfying* (i) *and* (ii) *in Proposition* 3.2. *Then $\nu\colon \mathcal{B}_{\mathrm{b}} \to L^1 \cap L^\infty$ is a vector measure and the space $L^1(\nu)$ is order isomorphic to $L^1_\omega \cap L^1_\xi$.*

Proof. The hypothesis and the monotonicity of K imply, by Proposition 3.2, that $\nu\colon \mathcal{B}_{\mathrm{b}} \to L^1 \cap L^\infty$ is a vector measure. Let $L^1_\omega \cap L^1_\xi$ be endowed with the norm $\|f\|_{\omega,\xi} = \max\{\|f\|_\omega, \|f\|_\xi\}$ and the m_ψ-a.e. order, where m_ψ is the Lebesgue measure with density $\psi = \max\{\omega, \xi\}$. Note that from Proposition 3.1(b), for $f \in L^1(\nu)$ we have $\int f\,d\nu = \int f(y)K(\cdot\,, y)\,dy$. Given a $\mathcal{B}_{\mathrm{b}}$-simple function f, we have $\int |f|\,d\nu \in L^1 \cap L^\infty$. Then

$$\|f\|_\omega = \int_0^\infty |f(y)|\omega(y)\,dy = \int_0^\infty\int_0^\infty |f(y)|K(x,y)\,dy\,dx \tag{2}$$
$$= \int_0^\infty \Big(\int |f|\,d\nu\Big)(x)\,dx = \Big\|\int |f|\,d\nu\Big\|_1.$$

Since K is decreasing,

$$\|f\|_\xi = \int_0^\infty |f(y)|\xi(y)\,dy = \operatorname*{ess\,sup}_{x\ge 0} \int_0^\infty |f(y)|K(x,y)\,dy = \Big\|\int |f|\,d\nu\Big\|_\infty.$$

Hence, $\|f\|_{\omega,\xi} = \|\int |f|\,d\nu\|_{L^1\cap L^\infty}$. In particular, m_ψ-null and ν-null sets coincide. From (1) it follows that $\frac{1}{2}\|f\|_\nu \le \|f\|_{\omega,\xi} \le \|f\|_\nu$. Hence, $L^1(\nu)$ is order isomorphic to $L^1_\omega \cap L^1_\xi$. ∎

Remark 3.6. If a kernel K is decreasing in the weaker sense that $K_{x_1}(y) \ge K_{x_2}(y)$ for m-a.e. $y \in [0,\infty)$ whenever $x_1 \le x_2$, then there exists a decreasing kernel $\widetilde{K}$ (in the strong sense) which satisfies $\widetilde{K}(x,y) = K(x,y)$ for $m\otimes m$-a.e. (x,y), and so $\widetilde{K}$ and K produce the same operator T. It is enough to take $\widetilde{K}(x,y) = \sup_{r\in\mathbb{Q},\, r>x} K(r,y)$ and notice that, for every $x > 0$, we have $\widetilde{K}(x,y) \le K(x,y) \le \widetilde{K}(x^-,y)$ for m-a.e. $y \in [0,\infty)$.

REMARK 3.7. Under the conditions of Theorem 3.5, $\nu\colon \mathcal{B}_{\mathrm{b}} \to L^1$ is a vector measure. Moreover, for a $\mathcal{B}_{\mathrm{b}}$-simple function f, from (1) and (2) it follows that $\frac{1}{2}\|f\|_\nu \le \|f\|_\omega \le \|f\|_\nu$. Thus, $L^1(\nu)$ is order isomorphic to L^1_ω. Also, for all $f \in L^1_\omega$, we find that $\|f\|_\omega = \|\int |f|\,d\nu\|_1$. Similarly, $\nu\colon \mathcal{B}_{\mathrm{b}} \to L^\infty$ is a vector measure such that $L^1(\nu)$ is order isomorphic to L^1_ξ and $\|f\|_\xi = \|\int |f|\,d\nu\|_\infty$ for all $f \in L^1_\xi$.

Now we consider the largest r.i. space $L^1 + L^\infty$. A further monotonicity property is needed.

THEOREM 3.8. *Let K be a decreasing admissible kernel satisfying* (i) *and* (ii) *in Proposition* 3.2. *Suppose K satisfies the following condition*: *there exists a constant $C > 0$ such that*

$$\int_0^1 K_y(x)\,dx \ge C \min\Big\{ \int_0^\infty K_y(x)\,dx,\ K(0^+, y)\Big\} \quad \textit{for all } y \ge 0. \tag{3}$$

Then $\nu\colon \mathcal{B}_{\mathrm{b}} \to L^1 + L^\infty$ is a vector measure and the space $L^1(\nu)$ is order isomorphic to $L^1_\omega + L^1_\xi$.

Proof. Consider the space $L^1_\omega + L^1_\xi$ of measurable functions f such that $f = g + h$ for some $g \in L^1_\omega$ and $h \in L^1_\xi$, endowed with the norm

$$\begin{aligned} \|f\|_{\omega,\xi} &= \inf\{\|g\|_\omega + \|h\|_\xi : f = g + h \text{ with } g \in L^1_\omega,\ h \in L^1_\xi\} \\ &= \int_0^\infty |f(y)| \min\{\omega(y), \xi(y)\}\,dy \end{aligned}$$

(see [3, (3.1.39), p. 307]), and the m_ψ-a.e. order, where $\psi = \min\{\omega, \xi\}$.

Let f be a $\mathcal{B}_{\mathrm{b}}$-simple function. Then $\|f\|_{\omega,\xi} \ge \|\int |f|\,d\nu\|_{L^1+L^\infty}$. To see this, let $f = g + h$ for $g \in L^1_\omega$ and $h \in L^1_\xi$. From Remark 3.7 we have $\|g\|_\omega = \|\int |g|\,d\nu\|_1$ and $\|h\|_\xi = \|\int |h|\,d\nu\|_\infty$. Then

$$\|g\|_\omega + \|h\|_\xi \ge \Big\| \int (|g| + |h|)\,d\nu \Big\|_{L^1+L^\infty} \ge \Big\| \int |f|\,d\nu \Big\|_{L^1+L^\infty}.$$

On the other hand, from (3) we have

$$\begin{aligned} \Big\| \int |f|\,d\nu \Big\|_{L^1+L^\infty} &= \int_0^1 \Big(\int |f|\,d\nu \Big)^*(s)\,ds \ge \int_0^1 \Big(\int |f|\,d\nu \Big)(x)\,dx \\ &= \int_0^\infty |f(y)| \int_0^1 K_y(x)\,dx\,dy \\ &\ge C \int_0^\infty |f(y)| \min\{\omega(y), \xi(y)\}\,dy = C\|f\|_{\omega,\xi}. \end{aligned}$$

Thus, (1) implies $C\|f\|_{\omega,\xi} \le \|f\|_\nu \le 2\|f\|_{\omega,\xi}$. In particular m_ψ and ν have the same null sets. So, $L^1(\nu)$ and $L^1_\omega + L^1_\xi$ are order isomorphic. ■

Let now X be a r.i. space on $[0,\infty)$. Recall that if a set function $\nu\colon \mathcal{B}_{\rm b} \to L^1 \cap L^\infty$ is a vector measure, so is $\nu\colon \mathcal{B}_{\rm b} \to X$. In order to describe the space $L^1(\nu)$ we need to recall the K-method of interpolation of Peetre. If (X_0, X_1) are Banach spaces continuously embedded in a common Hausdorff topological vector space, then the K-functional of $f \in X_0 + X_1$ is, for $t > 0$,

$$\mathcal{K}(t, f; X_0, X_1) = \inf\{\|f_0\| + t\|f_1\| : f = f_0 + f_1,\ f_0 \in X_0,\ f_1 \in X_1\}.$$

Assume $X_0 \cap X_1$ is dense in X_0. Let X be a r.i. space on $[0,\infty)$. Then $(X_0, X_1)_X$ is the space of all functions $f \in X_0 + X_1$ such that $\mathcal{K}'$, the derivative of the K-functional, satisfies $\mathcal{K}'(\cdot, f; X_0, X_1) \in X$. Endowed with the norm $\|f\|_{(X_0,X_1)_X} := \|\mathcal{K}'(\cdot, f; X_0, X_1)\|_X$, the space $(X_0, X_1)_X$ is an interpolation space between X_0 and X_1; see [1, Chp. V]. A r.i. space X on $[0,\infty)$ can be generated by this procedure as $(L^1[0,\infty), L^\infty[0,\infty))_X$.

Now we can prove the main result.

Theorem 3.9. *Let X be a r.i. space on $[0,\infty)$ with a.c. norm and K a decreasing admissible kernel satisfying* (i) *and* (ii) *in Proposition* 3.2. *Suppose K satisfies the following condition: there exists a constant $C > 0$ such that*

$$\int_0^t K_y(x)\,dx \ge C \min\Big\{\int_0^\infty K_y(x)dx,\ tK(0^+, y)\Big\} \quad \text{for all } t, y \ge 0. \tag{4}$$

Then $\nu\colon \mathcal{B}_{\rm b} \to X$ is a vector measure and the space $L^1(\nu)$ is order isomorphic to $(L^1_\omega, L^1_\xi)_X$.

Proof. For every $f \in L^1_\omega + L^1_\xi$, from [3, (3.1.39), p. 307], we have

$$\mathcal{K}(t, f; L^1_\omega, L^1_\xi) = \int_0^\infty |f(y)| \min\{\omega(y), t\xi(y)\}\,dy. \tag{5}$$

The space $(L^1_\omega, L^1_\xi)_X$ is a B.f.s. on $([0,\infty), \mathcal{B}_{\rm b}, m_\psi)$, with $\psi = \min\{\omega, \xi\}$. Moreover, $(L^1_\omega, L^1_\xi)_X$ has a.c. norm since X has a.c. norm (see Lemma 3.10 below). So, $\mathcal{B}_{\rm b}$-simple functions are dense in $(L^1_\omega, L^1_\xi)_X$.

Let f be a $\mathcal{B}_{\rm b}$-simple function. Applying condition (4) to (5) we have

$$\begin{aligned}\mathcal{K}(t, f; L^1_\omega, L^1_\xi) &\le \frac{1}{C}\int_0^\infty |f(y)| \int_0^t K_y(x)\,dx\,dy \\ &= \frac{1}{C}\int_0^t\int_0^\infty |f(y)| K(x,y)\,dy\,dx \le \frac{1}{C}\int_0^t \Big(\int |f|\,d\nu\Big)^*(s)\,ds.\end{aligned}$$

Moreover, $\mathcal{K}(t,f;L^1_\omega,L^1_\xi)\ge\int_0^t(\int|f|\,d\nu)^*(s)\,ds$, since for every $g\in L^1_\omega$ and $h\in L^1_\xi$ with $f=g+h$, from Remark 3.7, we have

$$\begin{aligned}\|g\|_\omega+t\|h\|_\xi&=\Big\|\int|g|\,d\nu\Big\|_1+t\Big\|\int|h|\,d\nu\Big\|_\infty\\&\ge\mathcal{K}\Big(t,\int(|g|+|h|)\,d\nu\,;L^1,L^\infty\Big)\ge\int_0^t\Big(\int|f|\,d\nu\Big)^*(s)\,ds,\end{aligned}$$

where we have used the fact that $\mathcal{K}(t,f;L^1,L^\infty)=\int_0^t f^*(s)\,ds$ for $f\in L^1+L^\infty$; see [1, Theorem V.1.6]. From [1, Theorem II.4.7], it follows that

$$\Big\|\int|f|\,d\nu\Big\|_X\le\|f\|_{(L^1_\omega,L^1_\xi)_X}=\|\mathcal{K}'(\cdot\,,f;L^1_\omega,L^1_\xi)\|_X\le\frac{1}{C}\Big\|\int|f|\,d\nu\Big\|_X,$$

and from (1), $C\|f\|_{(L^1_\omega,L^1_\xi)_X}\le\|f\|_\nu\le2\|f\|_{(L^1_\omega,L^1_\xi)_X}$. In particular, m_ψ and ν have the same null sets. Thus, $L^1(\nu)$ is order isomorphic to $(L^1_\omega,L^1_\xi)_X$. ∎

We now prove the technical lemma referred to in the previous proof.

LEMMA 3.10. *If X has a.c. norm then $(L^1_\omega,L^1_\xi)_X$ has a.c. norm, where ω and ξ are weights.*

Proof. Consider $(L^1_\omega,L^1_\xi)_X$ endowed with the m_ψ-a.e. order where $\psi=\min\{\omega,\xi\}$. To simplify notation let $\mathcal{K}(t,f)=\mathcal{K}(t,f;L^1_\omega,L^1_\xi)$. Let $f_n,f\in(L^1_\omega,L^1_\xi)_X$ with $0\le f_n\uparrow f$ m_ψ-a.e. Then $\mathcal{K}'(t,f_n)$ increases to $\mathcal{K}'(t,f)$, for all $t>0$. We now prove this. For any function g, we can write $\mathcal{K}'(t,g)$ as

$$\mathcal{K}'(t,g)=\lim_{h\to0^+}\frac{1}{h}\,(\mathcal{K}(t+h,g)-\mathcal{K}(t,g)).\tag{6}$$

In (5), writing $\varPhi(y,t)=\min\{\omega(y),t\xi(y)\}$ we have

$$\frac{1}{h}\,(\mathcal{K}(t+h,g)-\mathcal{K}(t,g))=\frac{1}{h}\int_0^\infty|g(y)|(\varPhi(t+h,y)-\varPhi(t,y))\,dy.\tag{7}$$

From (6) and (7) it follows that $\mathcal{K}'(t,f_n)\le\mathcal{K}'(t,f_{n+1})\le\mathcal{K}'(t,f)$ for all $t>0$. Since $\mathcal{K}'$ is decreasing, we have

$$\frac{1}{h}\,(\mathcal{K}(t+h,g)-\mathcal{K}(t,g))=\frac{1}{h}\int_t^{t+h}\mathcal{K}'(s,g)\,ds\le\mathcal{K}'(t,g).\tag{8}$$

From (5), it follows that for all $t>0$,

$$\mathcal{K}(t,f)=\lim_{n\to\infty}\mathcal{K}(t,f_n).\tag{9}$$

Thus, (8) and (9) imply

$$\begin{aligned}\frac{1}{h}\,(\mathcal{K}(t+h,f)-\mathcal{K}(t,f))&=\lim_{n\to\infty}\frac{1}{h}\,(\mathcal{K}(t+h,f_n)-\mathcal{K}(t,f_n))\\&\le\lim_{n\to\infty}\mathcal{K}'(t,f_n)\le\mathcal{K}'(t,f).\end{aligned}$$

Taking the limit as $h \to 0^+$, we obtain $\mathcal{K}'(t, f_n) \uparrow \mathcal{K}'(t, f)$ for all $t > 0$. Since X has a.c. norm, $\mathcal{K}'(\cdot, f_n)$ converges in X to $\mathcal{K}'(\cdot, f)$. Since $0 \le f_n \le f$, equation (7) shows that $\mathcal{K}'(t, f) - \mathcal{K}'(t, f_n) = \mathcal{K}'(t, f - f_n)$. Hence, f_n converges to f in $(L^1_\omega, L^1_\xi)_X$. ∎

REMARK 3.11. Considering Example 3.4, direct computation shows that the admissible kernel K for $\lambda \le 0$ satisfies condition (4) in Theorem 3.9. Therefore, for a r.i. space X on $[0, \infty)$ with a.c. norm and for the vector measure $\nu\colon \mathcal{B}_{\mathrm{b}} \to X$ associated to K, we have $L^1(\nu) = (L^1_\omega, L^1_\xi)_X$, where $\omega(y) = (1 - e^{-\lambda y})/\lambda$ and $\xi(y) = e^{-\lambda y}$.

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