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On the (C,α) uniform ergodic theorem

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Abstract. We improve a recent result of T. Yoshimoto about the uniform ergodic theorem with Cesàro means of order α. We give a necessary and sufficient condition for the (C,α) uniform ergodicity with $\alpha > 0$.

Introduction. In his classical paper [D], N. Dunford obtained several theorems about convergence of $(f_n(T))_{n\in\mathbb{N}}$, where T is a bounded linear operator on a Banach space and $(f_n)_{n\in\mathbb{N}}$ is a sequence of complex-valued functions, each of which is holomorphic on some open neighborhood of $\sigma(T)$. Different kinds of convergence (namely, convergence in $B(X)$, strong and weak convergence) were treated.

In connection with this, E. Hille [H] obtained, as an application of Abelian and Tauberian theorems, the uniform ergodic theorem as stated below with a view to relating the (C,α) ergodic theorem for an operator T and the properties of the resolvent $R(\cdot,T)$.

THEOREM A (Hille [H, Theorem 6]). *Let X be a Banach space and $T \in B(X)$. A necessary condition for the existence of an operator $E \in B(X)$ such that, for some fixed $\alpha > 0$,*

$$\left\| \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} T^k - E * \right\| \to 0 \qquad \text{as } n \to \infty \tag{1}$$

is that

$$\|(\lambda-1)R(\lambda,T) - E\| \to 0 \qquad \text{as } \lambda \to 1^+, \tag{2}$$

$$\|T^n\|/n^\alpha \to 0 \qquad \text{as } n \to \infty. \tag{3}$$

Conversely, if (3) is replaced by the power-boundedness of T, then (2) implies (1) for every $\alpha > 0$. Here, A_n^α, $n = 0,1,2,\ldots$, are the (C,α) coefficients of order α.

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In fact we have a particular interest in the case when the operator T is not necessarily power-bounded. More precisely, the question is whether the power-boundedness of the operator T is indispensable to deduce (1) from (2). A partial negative answer to this question was first given by M. Lin [L] and later by many other authors in the case $\alpha = 1$.

Recently T. Yoshimoto [Y] obtained an improvement of the above theorem by introducing *condition* (Y): $T^n/n^\omega \to 0$ as $n \to \infty$ where $\omega = \min(1, \alpha)$, together with (2): $(\lambda - 1)R(\lambda, T) \to 0$ as $\lambda \to 1^+$, to prove (1). And consequently, (1) is equivalent to conditions (2) and (3) if $0 < \alpha \le 1$.

In this paper we shall show that (1) is equivalent to (2) and (3) for every $\alpha > 0$ (Theorem 1), and we will give an example showing that condition (Y) is only a sufficient condition but not necessary when $\alpha > 1$.

Section 1 presents some preliminaries in order to make this paper as self-contained as possible. Section 2 is devoted to our main results. In Section 3, we give an example and corollaries.

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1. Preliminaries. In this section we recall some known results which we shall use in what follows. $B(X)$ denotes the Banach algebra of all bounded linear operators from a complex Banach space X into itself. For $T \in B(X)$ we denote the spectrum of T by $\sigma(T)$, the resolvent set of T by $\varrho(T) = \mathbb{C} \setminus \sigma(T)$, and the spectral radius of T by $r(T)$. It is well known that the resolvent function $R(\cdot, T) : \varrho(T) \ni \lambda \mapsto (\lambda I - T)^{-1} \in B(X)$, where I denotes the identity operator, is holomorphic on $\varrho(T)$.

By $\mathbb{N}$ and $\mathbb{Z}_+$ we denote the sets of all nonnegative and positive integers, respectively.

For real $\alpha > -1$ and integer $n \ge 0$, let A_n^α be the (C, α) *coefficient* of order α, which is defined by the generating function

$$\frac{1}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^\alpha t^n, \qquad 0 \le t < 1.$$

Explicitly, $A_n^\alpha = (\alpha + 1) \ldots (\alpha + n)/n!$. We check easily that

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} = \binom{\alpha+n}{n} = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)},$$

which is equivalent to $n^\alpha/\Gamma(\alpha+1)$ as $n \to \infty$.

The nth *Cesàro mean* of order α of the powers of T is defined by

$$M_n^\alpha = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} T^k.$$

For $\alpha = 1$ we find

$$M_n^1 = \frac{1}{n+1}\sum_{k=0}^{n} T^k,$$

the usual Cesàro mean.

For $T \in B(X)$, we denote the kernel and range of T by $N(T)$ and $R(T)$, respectively. We begin with the closed range theorem:

THEOREM 1.1 (see [T.L, 4.5.10]). *Let X, Y be Banach spaces and T a bounded linear operator from X into Y. If there exists a closed subspace Z of Y such that $R(T) \cap Z = \{0\}$ and $R(T) \oplus Z$ is closed, then $R(T)$ is closed.*

By a *projection* of a Banach space X, we mean an element P of $B(X)$ satisfying $P^2 = P$. We recall that if P is a projection of X, then $R(P)$ is a closed subspace of X and in addition $X = R(P) \oplus N(P)$. Conversely, for every direct-sum decomposition $X = Y \oplus Z$ where Y and Z are closed subspaces of X there exists a unique projection P of X such that $R(P) = Y$ and $N(P) = Z$; we call P the *projection of X onto Y along Z.*

We denote by $\alpha(T)$ and $\delta(T)$ the *ascent* and *descent* of T, respectively, defined by

$$\alpha(T) = \inf\{n \in \mathbb{N} \mid N(T^n) = N(T^{n+1})\},$$

$$\delta(T) = \inf\{n \in \mathbb{N} \mid R(T^n) = R(T^{n+1})\}.$$

Then $\alpha(T)$ and $\delta(T)$ belong to $\mathbb{N} \cup \{\infty\}$. We recall that if $\alpha(T) < \infty$ (respectively, $\delta(T) < \infty$), then $N(T^n) = N(T^{\alpha(T)})$ for every $n \geq \alpha(T)$ (respectively, $R(T^n) = R(T^{\delta(T)})$ for every $n \geq \delta(T)$). It is well known that finiteness of the ascent and descent of a bounded linear operator on a Banach space X is equivalent to a certain decomposition of X, as the following result shows:

THEOREM 1.2 (see [T.L, 5, 6.2, 6.3 and 6.4]). *Let X be a Banach space and let $T \in B(X)$. If both $\alpha(T)$ and $\delta(T)$ are finite, then $\alpha(T) = \delta(T)$ and $X = R(T^p) \oplus N(T^p)$ where p denotes the common value of $\alpha(T)$ and $\delta(T)$. Conversely, if the above decomposition holds for some integer $p \geq 1$, then $\alpha(T) = \delta(T) \leq p$.*

We conclude this section with an interesting result which shows a connection between the decomposition of a Banach space X and the uniform Abel summability of $T \in B(X)$.

LEMMA 1.3 ([H.P, Theorem 18.8.1]). *Let X be a Banach space and $T \in B(X)$. If there exists a sequence $(\lambda_n) \subset \varrho(T)$ such that*

(1) $$\lambda_n \to 1 \quad \text{as } n \to \infty,$$

(2) $$\|(\lambda_n - 1)R(\lambda_n, T) - E\| \to 0 \quad \text{as } n \to \infty,$$

where E is a bounded linear operator from X into itself, then $X = R(I-T) \oplus N(I-T)$ and E is the projection of X onto $N(I-T)$ along $R(I-T)$.

2. Main results

THEOREM 1. *Let T be a bounded linear operator on a Banach space X. There exists an operator $E \in B(X)$ such that, for some fixed $\alpha > 0$,*

$$\|M_n^\alpha - E\| \to 0 \quad \text{as } n \to \infty \tag{1}$$

if and only if

$$\|(\lambda - 1)R(\lambda, T) - E\| \to 0 \quad \text{as } \lambda \to 1^+, \tag{2}$$

$$\|T^n\|/n^\alpha \to 0 \quad \text{as } n \to \infty. \tag{3}$$

We begin with some auxiliary results.

DEFINITION 2.1. Let X be a Banach space and $T \in B(X)$. For $\alpha > 0$ and integer $l \geq 1$, we shall say that T satisfies *condition* $\delta(l, \alpha)$ if $\|(I-T)^l M_n^\alpha(T)\| \to 0$ as $n \to \infty$.

LEMMA 2.2. *Let T be a bounded linear operator on a Banach space X. If there exists an $E \in B(X)$ such that, for some fixed $\alpha > 0$,*

$$\|M_n^\alpha - E\| \to 0 \quad \text{as } n \to \infty, \tag{1}$$

then

$$\|(\lambda - 1)R(\lambda, T) - E\| \to 0 \quad \text{as } \lambda \to 1^+, \tag{2}$$

$$\sigma(T) \subset \overline{D(0,1)}, \tag{3}$$

(4) *T satisfies condition $\delta(l, \alpha)$ for some integer $l \geq 1$.*

Conversely, if (2)–(4) *are satisfied, then* (1) *holds.*

Proof. Assume that (1) holds. By Theorem A, (2) and (3) are satisfied ((3) of Theorem A implies (3) of the present lemma).

To prove (4), we choose a sequence (λ_n) with $|\lambda_n| > 1$ and $\lambda_n \to 1$ as $n \to \infty$. Using Lemma 1.3 we obtain the decomposition $X = R(I-T) \oplus N(I-T)$, and E is the projection of X onto $N(I-T)$ along $R(I-T)$. Then $(I-T)^l M_n^\alpha \to (I-T)^l E = 0$ for every integer $l \geq 1$. Hence (4) is satisfied.

Conversely, assume that (2)–(4) hold. Then $X = R(I-T) \oplus N(I-T)$ and from Theorems 1.1 and 1.2, $R(I-T)^n = R(I-T)$ is closed for every $n \geq 1$, so there exists a $k > 0$ such that for every $y \in R(I-T)^l$ there is an $x \in X$ such that $(I-T)^l x = y$ and $\|x\| \leq k\|y\|$.

For $x \in X$, we have $x = (I-E)x + Ex$ and $M_n^\alpha x - Ex = M_n^\alpha (I-E)x$. There is an $x_0 \in X$ such that $(I-T)^l x_0 = (I-E)x$ and $\|x_0\| \leq k\|(I-E)x\| \leq k\|I-E\| \cdot \|x\|$, thus

$$\|M_n^\alpha x - Ex\| = \|M_n^\alpha (I-T)^l x_0\| \leq \|(I-T)^l M_n^\alpha\| k \|I-E\| \cdot \|x\|.$$

Since $\|(I-T)^l M_n^\alpha\| \to 0$ as $n \to \infty$ it follows that $\|M_n^\alpha - E\| \to 0$ as $n \to \infty$.

In the following we shall consider the following extended concept.

For any real number α, we define

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \quad \text{for } n \geq 1, \quad A_0^\alpha = 1.$$

Then the equality

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}$$

remains valid for each real α and all $n = 0, 1, 2, \dots$ (see Lemma 2.3 below). For $\alpha \in \{-1, -2, \dots\}$, $A_n^\alpha = 0$ for every integer $n \geq -\alpha$.

Let $\alpha \in \mathbb{R} \setminus \{-1, -2, \dots\}$ and put

$$M_n^\alpha = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} T^k \quad \text{for } n = 0, 1, 2, \dots$$

We obtain the following lemma.

Lemma 2.3. (1) *$A_{n+1}^\alpha - A_n^\alpha = A_{n+1}^{\alpha-1}$ for any $\alpha \in \mathbb{R}$ and any integer $n \geq 0$.*

(2) *$\frac{\alpha+n+1}{n+1} M_{n+1}^\alpha - M_n^\alpha = \frac{\alpha}{n+1} M_{n+1}^{\alpha-1}$ for any $\alpha \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$.*

(3) *If α is a positive integer and $l = 1, \dots, \alpha$, or if α is a real positive non-integer and $l = 1, 2, \dots,$ then*

$$(T-I)^l M_n^\alpha = \frac{\alpha(\alpha-1)\dots(\alpha-l+1)}{(n+1)(n+2)\dots(n+l)} M_{n+l}^{\alpha-l} - P_{l-1}^n(T-I)$$

where

$$P_{l-1}^n(X) = \frac{\alpha}{n+1} X^{l-1} + \frac{\alpha(\alpha-1)}{(n+1)(n+2)} X^{l-2} + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-l+1)}{(n+1)(n+2)\dots(n+l)}.$$

Proof. (1) Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. Then $A_1^\alpha - A_0^\alpha = \alpha = A_1^{\alpha-1}$, and for $n \geq 1$,

$$A_{n+1}^\alpha - A_n^\alpha = \frac{(\alpha+1)\dots(\alpha+n+1)}{(n+1)!} - \frac{(\alpha+1)\dots(\alpha+n)}{n!} = A_{n+1}^{\alpha-1}.$$

(2) Let $\alpha \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$. Then

$$A_{n+1}^\alpha M_{n+1}^\alpha - A_n^\alpha M_n^\alpha = T^{n+1} + \sum_{k=0}^{n} (A_{n+1-k}^{\alpha-1} - A_{n-k}^{\alpha-1}) T^k$$

$$= T^{n+1} + \sum_{k=0}^{n} A_{n+1-k}^{\alpha-2} T^k = \sum_{k=0}^{n+1} A_{n+1-k}^{\alpha-2} T^k.$$

Dividing both sides of this equality by A_n^α, we obtain the desired result:

$$\frac{\alpha+n+1}{n+1}M_{n+1}^\alpha - M_n^\alpha = \frac{\alpha}{n+1}M_{n+1}^{\alpha-1}.$$

(3) Let α be a positive number.

(a) If α is an integer, then

$$\begin{aligned}(T-I)M_n^\alpha &= \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}T^{k+1} - M_n^\alpha \\ &= \frac{1}{A_n^\alpha}\sum_{k=0}^{n+1} A_{n+1-k}^{\alpha-1}T^{k} - M_n^\alpha - \frac{A_{n+1}^{\alpha-1}}{A_n^\alpha}\, I \\ &= \frac{\alpha+n+1}{n+1}\, M_{n+1}^\alpha - M_n^\alpha - \frac{\alpha}{n+1}\, I = \frac{\alpha}{n+1}\,(M_{n+1}^{\alpha-1} - I).\end{aligned}$$

If $\alpha = 1$ we are done. Next, if $\alpha \geq 2$ we apply $(T-I)^l$ to M_n^α for $l = 1, 2, \ldots, \alpha$ and we use (2) to obtain the desired result.

(b) If α is not an integer, then neither is $\alpha - l$ for each integer $l = 1, 2, \ldots,$ and $M_n^{\alpha-l}$ is well defined. In particular $\alpha - l$ is not in $\{0, -1, -2, \ldots\}$, so we can apply $(T-I)^l$ to M_n^α for any integer $l = 1, 2, \ldots$ and use (2) to obtain the corresponding result.

LEMMA 2.4. (a) *Let X be a Banach space and $T \in B(X)$. For every $\alpha > 0$, if $\|T^n\|/n^\alpha \to 0$ as $n \to \infty$, then* $\max_{k=0}^{n} \|T^k\|/n^\alpha \to 0$ *as $n \to \infty$.*

(b) *Let $(u_m^n)_{m,n}$ be a sequence of nonnegative numbers and $S_n = u_1^n + \ldots + u_n^n$ for $n = 1, 2, \ldots$ Then $S_n \to 0$ as $n \to \infty$ if and only if both $S_n^1 = u_1^n + \ldots + u_{[n/2]}^n \to 0$ and $S_n^2 = u_{[n/2]+1}^n + \ldots + u_n^n \to 0$ as $n \to \infty$, where $[\alpha]$ denotes the integer part of α.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. The condition is necessary by Theorem A. To prove that (2) and (3) imply (1), in view of Lemma 2.2 it is sufficient to show that T satisfies condition $\delta(l, \alpha)$ for some integer $l \geq 1$.

(a) If α is an integer, then

$$(T-I)^\alpha M_n^\alpha = \frac{\alpha!}{(n+1)(n+2)\ldots(n+\alpha)}\, M_{n+\alpha}^0 - P_{\alpha-1}^n(T-I).$$

It is clear that

$$\begin{aligned}P_{\alpha-1}^n(T-I) &= \frac{\alpha}{n+1}\,(T-I)^{\alpha-1} + \frac{\alpha(\alpha-1)}{(n+1)(n+2)}\,(T-I)^{\alpha-2} + \ldots \\ &\quad + \frac{\alpha!}{(n+1)(n+2)\ldots(n+\alpha)}\, I \to 0\end{aligned}$$

as $n \to \infty$. Thus $(T-I)^\alpha M_n^\alpha \to 0$ as $n \to \infty$ because $M_{n+\alpha}^0 = T^{n+\alpha}$ and $\|T^n\|/n^\alpha \to 0$ as $n \to \infty$.

(b) If α is not an integer, there exists a unique $\beta \in \mathbb{R}$, $0 < \beta < 1$, such that $\alpha = [\alpha] + \beta$. Now we prove that $(T-I)^l M_n^\alpha \to 0$ as $n \to \infty$ with $l = [\alpha]+1$. We have

$$(T-I)^{[\alpha]+1} M_n^\alpha = \frac{\alpha(\alpha-1)\ldots(\alpha-[\alpha])}{(n+1)(n+2)\ldots(n+[\alpha]+1)}\, M_{n+[\alpha]+1}^{\beta-1} - P_{[\alpha]}^n.$$

It is clear that

$$P_{[\alpha]}^n(T-I) = \frac{\alpha}{n+1}\,(T-I)^{[\alpha]} + \frac{\alpha(\alpha-1)}{(n+1)(n+2)}\,(T-I)^{[\alpha]-1} + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-[\alpha])}{(n+1)(n+2)\ldots(n+[\alpha]+1)}\, I \to 0$$

as $n \to \infty$. Thus $(T-I)^{[\alpha]+1} M_n^\alpha \to 0$ as $n \to \infty$ if and only if

$$\frac{\beta}{(n-[\alpha]+1)(n-[\alpha]+2)\ldots n(n+1)}\, M_{n+1}^{\beta-1} \to 0 \quad \text{as } n \to \infty.$$

Now, $(n-[\alpha]+1)(n-[\alpha]+2)\ldots n$ can be expressed as $q_{n,\alpha} n^{[\alpha]}$ with $q_{n,\alpha} \to 1$ as $n \to \infty$. Since

$$\begin{aligned}\frac{\beta}{n+1}\, M_{n+1}^{\beta-1} &= \frac{\beta+n+1}{n+1}\, M_{n+1}^{\beta} - M_n^{\beta} \\ &= \frac{1}{A_n^\beta}\Big[T^{n+1} + \sum_{k=0}^{n} (A_{n+1-k}^{\beta-1} - A_{n-k}^{\beta-1}) T^k\Big] \\ &= \frac{1}{A_n^\beta}\Big[T^{n+1} + \sum_{k=0}^{n} A_{n+1-k}^{\beta-2} T^k\Big] \\ &= \frac{1}{A_n^\beta}\, T^{n+1} + \frac{1}{A_n^\beta} \sum_{k=0}^{n} \frac{\beta-1}{n+1-k}\, A_{n-k}^{\beta-1} T^k.\end{aligned}$$

We have to show that this expression divided by $q_{n,\alpha} n^{[\alpha]}$ converges to zero as n tends to infinity. The first term $\|T^n\|/(q_{n,\alpha} n^{[\alpha]} A_n^\beta)$ is equivalent to $\Gamma(\beta+1)\|T^n\|/n^\alpha \to 0$ as $n \to \infty$. Thus it remains to show that

$$\frac{1}{q_{n,\alpha} n^{[\alpha]}} \cdot \frac{1}{A_n^\beta} \sum_{k=0}^{n} \frac{\beta-1}{n+1-k}\, A_{n-k}^{\beta-1} T^k \to 0 \quad \text{as } n \to \infty.$$

Put

$$I_1^n = \frac{1}{q_{n,\alpha} n^{[\alpha]}} \cdot \frac{1}{A_n^\beta} \sum_{k=0}^{[n/2]} \frac{\beta-1}{n+1-k}\, A_{n-k}^{\beta-1} T^k,$$

$$I_2^n = \frac{1}{q_{n,\alpha} n^{[\alpha]}} \cdot \frac{1}{A_n^\beta} \sum_{k=[n/2]+1}^{n} \frac{\beta-1}{n+1-k}\, A_{n-k}^{\beta-1} T^k.$$

Then

$$\|I_1^n\| \le \frac{2(1-\beta)}{q_{n,\alpha}} \cdot \frac{\max_{k=0}^n \|T^k\|}{n^{[\alpha]+1}} \to 0 \quad \text{as } n \to \infty.$$

Moreover,

$$I_2^n = \frac{1}{q_{n,\alpha} n^{[\alpha]}} \cdot \frac{1}{A_n^\beta} \sum_{k=[n/2]+1}^{n} \frac{\beta-1}{n+1-k} A_{n-k}^{\beta-1} T^k$$
$$= \frac{1}{q_{n,\alpha} n^{[\alpha]}} \cdot \frac{1}{A_n^\beta} \sum_{k=0}^{n-[n/2]-1} \frac{\beta-1}{k+1} A_k^{\beta-1} T^{n-k},$$

and so

$$\|I_2^n\| \le \frac{1-\beta}{q_{n,\alpha}} \cdot \frac{\max_{k=0}^n \|T^k\|}{n^{[\alpha]} A_n^\beta} \sum_{k=0}^{\infty} \frac{A_k^{\beta-1}}{k+1}.$$

The series $\sum_{k=0}^\infty A_k^{\beta-1}/(k+1)$ converges. Indeed, let $u_k = A_k^{\beta-1}/(k+1)$. Since

$$u_k = \frac{\beta(\beta+1)\dots(\beta-1+k)}{(k+1)k!} = \frac{\Gamma(\beta+k)}{\Gamma(k+2)\Gamma(\beta)},$$

u_k is equivalent to $1/(\Gamma(\beta)k^{2-\beta})$ as $k \to \infty$. The series $\sum_{k=1}^\infty 1/k^{2-\beta}$ converges and it follows that $\|I_2^n\| \to 0$ as $n \to \infty$. This completes the proof of Theorem 1.

If we look carefully at the above proof, we see that for fixed $\alpha > 0$,

$$\|T^n\|/n^\alpha \to 0 \overset{\text{(I)}}{\Longrightarrow} \begin{cases} \sigma(T) \subset D(0,1), \\ T \text{ satisfies condition } \delta(l,\alpha) \text{ for some } l \ge 1. \end{cases}$$

So, we summarize what we have proved as follows:

THEOREM 2. *Let T be a bounded linear operator in a Banach space X. There exists an operator $E \in B(X)$ such that, for fixed $\alpha > 0$,*

(1) $$\|M_n^\alpha - E\| \to 0 \quad \text{as } n \to \infty$$

if and only if

(2) $$\begin{aligned} &\text{(a) } \|(\lambda-1)R(\lambda,T) - E\| \to 0 && \text{as } \lambda \to 1^+, \\ &\text{(b) } \|T^n\|/n^\alpha \to 0 && \text{as } n \to \infty, \end{aligned}$$

if and only if

(3) $$\begin{aligned} &\text{(a) } \|(\lambda-1)R(\lambda,T) - E\| \to 0 && \text{as } \lambda \to 1^+, \\ &\text{(b) } \|(T-I)^l M_n^\alpha\| \to 0 && \text{as } n \to \infty \text{ for some } l \ge 1. \end{aligned}$$

Note that if T satisfies condition $\delta(l,\alpha)$ for some $l \ge 1$ then $\sigma(T) \subset D(0,1)$ and we will prove later that the converse of the implication (I) is not true in general, so (3)(b) in Theorem 2 is weaker than (2)(b).

3. Corollaries and an example

COROLLARY 3.1. *Let $\alpha > 0$ and $T \in B(X)$. If there exists an operator $E \in B(X)$ such that $\|M_n^\alpha - E\| \to 0$ as $n \to \infty$ then for every $\beta \geq \alpha$, $\|M_n^\beta - E\| \to 0$ as $n \to \infty$. Equivalently, if T is (C,α) uniformly ergodic for some $\alpha > 0$, then it is also uniformly ergodic for every $\beta \geq \alpha$.*

COROLLARY 3.2. *Let $\alpha > 0$, and let $T \in B(X)$ satisfy $\|T^n\|/n^\alpha \to 0$ as $n \to \infty$ and $\sup_n \|\sum_{k=0}^n A_{n-k}^{\alpha-1} T^k x\| < \infty$ for every $x \in \overline{R(I-T)}$. Then $R(I-T)$ is closed, and there exists an operator $E \in B(X)$ such that M_n^α converges to E in the uniform operator topology as $n \to \infty$.*

Now we shall give an example showing that condition (Y) is only sufficient but not necessary.

EXAMPLE. Let A be the bounded linear operator on the complex Banach space $X = C[0,1]$ defined by $Ax(t) = tx(t)$ for all $t \in [0,1]$. It is not hard to check that $\sigma(A) = [0,1]$ and $\|A^n\| = 1$ for $n = 0,1,2,\ldots$ Thus $\|A^n\|/n \to 0$ as $n \to \infty$ and 1 is not a pole of $R(\cdot, A)$. Consequently, $R(I-A)^k$ is not closed for every $k = 1,2,\ldots$ (see [M.Z, Theorem 1]). This is equivalent to saying that $M_n^1(A)$ does not converge. Now we consider the bounded linear operator B defined on the Banach space $Y = X \times X$ by $B = \left[\begin{smallmatrix} A & 0 \\ I & I \end{smallmatrix}\right]$. We check that $\sigma(B) = [0,1]$,

$$B^n = \begin{bmatrix} A^n & 0 \\ A^{n-1} + A^{n-2} + \ldots + I & I \end{bmatrix} \quad \text{for } n = 1,2,\ldots$$

and $R(I-B)$ is closed. Then, by the same reason as above, B^n/n cannot converge to zero as n goes to infinity. But it is clear that $B^n/n^2 \to 0$ as $n \to \infty$ since $\|A^n\| = 1$ for all $n \in \mathbb{N}$.

Let now T be the bounded linear operator on the Banach space $Y \times \mathbb{C}$ defined by $T = \left[\begin{smallmatrix} -B & 0 \\ 0 & I \end{smallmatrix}\right]$. Since $\sigma(-B) = [-1,0]$, we have $\sigma(T) = [-1,0] \cup \{1\}$, and

$$T^n = \begin{bmatrix} (-1)^n B^n & 0 \\ 0 & I \end{bmatrix} \quad \text{for } n = 1,2,\ldots$$

It is clear that T^n/n^2 converges to zero as n tends to infinity but T^n/n does not. Moreover $R(I-T) = R\left(\left[\begin{smallmatrix} I+B & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) = Y \times \{0\}$ and $N(I-T) = \{0\} \times \mathbb{C}$. Thus $R(I-T) \oplus N(I-T) = Y \times \mathbb{C}$, hence 1 is a pole of $R(\cdot, T)$ of order one (see [B, Theorem 1.2 or Theorem 1.3]), and therefore $\|(\lambda-1)R(\lambda,T) - E\| \to 0$ as $\lambda \to 1^+$ where E is the projection operator of X onto $N(I-T)$ along $R(I-T)$. Since $T^n/n^2 \to 0$ as $n \to \infty$ it follows from Theorem 1 that $\|M_n^2(T) - E\| \to 0$ as $n \to \infty$.

The following remark gives more information about the uniform (C,α) ergodicity of the bounded linear operator T in the above example.

REMARK. We can check that for any $\alpha > 1$, $\|T^n\|/n^\alpha$ converges to zero as n tends to infinity, and for any $0 < \alpha \le 1$, $\|T^n\|/n^\alpha$ does not converge. Since 1 is a pole of order one of the resolvent $R(\lambda, T)$, Theorem 1 ensures that there exists an $E \in B(X)$ such that $\|M_n^\alpha - E\| \to 0$ as $n \to \infty$ for any $\alpha > 1$, but M_n^α cannot converge in $B(X)$ for any $0 < \alpha \le 1$.

Now we shall prove that condition (3)(b) in Theorem 2 does not imply (2)(b). We consider the operator $B = \left[\begin{smallmatrix} A & 0 \\ I & I \end{smallmatrix}\right]$ used in the above example; we will check that if $\alpha > 1$, then $\|B^n\|/n^\alpha \to 0$ as $n \to \infty$, and if $0 < \alpha \le 1$, then $\|B^n\|/n^\alpha$ does not converge to zero. Take $0 < \alpha < 1$; from what we have just seen $\|B^n\|/n^\alpha$ cannot converge to zero; however, B satisfies condition $\delta(l, \alpha)$ for $l = 2$. Indeed,

$$(B - I)^2 M_n^\alpha(B) = \frac{\alpha(\alpha - 1)}{(n+1)(n+2)} M_{n+2}^{\alpha-2} - P_1^n(B - I).$$

Since $P_1^n(B - I) \to 0$ as $n \to \infty$, it suffices to prove that

$$\frac{1}{(n-1)n} M_n^{\alpha-2} = \frac{1}{(n-1)n} \cdot \frac{1}{A_n^{\alpha-2}} \sum_{k=0}^{n} A_{n-k}^{\alpha-3} B^k \to 0 \quad \text{as } n \to \infty.$$

Clearly

$$A_{n-k}^{\alpha-3} = \frac{(\alpha-2)(\alpha-1)\ldots(\alpha-3+n-k)}{(n-k)!} < 0 \quad \text{only if } k = n-1.$$

So we have

$$\sum_{k=0}^{n} \left| \frac{A_{n-k}^{\alpha-3}}{A_n^{\alpha-2}} \right| = \frac{1}{|A_n^{\alpha-2}|} \Big\{ \sum_{k \ne n-1} A_{n-k}^{\alpha-3} + |A_1^{\alpha-3}| \Big\}$$
$$= \left| \frac{A_n^{\alpha-2}}{|A_n^{\alpha-2}|} \left\{ \frac{1}{A_n^{\alpha-2}} \sum_{k \ne n-1} A_{n-k}^{\alpha-3} + \frac{|A_1^{\alpha-3}|}{A_n^{\alpha-2}} \right\} \right| = 1 + 2 \frac{|A_1^{\alpha-3}|}{|A_n^{\alpha-2}|}.$$

Given $\varepsilon > 0$, since $\|B^n\|/n^2 \to 0$ as $n \to \infty$, there exists N so large that $\|B^n\| \le \varepsilon n^2$ for all $n > N$. Then

$$\|M_n^{\alpha-2}(B)\| \le \frac{1}{|A_n^{\alpha-2}|} \sum_{k=0}^{N} |A_{n-k}^{\alpha-3}| \cdot \|B^k\| + \varepsilon n^2 \frac{1}{|A_n^{\alpha-2}|} \sum_{k=N+1}^{n} |A_{n-k}^{\alpha-3}|$$
$$\le (\max_{k=0}^{N} \|B^k\|)(N+1) \max_{k=0}^{N} \frac{|A_{n-k}^{\alpha-3}|}{|A_n^{\alpha-2}|} + \varepsilon n^2 \left(1 + 2 \frac{|A_1^{\alpha-3}|}{|A_n^{\alpha-2}|} \right).$$

For each $k = 0, 1, \ldots, N$, we have $|A_{n-k}^{\alpha-3}/A_n^{\alpha-3}| \to 0$ as $n \to \infty$, which yields

$$\sup_n \max_{0 \le k \le N} \frac{|A_{n-k}^{\alpha-3}|}{|A_n^{\alpha-2}|} = C_1 < \infty.$$

It follows that for every $n > N$,

$$\frac{1}{(n-1)n}\,\|M_n^{\alpha-2}(B)\| \le \frac{\max\limits_{k=0}^{N}\|B^k\|\cdot(N+1)C_1}{(n-1)n} + \varepsilon\,\frac{n^2}{(n-1)n}\left(1+2\,\frac{|A_1^{\alpha-3}|}{A_n^{\alpha-2}}\right).$$

Obviously

$$\sup_n \frac{n^2}{(n-1)n}\left(1+2\,\frac{|A_1^{\alpha-3}|}{A_n^{\alpha-2}}\right) = C_2 < \infty.$$

Thus

$$\frac{1}{(n-1)n}\|M_n^{\alpha-2}(B)\| \le \frac{\max\limits_{k=0}^{N}\|B^k\|\cdot(N+1)C_1}{(n-1)n} + \varepsilon C_2.$$

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