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On the functional equation defined by Lie's product formula

by

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Abstract. Let E be a real normed space and $\mathcal{A}$ a complex Banach algebra with unit. We characterize the continuous solutions $f : E \to \mathcal{A}$ of the functional equation $f(x+y) = \lim_{n\to\infty}(f(x/n)f(y/n))^n$.

Let $\mathcal{A}$ be a complex Banach algebra with unit $\mathbf{1}$. In this setting the famous Lie product formula reads

$$\exp(a+b) = \lim_{n\to\infty}(\exp(a/n)\exp(b/n))^n \quad (a,b \in \mathcal{A}); \tag{1}$$

see [4, Theorem VIII.29] for matrices and a proof which also holds for Banach algebras, and Trotter's version for semigroups [6].

Let $p \in \mathcal{A}$ be a projection (that is, $p^2 = p$), and consider the complex Banach algebra $p\mathcal{A}p$ which has unit p. The exponential function in $p\mathcal{A}p$ will be denoted by $\exp_p$. Now, let E be a real normed space, let $A : E \to p\mathcal{A}p$ be a continuous and linear mapping (here $p\mathcal{A}p$ is considered as a real vector space), and set

$$f(x) := \exp_p(A(x)) = p\exp(A(x))p \quad (x \in E). \tag{2}$$

As an immediate consequence of (1) the function f is a continuous solution of the functional equation

$$f(x+y) = \lim_{n\to\infty}(f(x/n)f(y/n))^n \quad (x,y \in E). \tag{3}$$

In this paper we prove conversely that all continuous solutions of (3) are of type (2). More precisely we have

THEOREM 1. *Let $f : E \to \mathcal{A}$ be a continuous function which satisfies* (3). *Then $p = f(0)$ is a projection, and there exists a unique continuous linear mapping $A : E \to p\mathcal{A}p$ such that*

$$f(x) = \exp_p(A(x)) \quad (x \in E).$$

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In the proof of Theorem 1 we use the following proposition. Note that $\sigma(a)$ denotes the spectrum of $a \in \mathcal{A}$.

PROPOSITION 1. *Let $a, b \in \mathcal{A}$.*

(i) *If $a^3 = a$ and $\sigma(a) \subseteq \{0,1\}$, then $a^2 = a$.*
(ii) *If $\exp(a) = \exp(b)$ and $\|a\| < \pi$, then $ab = ba$.*
(iii) *If $\exp(ta) = \mathbf{1}$ $(t > 0)$, then $a = 0$.*

Proof. (i) and (ii) follow from [2, Propositions 8.11 and 18.12], respectively, and (iii) follows by differentiation. ∎

Proof of Theorem 1.

STEP 1. *We have*

$$p^2 = p. \tag{4}$$

Proof. From (3) we obtain

$$f(0) = \lim_{n\to\infty} f(0)^{2n} \;\Rightarrow\; f(0)^3 = \lim_{n\to\infty} f(0)^{2n+2} = f(0).$$

Hence $\sigma(f(0)) \subseteq \{-1,0,1\}$. Now, assume $-1 \in \sigma(f(0))$. Choose open sets $U, V \subseteq \mathbb{C}$ such that $U \cap V = \emptyset$, $-1 \in U$, and $0, 1 \in V$. Then $\sigma(f(0)) \subseteq U \cup V$ and $\sigma(f(0)) \cap U \neq \emptyset$. Since $f(0)^{2n} \to f(0)$ as $n \to \infty$, Theorem 3.4.4 in [1] proves

$$\sigma(f(0)^{2n}) \cap U \neq \emptyset$$

for n sufficiently large. But $\sigma(f(0)^{2n}) \subseteq \{0,1\} \subseteq V$, a contradiction. Therefore $\sigma(f(0)) \subseteq \{0,1\}$ and (4) follows from Proposition 1(i).

STEP 2. *We have*

$$f(x) = pf(x) = f(x)p = pf(x)p \qquad (x \in E); \tag{5}$$

in particular $f(x) \in p\mathcal{A}p$ $(x \in E)$.

Proof. According to (3),

$$\begin{aligned} pf(x) &= f(0)f(x+0) = f(0)\lim_{n\to\infty}(f(x/n)f(0))^n \\ &= (\lim_{n\to\infty}(f(0)f(x/n))^n)f(0) = f(0+x)f(0) = f(x)p \end{aligned}$$

for each $x \in E$. Thus,

$$\begin{aligned} f(x) = f(x+0) &= \lim_{n\to\infty} f(x/n)^n f(0)^n \overset{(4)}{=} \lim_{n\to\infty} f(x/n)^n f(0)^{n+1} \\ &= (\lim_{n\to\infty} f(x/n)^n f(0)^n)f(0) = f(x+0)f(0) = f(x)p, \end{aligned}$$

and we have (5). In particular, if $p = 0$ then $f(x) = 0$ $(x \in E)$.

STEP 3. *For $x \in E$ and $m \in \mathbb{N}$,*

$$f(mx) = f(x)^m. \tag{6}$$

Proof. Note that by (5),

$$f(x) = \lim_{n\to\infty} f(x/n)^n \quad (x \in E).$$

First consider $m = 2$. Again from (3) we obtain

$$f(2x) = f(x+x) = \lim_{n\to\infty} f(x/n)^{2n} = \lim_{n\to\infty} (f(x/n)^n)^2 = f(x)^2.$$

Now, let $m > 2$ and suppose $f(mx) = f(x)^m$ $(x \in E)$. Then

$$\begin{aligned} f((m+1)x) &= \lim_{n\to\infty} (f(mx/n)f(x/n))^n = \lim_{n\to\infty} (f(x/n)^m f(x/n))^n \\ &= \lim_{n\to\infty} (f(x/n)^n)^{m+1} = f(x)^{m+1}. \end{aligned}$$

Thus (6) holds by induction.

Next, for each $x \in E$ let $f_x : \mathbb{R} \to \mathcal{A}$ be defined by

$$f_x(\alpha) := f(\alpha x).$$

STEP 4. *We have*

$$f_x(\alpha)f_x(\beta) = f_x(\beta)f_x(\alpha) \quad (\alpha, \beta \geq 0,\ x \in E). \tag{7}$$

Proof. Let $x \in E$ and $m, n, r, s \in \mathbb{N}$. Now

$$\begin{aligned} f_x(1/r)f_x(1/s) = f(x/r)f(x/s) &= f\left(s\,\frac{x}{rs}\right) f\left(r\,\frac{x}{rs}\right) \\ &\overset{(6)}{=} f\left(\frac{x}{rs}\right)^{s+r} = f_x(1/s)f_x(1/r). \end{aligned}$$

Hence

$$\begin{aligned} f_x(m/r)f_x(n/s) &= f\left(m\,\frac{x}{r}\right) f\left(n\,\frac{x}{s}\right) \overset{(6)}{=} f(x/r)^m f(x/s)^n = f(x/s)^n f(x/r)^m \\ &\overset{(6)}{=} f\left(n\,\frac{x}{s}\right) f\left(m\,\frac{x}{r}\right) = f_x(n/s)f_x(m/r). \end{aligned}$$

Therefore (7) is valid for $\alpha, \beta \in \mathbb{Q} \cap [0, \infty)$, hence for $\alpha, \beta \in [0, \infty)$, since f is continuous.

STEP 5. *We have*

$$f_x(\alpha + \beta) = f_x(\alpha)f_x(\beta) \quad (\alpha, \beta \geq 0,\ x \in E). \tag{8}$$

Proof. For $\alpha, \beta \geq 0$,

$$\begin{aligned} f_x(\alpha+\beta) = f(\alpha x + \beta x) &\overset{(3)}{=} \lim_{n\to\infty} \left(f\left(\alpha\,\frac{x}{n}\right) f\left(\beta\,\frac{x}{n}\right)\right)^n \\ &\overset{(7)}{=} \lim_{n\to\infty} f\left(\alpha\,\frac{x}{n}\right)^n f\left(\beta\,\frac{x}{n}\right)^n \overset{(6)}{=} \lim_{n\to\infty} f(\alpha x)f(\beta x) = f_x(\alpha)f_x(\beta). \end{aligned}$$

STEP 6. *The limit*

$$A(x) := \lim_{\alpha\to 0+} \frac{1}{\alpha} \left(f_x(\alpha) - p\right) \tag{9}$$

exists for each $x \in E$. Moreover

$$A(x) \in p\mathcal{A}p \qquad (x \in E), \tag{10}$$

and

$$f(\alpha x) = p + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} A(x)^n \qquad (\alpha \geq 0). \tag{11}$$

Note that in particular for $\alpha = 1$ we have

$$f(x) = p + \sum_{n=1}^{\infty} \frac{A(x)^n}{n!} = \exp_p(A(x)) = p\exp(A(x))p \qquad (x \in E). \tag{12}$$

Proof. Since $f_x : [0, \infty) \to \mathcal{A}$ is a continuous solution of the functional equation in (8), the existence of the limit in (9) and the equation (10) follow from [3, Theorem 9.4.2]. Now, (10) follows from (5) and (9).

STEP 7. *We have*

$$A(\beta x) = \beta A(x) \qquad (\beta \geq 0,\, x \in E). \tag{13}$$

Proof. Obviously (13) holds for $\beta = 0$. For $\beta > 0$,

$$A(\beta x) = \lim_{\alpha \to 0+} \frac{1}{\alpha}\,(f(\alpha\beta x) - p) = \lim_{\alpha \to 0+} \frac{\beta}{\alpha\beta}\,(f_x(\alpha\beta) - p) = \beta A(x).$$

STEP 8. *For $x, y \in E$,*

$$\exp_p(A(x+y)) = \exp_p(A(x) + A(y)), \tag{14}$$

and

$$A(x+y)(A(x) + A(y)) = (A(x) + A(y))A(x+y). \tag{15}$$

Proof. Fix $x, y \in E$ and let $\alpha > 0$. Set

$$a := A(\alpha(x+y)), \qquad b := A(\alpha x) + A(\alpha y).$$

Then, by Lie's product formula, and by (12) and (13),

$$\begin{aligned} \exp_p(b) &= \lim_{n\to\infty} (\exp_p(A(\alpha x/n)) \exp_p(A(\alpha y/n)))^n \\ &\overset{(3)}{=} \lim_{n\to\infty} (f(\alpha x/n) f(\alpha y/n))^n = f(\alpha x + \alpha y) \\ &= \exp_p(A(\alpha x + \alpha y)) = \exp_p(a). \end{aligned}$$

For $\alpha = 1$ we obtain (14), and by choosing $\alpha > 0$ such that

$$\|a\| = \alpha\|A(x+y)\| < \pi,$$

Proposition 1(ii) proves $ab = ba$, hence (15).

STEP 9. *We have*

$$A(x+y) = A(x) + A(y) \qquad (x, y \in E). \tag{16}$$

Proof. According to (14) and (15) we have

$$\exp_p(A(x+y)-(A(x)+A(y))) = p \quad (x, y \in E).$$

Fix $x, y \in E$. By (13),

$$\exp_p(t(A(x+y)-(A(x)+A(y)))) = p \quad (t > 0),$$

and Proposition 1(iii) proves (16).

STEP 10. *We have*

$$A(\alpha x) = \alpha A(x) \quad (\alpha \in \mathbb{R},\, x \in E). \tag{17}$$

Proof. Fix $x \in E$. Then

$$\exp_p(t(A(x)+A(-x))) \overset{(13)}{=} \exp_p(A(tx)+A(-tx)) \overset{(16)}{=} \exp_p(A(tx-tx)) = p$$

for each $t > 0$. Again, $A(-x) = -A(x)$ follows from Proposition 1(iii). In combination with (13) this gives (17).

STEP 11. *The linear mapping $A : E \to p\mathcal{A}p$ is continuous.*

Proof. It is sufficient to prove that A is continuous at 0. Assume the contrary. Then there is a sequence (x_n) in E with $\|x_n\| = 1$ ($n \in \mathbb{N}$) and $\|A(x_n)\| \to \infty$ ($n \to \infty$). Set

$$y_n = \frac{x_n}{3\|A(x_n)\|}, \qquad z_n = \frac{A(x_n)}{3\|A(x_n)\|}.$$

We have

$$f(y_n) = \exp_p(A(y_n)) = \exp_p(z_n) \to p \quad (n \to \infty),$$

because $y_n \to 0$ ($n \to \infty$). Since $\|\exp_p(z_n) - p\| \le 1/2 < 1$, we conclude

$$z_n = \log_p(\exp_p(z_n)) \to 0 \quad (n \to \infty),$$

a contradiction. Here $\log_p$ denotes the power series

$$\log_p(p+a) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, a^k \quad (a \in p\mathcal{A}p,\, \|a\| < 1).$$

Finally, concerning the uniqueness of A, let $B : E \to p\mathcal{A}p$ be a continuous linear operator such that

$$f(x) = \exp_p(B(x)) \quad (x \in E).$$

Then

$$\frac{1}{\alpha}\,(f(\alpha x) - p) = \frac{1}{\alpha}\,(\exp_p(\alpha(B(x))) - p) \to B(x) \quad (\alpha \to 0+).$$

According to (9), $A(x) = B(x)$ ($x \in E$). ∎

As an application of Theorem 1 we may characterize in terms of A those continuous solutions of (3) which satisfiy the exponential equation of Cauchy

$$f(x+y) = f(x)f(y) \quad (x, y \in E). \tag{18}$$

Corollary 1. *Let $f : E \to \mathcal{A}$ be a continuous solution of* (3), *and let p and $A : E \to p\mathcal{A}p$ be as in Theorem* 1. *Then* (18) *holds if and only if*

$$A(x)A(y) = A(y)A(x) \quad (x, y \in E). \tag{19}$$

Proof. If (19) holds then clearly

$$f(x+y) = \exp_p(A(x) + A(y)) = \exp_p(A(x)) \exp_p(A(y)) = f(x)f(y)$$

for $x, y \in E$.

Now, let (18) be valid. Then $f(x)f(y) = f(y)f(x)$, hence

$$\exp_p(A(x)) \exp_p(A(y)) = \exp_p(A(y)) \exp_p(A(x)) \quad (x, y \in E).$$

Fix $x, y \in E$ and let $\alpha > 0$ be such that

$$\max\{\|A(\alpha x)\|, \|A(\alpha y)\|\} < \pi.$$

According to the result in [5],

$$A(\alpha x)A(\alpha y) = A(\alpha y)A(\alpha x),$$

from which (19) follows. ∎

Remark. Theorem 1 is also valid if $\mathcal{A}$ is a real Banach algebra with unit $\mathbf{1}$. In this case apply the complex version to the complexification $\mathcal{A}_{\mathbb{C}}$ of $\mathcal{A}$ and note that $p = f(0) \in \mathcal{A}$ and that A maps E to $p\mathcal{A}p$ according to (9).

As an example consider $E = \mathbb{R}$ and assume that $f : \mathbb{R} \to \mathcal{A}$ is a continuous solution of (3) with $f(0)$ invertible. Then by Theorem 1, $f(0) = \mathbf{1}$ and there is a unique $a \in \mathcal{A}$ such that

$$f(x) = \exp(xa) \quad (x \in \mathbb{R}).$$

Here f is a solution of (18).

On the other hand consider $E = \mathbb{R}^2$ and again assume that $f : \mathbb{R}^2 \to \mathcal{A}$ is a continuous solution of (3) with $f(0)$ invertible. Then there exist unique $a, b \in \mathcal{A}$ such that

$$f((x_1, x_2)) = \exp(x_1 a + x_2 b) \quad ((x_1, x_2) \in \mathbb{R}^2).$$

Here, f is a solution of (18) if and only if $ab = ba$.

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