Parabolic potentials and wavelet transforms with the generalized translation

by

Ilham A. Aliev (Antalya) and Boris Rubin (Jerusalem)

Abstract. Parabolic wavelet transforms associated with the singular heat operators $-\Delta_\gamma + \partial/\partial t$ and $I - \Delta_\gamma + \partial/\partial t$, where $\Delta_\gamma = \sum_{k=1}^{n} \partial^2/\partial x_k^2 + (2\gamma/x_n)\partial/\partial x_n$, are introduced. These transforms are defined in terms of the relevant generalized translation operator. An analogue of the Calderón reproducing formula is established. New inversion formulas are obtained for generalized parabolic potentials representing negative powers of the singular heat operators.

1. Introduction. The continuous wavelet transforms

$$(1.1) \qquad f(x) \mapsto Wf(x,t) = \frac{1}{t^n} \int_{\mathbb{R}^n} f(y) w\left(\frac{|x-y|}{t}\right) dy,$$

where $x \in \mathbb{R}^n$, $t \in (0,\infty)$, $\int_{\mathbb{R}^n} w(|x|)\, dx = 0$, and the corresponding Calderón-type reproducing formula

$$(1.2) \qquad \int_0^\infty Wf(x,t)\, \frac{dt}{t} = c_w f(x),$$

were studied by many authors in connection with various applications (see, e.g., [14] for further references). The more general formula

$$(1.3) \qquad \int_0^\infty Wf(x,t)\, \frac{dt}{t^{1+\alpha}} = c_{\alpha,w}(-\Delta)^{\alpha/2} f(x), \qquad \alpha \in \mathbb{C},$$

gives integral representations of complex powers of $-\Delta$ where Δ is the Laplacian in $\mathbb{R}^n$. In the present article we extend these results to the case when

Δ is replaced by the singular heat operators

$$(1.4) \qquad -\Delta_\gamma + \frac{\partial}{\partial t} \quad \text{and} \quad I - \Delta_\gamma + \frac{\partial}{\partial t}, \qquad \Delta_\gamma = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{2\gamma}{x_n} \frac{\partial}{\partial x_n}.$$

We introduce new wavelet transforms which reflect anisotropy and the character of the singularity in (1.4) simultaneously. The problem was inspired, on the one hand, by the earlier papers of A. Weinstein, M. N. Olevskiĭ and other authors (see, e.g., [24, 25]) related to singular PDE and the corresponding generalized axially symmetric problems. On the other hand, our study was motivated by needs of Fourier–Bessel harmonic analysis which employs the notion of the generalized translation operator and goes back to S. Bochner, J. Delsarte, B. M. Levitan, and others (see [8–11, 20, 21, 23] for further details and references). Discrete wavelet systems were studied by many authors; see, e.g., the nice book of P. Wojtaszczyk [26] and references therein.

The plan of the paper is as follows. Some necessary definitions and auxiliary facts are given in Section 2. In particular, we introduce new notions of "generalized parabolic wavelet transforms" associated with the singular heat operators (1.4). In Section 3 we obtain inversion formulas (Calderón-type reproducing formulas) for these transforms. The generalized parabolic potentials associated with (1.4) and defined in Fourier–Bessel terms by

$$(1.5) \qquad (H_\gamma^\alpha f)^\wedge(x,t) = (|x|^2 + it)^{-\alpha/2} f^\wedge(x,t),$$

$$(1.6) \qquad (\mathcal{H}_\gamma^\alpha f)^\wedge(x,t) = (1 + |x|^2 + it)^{-\alpha/2} f^\wedge(x,t),$$

are studied in Section 4. We obtain wavelet-type representations of these potentials, like (1.3), and prove the relevant (L^p-L^q)-estimates of the Hardy–Littlewood–Sobolev type. Section 5 is devoted to the inversion of the generalized parabolic potentials by means of the relevant continuous wavelet transforms introduced in Section 2.

Historical remarks. Parabolic potentials associated with ordinary (nonsingular) heat operators were introduced by B. F. Jones and C. H. Sampson [7, 17]. Various properties of these potentials and the relevant anisotropic function spaces of the Sobolev type were studied by R. Bagby, V. R. Gopala Rao, S. Chanillo, V. A. Nogin and B. S. Rubin (see [12, 16] and references therein). Continuous wavelet transforms corresponding to these potentials were introduced in the authors' paper [4]. Singular parabolic equations were studied by many authors (see, e.g., [5, 22]). The relevant parabolic potentials were introduced by I. Aliev [1, 2]. The wavelet approach to these potentials, developed below, seems to be new. In subsequent publications we plan to give some applications of our results to singular parabolic equations.

2. Preliminaries. Let $x = (x_1, \ldots, x_{n-1}, x_n) \equiv (x', x_n) \in \mathbb{R}^N$; $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$; $\dot{\mathbb{R}}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$; $d\nu(x) = x_n^{2\gamma} \, dx$, $\gamma > 0$. The Fourier–Bessel transform of a sufficiently nice function $f : \mathbb{R}^n_+ \to \mathbb{C}$ and its inverse are defined by

$$(2.1) \qquad f^\wedge(x) \equiv (F_\gamma f)(x) = \int\limits_{\mathbb{R}^n_+} f(y) e^{-ix' \cdot y'} j_{\gamma - 1/2}(x_n y_n) \, d\nu(y),$$

$$F_\gamma^{-1} f(x) = c_\gamma^2(n)(F_\gamma f)(-x', x_n), \qquad x = (x', x_n) \in \mathbb{R}^n_+,$$

$$(2.2) \qquad c_\gamma(n) = [2^{\gamma - 1/2}(2\pi)^{(n-1)/2} \Gamma(\gamma + 1/2)]^{-1},$$

where $j_\lambda(z) = 2^\lambda \Gamma(\lambda + 1) z^{-\lambda} J_\lambda(z)$, $\lambda > -1/2$, is the normalized Bessel function such that $j_\lambda(0) = 1$ (see, e.g., [8, 9]). The same definition was applied in (1.5) and (1.6) to functions on $\mathbb{R}^n_+ \times \mathbb{R}^1 = \{(x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1} : x_n > 0\}$.

We need the following weighted L^p-spaces, $1 \leq p < \infty$:

$$X_p \equiv L^p(\mathbb{R}^n_+; d\nu(x)) = \Big\{ g : \|g\|_{X_{p,\gamma}} = \Big(\int\limits_{\mathbb{R}^n_+} |g(x)|^p \, d\nu(x) \Big)^{1/p} < \infty \Big\},$$

$$Y_p \equiv L^p(\mathbb{R}^n_+ \times \mathbb{R}^1; d\nu(x)dt) = \Big\{ f : \Big(\int\limits_{\mathbb{R}^n_+ \times \mathbb{R}^1} |f(x,t)|^p \, d\nu(x) \, dt \Big)^{1/p} < \infty \Big\}.$$

In the case $p = \infty$ we identify L^∞ with C^0, the corresponding space of continuous functions vanishing at infinity.

Given a finite (signed) Borel measure $m \equiv m(y, \tau)$, $y \in \dot{\mathbb{R}}^n_+$, $\tau \in \mathbb{R}^1$, and $\eta > 0$, we define an anisotropic dilation $m_{\sqrt{\eta}, \eta}$ of m by

$$(2.3) \qquad \int\limits_{\dot{\mathbb{R}}^n_+ \times \mathbb{R}^1} \omega(y, \tau) \, dm_{\sqrt{\eta}, \eta}(y, \tau) = \int\limits_{\dot{\mathbb{R}}^n_+ \times \mathbb{R}^1} \omega(\sqrt{\eta}\, y, \eta \tau) \, dm(y, \tau)$$

for $\omega \in C^0 \equiv C^0(\dot{\mathbb{R}}^n_+ \times \mathbb{R}^1)$. In particular, for $dm(y, \tau) = \varphi(y, \tau) d\nu(y) d\tau$ with $\varphi \in Y_1$, we have

$$dm_{\sqrt{\eta}, \eta}(y, \tau) = \varphi_{\sqrt{\eta}, \eta}(y, \tau) d\nu(y) d\tau,$$

$$\varphi_{\sqrt{\eta}, \eta}(y, \tau) = (\sqrt{\eta})^{-(n+2\gamma+2)} \varphi\Big(\frac{y}{\sqrt{\eta}}, \frac{\tau}{\eta} \Big).$$

For $x \in \mathbb{R}^n_+$, $y \in \dot{\mathbb{R}}^n_+$, and $t, \tau \in \mathbb{R}^1$, the *generalized translation* of $f : \mathbb{R}^n_+ \times \mathbb{R}^1 \to \mathbb{C}$ is defined by

$$(2.4) \quad T^{y,\tau} f(x,t) = \frac{\Gamma(\gamma + 1/2)}{\Gamma(\gamma) \Gamma(1/2)} \int\limits_0^\pi f(x' - y'; \sqrt{x_n^2 - 2x_n y_n \cos\beta + y_n^2}; t - \tau)$$

$$\times \sin^{2\gamma - 1} \beta \, d\beta$$

(cf. [8, 9, 23]). Here we actually deal with the ordinary translation in x' and t, and with the generalized translation in x_n. For the relevant one-dimensional operator

$$S^\varrho g(r) = \frac{\Gamma(\gamma + 1/2)}{\Gamma(\gamma)\Gamma(1/2)} \int_0^\pi g(\sqrt{r^2 - 2r\varrho\cos\beta + \varrho^2})\sin^{2\gamma-1}\beta\, d\beta$$

the following relations are known [9]:

(2.5) $\qquad S^\varrho g(r) = S^r g(\varrho), \quad S^\varrho S^\tau g(r) = S^\tau S^\varrho g(r),$

(2.6) $\qquad S^\varrho g(r) = S^{-\varrho} g(r), \quad S^0 g(r) = g(r),$

(2.7) $\qquad \displaystyle\int_0^\infty f(r)S^r g(\varrho) r^{2\gamma}\, dr = \int_0^\infty S^r f(\varrho)g(r) r^{2\gamma}\, dr.$

Note also that for $1 \le p \le \infty$,

(2.8) $\qquad\qquad\qquad\qquad \|T^{y,\tau}f\|_{Y_p} \le \|f\|_{Y_p},$

(2.9) $\qquad \|T^{y,\tau}f - f\|_{Y_p} \to 0 \quad \text{as } (y,\tau) \to (0,0) \in \dot{\mathbb{R}}_+^n \times \mathbb{R}^1,$

Y_∞ being identified with C^0 (see, e.g., [10]).

The *generalized convolution* associated with the generalized translation operator (2.4) is defined by

(2.10) $\quad (f \otimes g)(x,t) = \displaystyle\int_{\dot{\mathbb{R}}_+^n \times \mathbb{R}^1} g(y,\tau)\, T^{y,\tau}f(x,t)\, d\nu(y)\, d\tau, \quad (x,t) \in \mathbb{R}_+^n \times \mathbb{R}^1.$

By using (2.8) and the Riesz–Thorin interpolation theorem it is not difficult to prove the corresponding Young inequality

(2.11) $\qquad \|f \otimes g\|_{Y_r} \le \|f\|_{Y_p}\|g\|_{Y_q}, \quad 1 \le p,q,r \le \infty, \ \dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r} + 1.$

Similarly to (2.10), for a finite Borel measure $m = m(y,\tau)$ on $\dot{\mathbb{R}}_+^n \times \mathbb{R}^1$ we define

(2.12) $\quad (f \otimes m)(x,t) = \displaystyle\int_{\dot{\mathbb{R}}_+^n \times \mathbb{R}^1} T^{y,\tau}f(x,t)\, dm(y,\tau), \quad (x,t) \in \mathbb{R}_+^n \times \mathbb{R}^1.$

DEFINITION 2.1. Given a finite (signed) Borel measure m such that $m(\dot{\mathbb{R}}_+^n \times \mathbb{R}^1) = 0$, a *generalized anisotropic continuous wavelet transform* of $f : \mathbb{R}_+^n \times \mathbb{R}^1 \to \mathbb{C}$ is defined by

(2.13) $\quad V_m f(x,t;\eta) = (f \otimes m_{\sqrt{\eta},\eta})(x,t) \equiv \displaystyle\int_{\dot{\mathbb{R}}_+^n \times \mathbb{R}^1} T^{\sqrt{\eta}y,\eta\tau} f(x,t)\, dm(y,\tau),$

$$(x,t) \in \mathbb{R}_+^n \times \mathbb{R}^1, \ \eta > 0.$$

The choice of the "wavelet measure" m is at our disposal. We specify m as follows. Let us introduce the *generalized Gauss–Weierstrass kernel*

$$(2.14) \quad W_\gamma(y,s) = c_\gamma(n)(2s)^{-(n+2\gamma)/2} \exp(-|y|^2/(4s)), \quad y \in \mathbb{R}^n_+, \ s > 0,$$

$c_\gamma(n)$ being defined by (2.2) (see [20] for $n = 1$ and [2] for any $n \geq 1$).

LEMMA 2.2. (a) $(F_\gamma W_\gamma(\cdot, s))(x) = \exp(-s|x|^2)$.

(b) $W_\gamma(y,s)$ *is an anisotropic* $(1/2, 1)$*-homogeneous function of order* $k = -\gamma - n/2$, *i.e.*

$$(2.15) \qquad W_\gamma(\lambda^{1/2}y, \lambda s) = \lambda^{-\gamma-n/2} W_\gamma(y,s), \quad \forall y \in \mathbb{R}^n_+, \ s > 0, \ \lambda > 0.$$

In particular,

$$(2.16) \qquad\qquad W_\gamma(\lambda^{1/2}y, \lambda) = \lambda^{-\gamma-n/2} W_\gamma(y, 1).$$

(c) *We have*

$$(2.17) \qquad\qquad \|W_\gamma(\cdot, s)\|_{X_1} \equiv \int_{\mathbb{R}^n_+} W_\gamma(y,s)\, d\nu(y) = 1.$$

(d) *Moreover,*

$$(2.18) \quad (W_\gamma(\cdot, \tau) \otimes W_\gamma(\cdot, t))(x)$$
$$\equiv \int_{\mathbb{R}^n_+} W_\gamma(y, \tau) T^y W_\gamma(x, t)\, d\nu(y) = W_\gamma(x, t + \tau),$$

T^y *being the generalized translation operator in the* x*-variable* (cf. (2.4)).

Proof. (a) is a consequence of the formulas

$$\int_{\mathbb{R}^n} e^{-s|x|^2} e^{-ix \cdot y}\, dx = (\pi/s)^{n/2} \exp(-|y|^2/(4s)),$$

$$\int_0^\infty e^{-sr^2} r^{\gamma+1} J_\gamma(tr)\, dr = t^\gamma (2s)^{-\gamma-1} \exp(-t^2/(4s));$$

(b) is obvious; (c) follows from (a); (d) is clear in terms of the Fourier–Bessel transform because $F_\gamma(\varphi \otimes \psi) = (F_\gamma \varphi)(F_\gamma \psi)$. ∎

Given a function $g : \mathbb{R}^n_+ \to \mathbb{C}$, we introduce the *generalized Gauss–Weierstrass integral*

$$(2.19) \qquad (G_\gamma g)(x, s) = (g \otimes W_\gamma(\cdot, s))(x) = \int_{\mathbb{R}^n_+} T^y g(x) W_\gamma(y, s)\, d\nu(y),$$

and the γ-*maximal function*

$$(2.20) \qquad M_\gamma g(x) = \sup_{r>0} \frac{1}{r^{n+2\gamma}\omega(n,\gamma)} \int_{B_r^+} |T^x g(y)|\, d\nu(y),$$

where

$$B_r^+ = \{y : y \in \mathbb{R}_+^n, \ |y| \leq r\}, \quad \omega(n, \gamma) = \int\limits_{B_1^+} d\nu(y),$$

so that

$$(2.21) \qquad \qquad \|M_\gamma g\|_{X_\varrho} \leq c\|g\|_{X_\varrho}, \quad 1 < p \leq \infty,$$

$$(2.22) \qquad \qquad \sup_{s>0} |G_\gamma g(x, s)| \leq M_\gamma g(x)$$

(see [20, 21] and [3, p. 370]).

Now we specify m in Definition 2.1. Let μ be a finite Borel measure on $\mathbb{R}^1$ such that $\mu(\mathbb{R}^1) = 0$ and $\operatorname{supp}\mu \subset [0, \infty)$. Define $\mu' = \mu - \mu(\{0\})\delta$, where $\delta = \delta(\tau)$ is the unit mass at $\tau = 0$, and set

$$(2.23) \qquad m = \mu(\{0\})\delta(y, \tau) + m', \quad dm'(y, \tau) = W_\gamma(y, \tau)d\nu(y)d\mu'(\tau),$$

where $\delta(y, \tau)$ is the unit mass at $(0, 0) \in \dot{\mathbb{R}}_+^n \times \mathbb{R}^1$ and $W_\gamma(y, \tau)$ is extended by zero to $\tau < 0$. Clearly, m is finite and by (2.17),

$$m(\dot{\mathbb{R}}_+^n \times \mathbb{R}^1) = \int\limits_{\mathbb{R}^1} d\mu(\tau) \int\limits_{\mathbb{R}_+^n} W_\gamma(y, \tau)\, d\nu(y) = \int\limits_{\mathbb{R}^1} d\mu(\tau) = 0.$$

DEFINITION 2.3. The anisotropic wavelet transform (2.13) with m defined by (2.23) will be called a *generalized parabolic wavelet transform* and denoted by $V_\mu f(x, t; \eta)$. Thus

$$(2.24) \qquad V_\mu f(x, t; \eta) = \int\limits_{\mathbb{R}_+^n \times [0, \infty)} T^{\sqrt{\eta}y, \eta\tau} f(x, t)W_\gamma(y, \tau)\, d\nu(y)\, d\mu(\tau)$$

or, by (2.16) and (2.17),

$$(2.25) \quad V_\mu f(x, t; \eta) = \int\limits_{\mathbb{R}_+^n \times [0, \infty)} T^{\sqrt{\eta\tau}y, \eta\tau} f(x, t)W_\gamma(y, 1)\, d\nu(y)\, d\mu(\tau)$$

$$= \mu(\{0\})f(x, t)$$

$$+ \int\limits_{\mathbb{R}_+^n \times (0, \infty)} T^{\sqrt{\eta\tau}y, \eta\tau} f(x, t)W_\gamma(y, 1)\, d\nu(y)\, d\mu(\tau).$$

DEFINITION 2.4. The transform

$$(2.26) \quad \mathcal{V}_\mu f(x, t; \eta) = \int\limits_{\mathbb{R}_+^n \times [0, \infty)} T^{\sqrt{\eta}y, \eta\tau} f(x, t)W_\gamma(y, \tau)e^{-\eta\tau}\, d\nu(y)\, d\mu(\tau)$$

$$\overset{(2.16)}{=} \int\limits_{\mathbb{R}_+^n \times [0, \infty)} T^{\sqrt{\eta\tau}y, \eta\tau} f(x, t)W_\gamma(y, 1)e^{-\eta\tau}\, d\nu(y)\, d\mu(\tau)$$

will be called a *weighted generalized parabolic wavelet transform* of f.

In the following we use the convention $\int_a^b \varphi(t)\, d\mu(t) = \int_{[a, b)} \varphi(t)\, d\mu(t)$.

3. Inversion of generalized parabolic wavelet transforms (the Calderón-type reproducing formula)

THEOREM 3.1. *Let $V_\mu f(x, t; \eta)$ be the wavelet transform* (2.24), $f \in Y_p$, $1 < p \le \infty$ *(recall that Y_∞ is identified with C^0). Suppose that*

$$(3.1) \qquad \mu([0, \infty)) = 0, \quad \int_0^\infty |\log \tau| \, d|\mu|(\tau) < \infty,$$

$$(3.2) \qquad c_\mu = \int_0^\infty \log \frac{1}{\tau} \, d\mu(\tau).$$

Then

$$(3.3) \qquad \int_0^\infty V_\mu f(x, t; \eta) \frac{d\eta}{\eta} \equiv \lim_{\varepsilon \to 0, \varrho \to \infty} \int_\varepsilon^\varrho V_\mu f(x, t; \eta) \frac{d\eta}{\eta} = c_\mu f(x, t),$$

the limit being understood in the Y_p-norm. If moreover $1 < p < \infty$ and

$$(3.4) \qquad \int_0^1 |\log \tau|^\delta \, d|\mu|(\tau) < \infty \quad \text{for some } \delta > 1,$$

then the limit in (3.3) *can be interpreted pointwise for almost all $(x, t) \in \mathbb{R}_+^n \times \mathbb{R}^1$. The same statements hold for the weighted transform* (2.26).

Proof. Define

$$(3.5) \qquad I_{\varepsilon, \varrho}(\mu, f) = \int_\varepsilon^\varrho V_\mu f(x, t; \eta) \frac{d\eta}{\eta}, \qquad 0 < \varepsilon < \varrho < \infty.$$

We have

$$(3.6) \qquad I_{\varepsilon, \varrho}(\mu, f) = f \otimes K_\varepsilon - f \otimes K_\varrho,$$

$$(3.7) \qquad K_\varepsilon \equiv K_\varepsilon(y, \tau) = W_\gamma(y, \tau) k_\varepsilon(\tau),$$

$$(3.8) \qquad k_\varepsilon(\tau) = \frac{1}{\varepsilon} k\left(\frac{\tau}{\varepsilon}\right), \quad k(\tau) = \frac{1}{\tau} \int_0^\tau d\mu(s) = \frac{1}{\tau} \mu([0, \tau))$$

(and similarly for K_ϱ). Indeed, by (3.1), $\mu(\{0\}) = 0$ so that

$$I_{\varepsilon, \varrho}(\mu, f)(x, t) \overset{(2.24)}{=} \int_0^\infty d\mu(\tau) \int_\varepsilon^\varrho \frac{d\eta}{\eta} \int_{\mathbb{R}_+^n} T^{\sqrt{\eta} y, \eta \tau} f(x, t) W_\gamma(y, \tau) \, d\nu(y)$$

$$\overset{(2.15)}{=} \int_0^\infty d\mu(\tau) \int_{\varepsilon\tau}^{\varrho\tau} \frac{ds}{s} \int_{\mathbb{R}_+^n} T^{z, s} f(x, t) W_\gamma(z, s) \, d\nu(z)$$

$$= \int\limits_0^\infty \frac{ds}{s} \int\limits_{s/\varrho}^{s/\varepsilon} d\mu(\tau) \int\limits_{\mathbb{R}_+^n} T^{z,s} f(x,t) W_\gamma(z,s) \, d\nu(z)$$

$$= \int\limits_{\mathbb{R}_+^n \times (0,\infty)} T^{z,s} f(x,t) W_\gamma(z,s) \left[\frac{1}{s} \int\limits_{s/\varrho}^{s/\varepsilon} d\mu(\tau) \right] d\nu(z) \, d(s),$$

which gives (3.6).

Let $\varepsilon \to 0$ and $\varrho \to \infty$ in (3.6). By (2.15),

$$(f \otimes K_\varepsilon)(x,t) = \int\limits_{\mathbb{R}_+^n \times (0,\infty)} T^{\sqrt{\varepsilon}y, \varepsilon\tau} f(x,t) W_\gamma(y,\tau) k(\tau) \, d\nu(y) \, d\tau.$$

It is known that (3.1) implies $k \in L_1(0,\infty)$ and $\int_0^\infty k(\tau) \, d\tau = c_\mu$ (see [13], p. 188). Hence, by (2.17),

$$\|f \otimes K_\varepsilon - c_\mu f\|_{Y_p} \leq \int\limits_{\mathbb{R}_+^n \times (0,\infty)} W_\gamma(y,\tau) |k(\tau)| \cdot \|T^{\sqrt{\varepsilon}y,\varepsilon\tau} f - f\|_{Y_p} \, d\nu(y) \, d\tau.$$

Owing to (2.19), (2.9) and the Lebesgue dominated convergence theorem, this yields $\lim_{\varepsilon \to 0} \|f \otimes K_\varepsilon - c_\mu f\|_{Y_p} = 0$, $1 \leq p \leq \infty$. To complete the proof of the first statement we note that $\lim_{\varrho \to \infty} \|f \otimes K_\varrho\|_{Y_p} = 0$. Indeed, by (2.8) and (2.17), we have

$$(3.9) \qquad \|f \otimes K_\varrho\|_{Y_p} \leq \left\| \int\limits_0^\infty |k(\tau)| g(t - \varrho\tau) \, d\tau \right\|_{L^p(\mathbb{R}_t^1)},$$

where $g(t) = \|f(\cdot,t)\|_{X_p} \in L^p(\mathbb{R}^1)$, and the result follows in a standard way (see, e.g., Theorem 1.15 of [13]).

Let us prove the a.e. convergence in (3.3). By (2.19),

$$\begin{aligned}
|(f \otimes K_\varepsilon)(x,t)| &\leq \int\limits_0^\infty |k(\tau)| \, d\tau \int\limits_{\mathbb{R}_+^n} W_\gamma(y,\varepsilon\tau) T_x^y |f|(x, t - \varepsilon\tau) \, d\nu(y) \\
&= \int\limits_0^\infty |k(\tau)| G_\gamma(|f(\cdot, t - \varepsilon\tau)|)(x, \varepsilon\tau) \, d\tau \\
&\leq \int\limits_0^\infty |k(\tau)| \sup_{s>0} G_\gamma(|f(\cdot, t - \varepsilon\tau)|)(x,s) \, d\tau \\
&\overset{(2.22)}{\leq} \int\limits_0^\infty |k(\tau)| M_\gamma[|f(\cdot, t - \varepsilon\tau)|](x) \, d\tau.
\end{aligned}$$

By (3.1) and (3.4), the function $k(t) = t^{-1} \int_0^t d\mu(\tau) \equiv -t^{-1} \int_t^\infty d\mu(\tau)$ has a

decreasing integrable majorant, and therefore

$$(3.10) \qquad F(x,t) := \sup_{\varepsilon > 0} |(f \otimes K_\varepsilon)(x,t)| \le c(Mu_x)(t),$$

where $u_x(t) = M_\gamma[|f(\cdot,t)|](x)$, with the standard one-dimensional Hardy–Littlewood maximal function M in the t-variable [19, p. 59]. Hence

$$\|F(x,t)\|_{Y_p} = \| \, \|F(x,\cdot)\|_{L^p(\mathbb{R}^1)}\|_{X_p} \overset{(3.10)}{\le} c\| \, \|u_x(\cdot)\|_{L^p(\mathbb{R}^1)}\|_{X_p}$$

$$= c\| \, \|u_x(t)\|_{X_p}\|_{L^p(\mathbb{R}^1)} \overset{(2.21)}{\le} c\| \, \|f(\cdot,t)\|_{X_p}\|_{L^p(\mathbb{R}^1)} = c\|f\|_{Y_p}$$

(here and below the constant c may be different at each occurrence). Thus

$$(3.11) \qquad \|\sup_{\varepsilon > 0} |f \otimes K_\varepsilon|\|_{Y_p} \le c\|f\|_{Y_p}, \quad 1 < p \le \infty.$$

Owing to this estimate and taking into account that

$$(f \otimes K_\varepsilon)(x,t) \equiv \int_{\mathbb{R}^n_+ \times (0,\infty)} W_\gamma(y,\tau)k(\tau)T^{\sqrt{\varepsilon}y,\varepsilon\tau}f(x,t)\,d\nu(y)\,d\tau \to f(x,t)$$

pointwise as $\varepsilon \to 0$ for any $f \in C^0(\mathbb{R}^n_+ \times \mathbb{R}^1)$ (this set of functions is dense in Y_p), we get $\lim_{\varepsilon \to 0}^{\text{a.e.}}(f \otimes K_\varepsilon)(x,t) = f(x,t)$ for all $f \in Y_p$. By the same reason, $(f \otimes K_\varrho)(x,t) \to 0$ a.e. as $\varrho \to \infty$ (for nice f this follows from (3.9) with $p = \infty$).

The validity of the theorem for weighted wavelet transforms can be proven following the same lines by taking into account that

$$\int_\varepsilon^\varrho \mathcal{V}_\mu f(x,t;\eta)\,\frac{d\eta}{\eta} = f \otimes \widetilde{K}_\varepsilon - f \otimes \widetilde{K}_\varrho, \quad \widetilde{K}_\varepsilon(x,t) = e^{-t}W_\gamma(x,t)k_\varepsilon(t),$$

where $k_\varepsilon(t) = \varepsilon^{-1}k(t/\varepsilon)$ (similarly for $\widetilde{K}_\varrho$; cf. (3.6)). In particular,

$$\|f \otimes \widetilde{K}_\varepsilon - c_\mu f\|_{Y_p}$$

$$\le \int_{\mathbb{R}^n_+ \times (0,\infty)} e^{-\varepsilon\tau}W_\gamma(y,\tau)k(\tau)\|T^{\sqrt{\varepsilon}y,\varepsilon\tau}f(x,t) - f(x,t)\|_{Y_p}\,d\nu(y)\,d\tau$$

$$+ \|f\|_{Y_p} \int_{\mathbb{R}^n_+ \times (0,\infty)} |e^{-\varepsilon\tau} - 1|W_\gamma(y,\tau)k(\tau)\,d\nu(y)\,d\tau \to 0 \quad \text{as } \varepsilon \to 0.$$

We leave other details to the interested reader. ∎

4. Parabolic potentials associated with the Laplace–Bessel differential operator. The generalized parabolic potentials $H_\gamma^\alpha f$ and $\mathcal{H}_\gamma^\alpha f$, defined in terms of the Fourier–Bessel transform by (1.5) and (1.6), can be

represented as integral operators

$$(4.1) \quad H_\gamma^\alpha f(x,t) = \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}_+^n} \int_0^\infty \tau^{\alpha/2-1} W_\gamma(y,\tau) T^{y,\tau} f(x,t) \, d\nu(y) \, d\tau,$$

$$(4.2) \quad \mathcal{H}_\gamma^\alpha f(x,t) = \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}_+^n} \int_0^\infty \tau^{\alpha/2-1} e^{-\tau} W_\gamma(y,\tau) T^{y,\tau} f(x,t) \, d\nu(y) \, d\tau$$

involving the generalized translation (2.4). To see this it suffices to apply the Fourier–Bessel transform (the Fourier transform in x' and t, and the Bessel transform in x_n) to both sides of (4.1) and (4.2), and make use of the statement (a) of Lemma 2.2. By setting

$$h_\alpha(x,t) = \frac{1}{\Gamma(\alpha/2)} t_+^{\alpha/2-1} W_\gamma(x,t), \quad \widetilde{h}_\alpha(x,t) = e^{-t} h_\alpha(x,t),$$

we have $H_\gamma^\alpha f(x,t) = (h_\alpha \otimes f)(x,t)$ and $\mathcal{H}_\gamma^\alpha f(x,t) = (\widetilde{h}_\alpha \otimes f)(x,t)$.

The following statement is an analogue of the Hardy–Littlewood–Sobolev theorem (see [6] in the case $\gamma = 0$).

THEOREM 4.1. (i) *Let $1 < p < q < \infty$, $\alpha > 0$. The inequality*

$$\|H_\gamma^\alpha f\|_{Y_q} \le c \, \|f\|_{Y_p}, \quad c = c(p,q,n,\gamma),$$

holds for any $f \in Y_p$ if and only if $\alpha = (n + 2\gamma - 2)(p^{-1} - q^{-1})$.
(ii) *Define $m\{E\} = \int_E d\nu(x) \, dt$ for $E \subset \mathbb{R}_+^n \times \mathbb{R}^1$. If $f \in Y_1$, then*

$$m\{(x,t) : |H_\gamma^\alpha f(x,t)| > \lambda\} \le \left(\frac{c\|f\|_{Y_1}}{\lambda} \right)^q$$

for any $\lambda > 0$ and $q > 1$ such that $1/q = 1 - \alpha/(n + 2\gamma - 2)$.
(iii) *For any $1 \le p \le \infty$ and $\alpha > 0$, $\|\mathcal{H}_\gamma^\alpha f\|_{Y_p} \le \|f\|_{Y_p}$.*

Proof. The necessity in (i) is a consequence of the standard homogeneity argument (cf. [18, p. 118]). Further, let

$$k \equiv k(x,t) = t^{(\alpha - n - 2\gamma)/2 - 1} \exp(-|x|^2/(4t)), \quad x \in \mathbb{R}_+^n, \ t > 0.$$

We split k as $k = k_1 + k_2$, where

$$k_1 = \begin{cases} k, & t \le \mu, \\ 0, & t > \mu, \end{cases} \qquad k_2 = \begin{cases} 0, & t \le \mu, \\ k, & t > \mu. \end{cases}$$

Set $m_0 = m\{k \otimes f > 2\lambda\}$ and $m_i = m\{k_i \otimes f > \lambda\}$, $i = 1, 2$, and assume $f \ge 0$, $\|f\|_{Y_1} = 1$. Then

(4.3) $\qquad m_0 \le m_1 + m_2, \qquad m_1 \le \lambda^{-p}\|k_1 \otimes f\|_{Y_p}^p \overset{(2.11)}{\le} \lambda^{-p}\|k_1\|_{Y_1}^p,$

(4.4) $\qquad \|k_1\|_{Y_1} = \int\limits_0^\mu t^{(\alpha-n-2\gamma)/2-1}\,dt \int\limits_{\mathbb{R}_+^n} e^{-|x|^2/(4t)} x_n^{2\gamma}\,dx$

$$= c_{n,\gamma} \int\limits_0^\mu t^{\alpha/2-1}\,dt = c_1 \mu^{\alpha/2}.$$

Further, by Hölder's inequality,

$$\|k_2 \otimes f\|_{Y_\infty} \le \|k_2\|_{Y_{p'}} = \Big(\int\limits_\mu^\infty \int\limits_{\mathbb{R}_+^n} [t^{(\alpha-n-2\gamma)/2-1}e^{-|x|^2/(4t)}]^{p'} x_n^{2\gamma}\,dx\Big)^{1/p'}$$

$$= c_{p,n,\gamma}\Big[\int\limits_\mu^\infty t^{(\alpha-n-2\gamma-2)p'/2+(n+2\gamma)/2}\,dt\Big]^{1/p'}$$

$$= c_2 \mu^\beta, \qquad 1/p' + 1/p = 1,$$

$$\beta = \Big(\frac{\alpha-n-2\gamma-2}{2}p' + \frac{n+2\gamma}{2} + 1\Big)\frac{1}{p'} = -\frac{n+2\gamma+2}{2q}.$$

Setting $c_2\mu^\beta = \lambda$, we have $\|k_2 \otimes f\|_{Y_\infty} \le \lambda$ and therefore $m_2 = 0$ (see (4.3)). Hence

$$m_0 \overset{(4.3)}{\le} m_1 \overset{(4.4)}{\le} c\lambda^{-p}(\mu^{\alpha/2})^p = c\lambda^{-p}\lambda^{\alpha p/2\beta} = c\lambda^{-q} = c(\|f\|_{Y_1}/\lambda)^q,$$

and (ii) is proved. By the Marcinkiewicz interpolation theorem this implies the sufficiency in (i). (iii) is a consequence of Young's inequality (2.11) and the fact that $\|\widetilde{h}_\alpha\|_{Y_1} = 1$. ∎

REMARK. For the ordinary Jones–Sampson parabolic potentials, corresponding to $\gamma = 0$, an analogue of Theorem 4.1 was stated by V. R. Gopala Rao [6, p. 989]. It is our impression that the proof of the weak $(1,q)$-estimate indicated very briefly in [6] does not yield the required result. In fact, the kernel of the parabolic potential should be split in the t-variable (as in our proof) rather than in x, as suggested in [6] following Stein's argument [18] applicable to Riesz potentials.

The potentials $H_\gamma^\alpha f$ and $\mathcal{H}_\gamma^\alpha f$ can be represented via the relevant wavelet transforms (2.25), (2.26).

LEMMA 4.2. *Let μ be a Borel measure on $[0,\infty)$ such that*

$$\int\limits_0^\infty \frac{d|\mu|(\tau)}{t^{\alpha/2}} < \infty \quad and \quad c_{\alpha,\mu} = \int\limits_0^\infty \frac{d\mu(\tau)}{\tau^{\alpha/2}} \ne 0, \qquad \alpha > 0.$$

Then

$$(4.5) \qquad H_\gamma^\alpha f(x,t) = \frac{c_{\alpha,\mu}^{-1}}{\Gamma(\alpha/2)} \int\limits_0^\infty V_\mu f(x,t;\eta) \, \frac{d\eta}{\eta^{1-\alpha/2}},$$

$$(4.6) \qquad \mathcal{H}_\gamma^\alpha f(x,t) = \frac{c_{\alpha,\mu}^{-1}}{\Gamma(\alpha/2)} \int\limits_0^\infty \mathcal{V}_\mu f(x,t;\eta) \, \frac{d\eta}{\eta^{1-\alpha/2}},$$

provided that the potentials on the left-hand sides are absolutely convergent (i.e. $H_\gamma^\alpha|f|(x,t) < \infty$ and $\mathcal{H}_\gamma^\alpha|f|(x,t) < \infty$).

Proof. Owing to (2.25), a simple calculation yields

$$\int\limits_0^\infty \frac{V_\mu f(x,t;\eta)}{\eta^{1-\alpha/2}} \, d\eta = \int\limits_{\mathbb{R}_+^n \times [0,\infty)} W_\gamma(y,1) \, d\nu(y) \, d\mu(\tau) \int\limits_0^\infty T^{\sqrt{\eta\tau}y,\eta\tau} f(x,t) \, \frac{d\eta}{\eta^{1-\alpha/2}}$$

$$= \int\limits_{\mathbb{R}_+^n \times (0,\infty)} W_\gamma(y,1) T^{\sqrt{s}y,s} f(x,t) \, \frac{d\nu(y) \, ds}{s^{1-\alpha/2}} \int\limits_0^\infty \frac{d\mu(\tau)}{\tau^{\alpha/2}}.$$

By (2.16) this implies (4.5). The proof of (4.6) is similar and based on (2.26). ∎

5. Inversion of generalized parabolic potentials by means of continuous wavelet transforms. By comparing (4.5) and (4.6) with (1.5) and (1.6), it is natural to expect that explicit inverses of H_γ^α and $\mathcal{H}_\gamma^\alpha$ can be obtained from (4.5) and (4.6) if one replaces (formally) α by $-\alpha$. Indeed, we have the following

THEOREM 5.1. *Let μ be a Borel measure supported by $[0,\infty)$ and satisfying the following conditions:*

$$(5.1) \qquad \int\limits_0^\infty t^j \, d\mu(t) = 0, \quad j = 0,1,\ldots,[\alpha/2] \ \textit{(the integral part of $\alpha/2$)};$$

$$(5.2) \qquad \int\limits_0^\infty t^\beta \, d|\mu|(t) < \infty \quad \textit{for some } \beta > \alpha/2.$$

If $\varphi = H_\gamma^\alpha f$, $f \in Y_p$, $1 < p < \infty$, $0 < \alpha < (n+2\gamma+2)/p$, then

$$(5.3) \qquad \int\limits_0^\infty V_\mu \varphi(x,t;\eta) \, \frac{d\eta}{\eta^{1+\alpha/2}} := \lim_{\varepsilon \to 0} \int\limits_\varepsilon^\infty (\ldots) = d_{\alpha,\mu} f(x,t),$$

where

$$(5.4) \qquad d_{\alpha,\mu} = \begin{cases} \Gamma(-\alpha/2) \int\limits_0^\infty t^{\alpha/2} \, d\mu(t) & \textit{if } \alpha/2 \notin \mathbb{N}, \\[2mm] \dfrac{(-1)^{1+\alpha/2}}{(\alpha/2)!} \int\limits_0^\infty t^{\alpha/2} \log t \, d\mu(t) & \textit{if } \alpha/2 \in \mathbb{N}. \end{cases}$$

The limit in (5.3) *is interpreted in the* Y_p-*norm. The same holds for all* $\alpha > 0$ *and* $1 \le p \le \infty$ (*recall the convention* $Y_\infty \sim C^0$) *provided that* H_γ^α *and* V_μ *are replaced by* $\mathcal{H}_\gamma^\alpha$ *and* $\mathcal{V}_\mu$, *respectively.*

Proof. Let us first show that $V_\mu H_\gamma^\alpha f(x, t; \eta) = (f \otimes h_\eta^{\alpha/2})(x, t)$, where

$$(5.5) \qquad h_\eta^{\alpha/2}(x, t) = \eta^{\alpha/2 - 1} W_\gamma(x, t)(I_{0+}^{\alpha/2} \mu)(t/\eta)$$

and

$$(I_{0+}^{\alpha/2} \mu)(t) = \frac{1}{\Gamma(\alpha/2)} \int_0^t (t - \tau)^{\alpha/2 - 1} \, d\mu(\tau)$$

is the Riemann–Liouville fractional integral of μ. By (2.25) and (4.1), owing to the properties (2.5)–(2.7) of the generalized translation in the ξ_n-variable, we have

$$V_\mu \varphi(x, t; \eta) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty d\mu(\tau) \int_{\mathbb{R}_+^n} W_\gamma(y, 1) \, d\nu(y) \int_0^\infty s^{\alpha/2 - 1} \, ds$$
$$\times \int_{\mathbb{R}_+^n} W_\gamma(\xi, s) T_{x,t}^{\sqrt{\eta\tau} y, \eta\tau} T^{\xi, s} f(x, t) \, d\nu(\xi)$$

$$= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty d\mu(\tau) \int_{\mathbb{R}_+^n} W_\gamma(y, 1) \, d\nu(y) \int_{\eta\tau}^\infty (\zeta - \eta\tau)^{\alpha/2 - 1} \, d\zeta$$
$$\times \int_{\mathbb{R}_+^n} [T^{\sqrt{\eta\tau} y, \eta\tau} W_\gamma(\xi, \zeta)] T^{\xi, \zeta} f(x, t) \, d\nu(\xi)$$

$$= \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}_+^n \times (0, \infty)} T^{\xi, \zeta} f(x, t) \, d\nu(\xi) \, d\zeta$$
$$\times \int_0^\infty (\zeta - \eta\tau)_+^{\alpha/2 - 1} I(\xi, \zeta, \tau) \, d\mu(\tau),$$

where

$$I(\xi, \zeta, \tau) = \int_{\mathbb{R}_+^n} W_\gamma(y, 1) T^{\sqrt{\eta\tau} y, \eta\tau} W_\gamma(\xi, \zeta) \, d\nu(y).$$

By setting $y = z(\eta\tau)^{-1/2}$ and $d\nu(y) = (\eta\tau)^{-\gamma - n/2} d\nu(z)$, and taking into account that $(\eta\tau)^{-\gamma - n/2} W_\gamma(z/\sqrt{\eta\tau}, 1) = W_\gamma(z, \eta\tau)$ (see (2.16)), according to (2.18) we have

$$I(\xi, \zeta, \tau) = \int_{\mathbb{R}_+^n} W_\gamma(z, \eta\tau) T^{z, \eta\tau} W_\gamma(\xi, \zeta) \, d\nu(z) = W_\gamma(\xi, \zeta).$$

Hence

$$V_\mu \varphi(x, t; \eta) = \frac{1}{\Gamma(\alpha/2)} \int\limits_{\mathbb{R}_n^+ \times (0,\infty)} W_\gamma(\xi, \zeta) T^{\xi,\zeta} f(x, t)$$

$$\times \left(\int\limits_0^\infty (\zeta - \eta\tau)_+^{\alpha/2-1} \, d\mu(\tau) \right) d\nu(\xi) \, d\zeta,$$

which gives (5.5). Further,

$$(5.6) \qquad \int\limits_\varepsilon^\infty V_\mu \varphi(x, t; \eta) \frac{d\eta}{\eta^{1+\alpha/2}} = \int\limits_{\mathbb{R}_+^n \times (0,\infty)} \Psi_\varepsilon(\xi, \zeta) T^{\xi,\zeta} f(x, t) \, d\nu(\xi) \, d\zeta,$$

where

$$\Psi_\varepsilon(\xi, \zeta) = \frac{1}{\Gamma(\alpha/2)} W_\gamma(\xi, \zeta) \int\limits_\varepsilon^\infty \frac{d\eta}{\eta^{1+\alpha/2}} \int\limits_0^\infty (\zeta - \eta\tau)_+^{\alpha/2-1} \, d\mu(\tau).$$

A simple calculation yields

$$\Psi_\varepsilon(\xi, \zeta) = \frac{1}{\varepsilon} \lambda_\alpha \left(\frac{\zeta}{\varepsilon} \right) W_\gamma(\xi, \zeta), \quad \lambda_\alpha(t) = \frac{1}{t}(I_{0+}^{\alpha/2+1} \mu)(t)$$

(cf. (3.7), (3.8)). Owing to (5.1) and (5.2), $\lambda_\alpha(t)$ has a decreasing integrable majorant and $\int_0^\infty \lambda_\alpha(t) \, dt = d_{\alpha,\mu}$, with $d_{\alpha,\mu}$ defined by (5.4) (see [15], Lemma 1.3). Hence (5.3) follows by the same arguments as in the proof of Theorem 3.1.

By analogy to (5.6), it can be shown that for $\varphi = \mathcal{H}_\gamma^\alpha f$,

$$\int\limits_\varepsilon^\infty \mathcal{V}_\mu \varphi(x, t; \eta) \frac{d\eta}{\eta^{1+\alpha/2}} = (\widetilde{\Psi}_\varepsilon \otimes f)(x, t),$$

where $\widetilde{\Psi}_\varepsilon(x, t) = e^{-t} W_\gamma(x, t)(1/\varepsilon)\lambda_\alpha(t/\varepsilon)$ and $\mathcal{V}_\mu \varphi(x, t; \eta)$ is defined by (2.26). The convergence $\widetilde{\Psi}_\varepsilon \otimes f \to d_{\alpha,\mu} f$ as $\varepsilon \to 0$ can be proven as the similar one in Theorem 3.1. ∎

EXAMPLE 5.2. Take a measure $\mu = \sum_{k=0}^l (-1)^k \binom{l}{k} \delta_k$, where $l > \alpha/2$ and $\delta_k = \delta_k(t)$ is the unit mass at $t = k$ $(k = 0, 1, \ldots, l)$. By (2.25),

$$V_\mu \varphi(x, t; \eta) = \int\limits_{\mathbb{R}_+^n} \left[\sum_{k=0}^l (-1)^k \binom{l}{k} T^{\sqrt{\eta k} y, \eta k} \varphi(x, t) \right] W_\gamma(y, 1) \, d\nu(y)$$

$$\text{(set } \sqrt{\eta}\, y = z \text{ and use (2.16))}$$

$$= \int\limits_{\mathbb{R}_+^n} \Delta_{z,\eta}^l \varphi(x, t) W_\gamma(z, \eta) \, d\nu(z),$$

where

$$\Delta_{z,\eta}^l \varphi(x,t) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} T^{\sqrt{k}z,k\eta} \varphi(x,t)$$

is an anisotropic finite difference of φ generated by the generalized translation operator. Furthermore,

$$\int_0^\infty t^j \, d\mu(t) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} k^j = 0, \quad \forall j = 0, 1, \ldots, l-1$$

(see [16], pp. 116–117), and (4.2) holds automatically. Thus Theorem 5.1 gives the inversion formula

$$\int_{\mathbb{R}_+^n \times (0,\infty)} \frac{\Delta_{z,\eta}^l H_\gamma^\alpha f(x,t)}{\eta^{1+\alpha/2}} W_\gamma(z,\eta) z_n^{2\gamma} \, dz \, d\eta = d_{\alpha,\mu} f(x,t)$$

with

$$d_{\alpha,\mu} = \int_0^\infty (1 - e^{-t})^l \frac{dt}{t^{1+\alpha/2}} \neq 0.$$

By the same argument one can obtain an inversion formula for $\mathcal{H}_\gamma^\alpha f$:

$$\int_{\mathbb{R}_+^n \times (0,\infty)} \frac{\widetilde{\Delta}_{z,\eta}^l \mathcal{H}_\gamma^\alpha f(x,t)}{\eta^{1+\alpha/2}} W_\gamma(z,\eta) z_n^{2\gamma} \, dz \, d\eta = d_{\alpha,\mu} f(x,t),$$

$$\widetilde{\Delta}_{z,\eta}^l \varphi(x,t) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} e^{-kt} T^{\sqrt{k}z,k\eta} \varphi(x,t).$$

These formulas were obtained in [12] for potentials with ordinary (euclidean) translation and in [1, 2] for the generalized potentials $H_\gamma^\alpha f$ and $\mathcal{H}_\gamma^\alpha f$.

References

[1] I. A. Aliev, *On parabolic potentials generated by a generalized shift operator*, in: Proc. 3rd Turkish-Azerbaijan Math. Symp., Trabzon, 1993, 14–16.

[2] —, *The properties and inversion of B-parabolic potentials*, in: Special Problems of Math. and Mech., "Bilik", Baku, 1992, 56–75 (in Russian).

[3] I. A. Aliev and S. Bayrakci, *On inversion of B-elliptic potentials by the method of Balakrishnan–Rubin*, Fract. Calc. Appl. Anal. 1 (1998), 365–384.

[4] I. A. Aliev and B. Rubin, *Parabolic wavelet transforms and Lebesgue spaces of parabolic potentials*, preprint, The Hebrew University of Jerusalem, 1999.

[5] O. Arena, *On a singular parabolic equation related to axially symmetric heat potentials*, Ann. Mat. Pura Appl. 105 (1975), 347–393.

[6] V. R. Gopala Rao, *A characterization of parabolic function spaces*, Amer. J. Math. 99 (1977), 985–993.

[7]   B. F. Jones, *Lipschitz spaces and heat equation*, J. Math. Mech. 18 (1968), 379–409.
[8]   I. A. Kipriyanov, *Singular Elliptic Boundary Value Problems*, Nauka, Fizmatlit, Moscow, 1997 (in Russian).
[9]   B. M. Levitan, *Bessel function expansions in series and Fourier integrals*, Uspekhi Mat. Nauk 6 (1951), no. 2, 102–143 (in Russian).
[10] J. Löfström and J. Peetre, *Approximation theorems connected with generalized translations*, Math. Ann. 181 (1969), 255–268.
[11] M. A. Mourou and K. Trimèche, *Calderón's reproducing formula associated with the Bessel operator*, J. Math. Anal. Appl. 219 (1998), 97–109.
[12] V. A. Nogin and B. S. Rubin, *Inversion and description of parabolic potentials with L_p-densities*, Dokl. Akad. Nauk SSSR 284 (1985), 535–538 (in Russian).
[13] B. Rubin, *Fractional Integrals and Potentials*, Addison-Wesley and Longman, 1996.
[14] —, *Calderón-type reproducing formula*, in: Encyclopaedia Math. Supplement II, Kluwer, 2000, 104–105; reprinted in Frac. Calc. Appl. Anal. 3 (2000), 103–106.
[15] —, *Fractional integrals and wavelet transforms related to the Blaschke–Levy representation and the spherical Radon transform*, Israel J. Math. 114 (1999), 1–27.
[16] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, 1993.
[17] C. H. Sampson, *A characterization of parabolic Lebesgue spaces*, Dissertation, Rice Univ., 1968.
[18] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.
[19] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, NJ, 1971.
[20] K. Stempak, *The Littlewood–Paley theory for the Fourier–Bessel transform*, preprint no. 45, Univ. of Wrocław, 1985.
[21] —, *La théorie de Littlewood–Paley pour la transformation de Fourier–Bessel*, C. R. Acad. Sci. Paris Sér. I 303 (1986), 15–18.
[22] S. A. Tersenov, *Parabolic Equations with Changing Time Direction*, Nauka, Sib. Otdel., Novosibirsk, 1985 (in Russian).
[23] K. Trimèche, *Generalized Wavelets and Hypergroups*, Gordon and Breach, New York, 1997.
[24] A. Weinstein, *Generalized axially symmetric potential theory*, Bull. Amer. Math. Soc. 59 (1953), 20–38.
[25] —, *Singular partial differential equations and their applications*, in: Fluid Dynamics and Applied Mathematics, J. B. Diaz and S. I. Pai (eds.), Gordon and Breach, New York, 1962, 229–249.
[26] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, Cambridge Univ. Press, 1997.

Department of Mathematics   Institute of Mathematics
Akdeniz University   The Hebrew University
07058 Antalya, Turkey   91904 Jerusalem, Israel
E-mail: ialiev@pascal.sci.akdeniz.edu.tr   E-mail: boris@math.huji.ac.il