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On ergodicity for operators with bounded resolvent in Banach spaces

by

Kirsti Mattila (Stockholm)

Abstract. We prove results on ergodicity, i.e. on the property that the space is a direct sum of the kernel of an operator and the closure of its range, for closed linear operators A such that $\|\alpha(\alpha - A)^{-1}\|$ is uniformly bounded for all $\alpha > 0$. We consider operators on Banach spaces which have the property that the space is complemented in its second dual space by a projection P. Results on ergodicity are obtained under a norm condition $\|I - 2P\| \, \|I - Q\| < 2$ where Q is a projection depending on the operator A. For the space of James we show that $\|I - 2P\| < 2$ where P is the canonical projection of the predual of the space. If $(T(t))_{t\geq 0}$ is a bounded strongly continuous and eventually norm continuous semigroup on a Banach space, we show that if the generator of the semigroup is ergodic, then, for some positive number δ, the operators $T(t) - I$, $0 < t < \delta$, are also ergodic.

1. Introduction. Let X be a complex Banach space. We consider operators on X which generate a bounded resolvent family.

Definition 1. If a closed linear operator A on X satisfies the conditions

- $\{\alpha \in \mathbb{R} : \alpha > 0\} \subset \rho(A)$,
- there is a constant $K > 0$ such that $\|\alpha(\alpha - A)^{-1}\| \leq K$ for all $\alpha > 0$,

then we say that A generates a (*uniformly*) *bounded resolvent family* (UBR).

The domain of an operator A in the class (UBR) is not necessarily assumed to be dense in X. We call an operator A *ergodic* if the Banach space can be expressed as the direct sum $X = \mathrm{Ker}(A) \oplus \overline{A(D(A))}$. There are many operators in the class (UBR) which do not have this property. For example the generator of the left translation group on $L^1(\mathbb{R})$ is not ergodic ([3, p. 344]). On the other hand in reflexive Banach spaces all operators in this class are ergodic (see Yosida [12, p. 217]). The main result of this paper is Theorem 1 which gives sufficient conditions for the ergodicity of operators

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in the class (UBR). A projection Q on the dual space will be connected to each operator A in the class (UBR). We assume in Theorem 1 that the Banach space X is complemented in its second dual. If P is a projection with $PX^{**} = \hat{X}$, we assume that $\|I-2P\|\,\|I-Q\| < 2$. If in addition the second adjoints of the resolvent operators commute with P, then A is ergodic.

In the case that P is an L-projection we show in Theorem 3 that if $\|Q\| \le 1$ and $\|I-Q\| < 2$ then A is ergodic. Theorem 2 is an extension of the ergodicity result for reflexive Banach spaces in [12]. In Section 3 we give an example of a semigroup with unbounded generator which satisfies the assumptions of Theorems 1 and 3. We also consider strongly continuous semigroups $(T(t))_{t\ge 0}$ with the property that, for some positive number δ, -1 is not an eigenvalue of $T(t)$ or $T(t)^*$ for $0 < t < \delta$. We show in Proposition 1 that then

$$\text{(1)} \qquad \operatorname{Ker}(T(t)-I) = \operatorname{Ker}(A) \quad \text{and} \quad \overline{(T(t)-I)X} = \overline{A(D(A))}$$

for $0 < t < \delta$.

Sato [9] proved that

$$\operatorname{Ker}(T(t)-I) \oplus \overline{(T(t)-I)X} = \operatorname{Ker}(A) \oplus \overline{A(D(A))}$$

for bounded semigroups such that $\|T(t)-I\| < 2$ for $0 < t < \delta$. Shaw [10] showed that the equalities (1) hold for bounded semigroups with bounded generator. There are examples in [10] and in [3, p. 345] of semigroups for which these subspaces are not equal. In Corollary 2 we show that the equalities (1) hold for any semigroup $(T(t))_{t\ge 0}$ which is strongly continuous and eventually norm continuous. If the semigroup is also bounded and the generator is ergodic, then there is a positive number δ such that $T(t)-I$ is ergodic for $0 < t < \delta$.

In Section 4 we consider the space of James ([1], [2]). We prove in Theorem 4 that the canonical projection P of its predual space satisfies the inequality $\|I-2P\| < 2$, which is a necessary assumption in Theorem 1. The *canonical projection* of a Banach space Y is the projection on Y^{***} corresponding to the decomposition $Y^{***} = \widehat{Y^*} \oplus (\hat{Y})^{\perp}$ with range $\widehat{Y^*}$, the canonical image of Y^*, and kernel $(\hat{Y})^{\perp}$, the annihilator of $\hat{Y}$. For example, for the space c_0 and for the Banach space of all compact operators on a Hilbert space, $\|I-2P\| = 1$. For further examples and references on spaces with this property see Harmand, Werner and Werner [6].

Let $B(X)$ be the space of all bounded linear operators on X. For a closed linear operator A on X, we denote by $\rho(A)$ the *resolvent set* of A, which is the set $\{\alpha \in \mathbb{C} : \alpha - A \text{ has bounded inverse in } X\}$, and the spectrum by $\sigma(A)$. The *resolvent operators* $(\alpha - A)^{-1}$ are denoted by $R(\alpha, A)$.

Example 1. A linear operator which generates a bounded strongly continuous semigroup generates a (UBR). In particular, if $S \in B(X)$ is *power-*

bounded, i.e. if $\|S^n\| \leq K$ for some constant K, $n = 0, 1, \ldots$, then $T = S - I$ generates a bounded semigroup. This can be proved by using an equivalent norm $|x| = \sup_{n\geq 0} \|S^n x\|$. Moreover, if A is an operator in the class (UBR) with dense domain in X, then A^*, the adjoint of A, is in the class (UBR) in the dual space X^*.

EXAMPLE 2. Let $A = -(\beta - B)^{-1}$ where B is in the class (UBR) on a Banach space X and $\beta > 0$. We assume that $\|t(t - B)^{-1}\| \leq M$ for $t > 0$. We can show that A is also in the class (UBR). Obviously A is bounded and injective. It is easy to show that for $\alpha > 0$, $\alpha - A$ has bounded inverse and $A(\alpha - A)^{-1} = -(\alpha\beta + 1 - \alpha B)^{-1}$. Further $\|A(\alpha - A)^{-1}\| \leq M/(\alpha\beta + 1) < M$ for all $\alpha > 0$. It follows that $\|\alpha(\alpha - A)^{-1}\| < M + 1$ for all $\alpha > 0$. The operator A is ergodic if and only if the domain of B is dense in X.

2. Ergodicity. In this section we assume that A is an operator in the class (UBR). Let Lim be a Banach limit on l^∞. We choose a sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers such that $\lim_{n\to\infty} \alpha_n = 0$. Let Q be the operator on X^* defined by

$$(Qf)(x) = \operatorname{Lim} f(W_n x)$$

for each $f \in X^*$ and $x \in X$, where $W_n = \alpha_n(\alpha_n - A)^{-1}$.

We shall need the following lemma, which states some straightforward properties of Q, and a corollary. For completeness, we include their proofs.

A *Banach limit* Lim is a continuous linear functional on l^∞ such that $\|\operatorname{Lim}\| = 1$, $\operatorname{Lim}(1, 1, 1, \ldots) = 1$ and $\operatorname{Lim} c_n = \operatorname{Lim} c_{n+1}$. It also satisfies $|\operatorname{Lim} c_n| \leq \operatorname{Lim} |c_n|$ and if $\{c_n\}$ converges to c, then $\operatorname{Lim} c_n = c$.

LEMMA 1. *The operator Q defined by $(Qf)(x) = \operatorname{Lim} f(W_n x)$ is a bounded linear projection onto* $\operatorname{Ker}((AR(\alpha, A))^*)$ *for each $\alpha > 0$. It commutes with the operators $R(\alpha, A)^*$. Hence $(AR(\alpha, A))^* X^* \subset \operatorname{Ker}(Q)$.*

Proof. The kernel of $(AR(\alpha, A))^*$ is independent of $\alpha > 0$. This follows from the resolvent equation ([3, p. 239]). Since $|(Qf)(x)| \leq \operatorname{Lim} |f(W_n x)| \leq K\|f\|\,\|x\|$, Q is bounded. As $n \to \infty$,

$$\|AW_n\| = \|\alpha_n A(\alpha_n - A)^{-1}\| = \|\alpha_n(\alpha_n(\alpha_n - A)^{-1} - I)\| \to 0.$$

Therefore $((AR(\alpha, A))^* Qf)(x) = \operatorname{Lim} f(W_n AR(\alpha, A)x) = 0$ for every $x \in X$ and $\alpha > 0$. Hence $(AR(\alpha, A))^* Qf = 0$; also $Q((AR(\alpha, A))^* f) = 0$ for every $f \in X^*$. Thus Q has the property

$$\alpha R(\alpha, A)^* Q = \alpha Q R(\alpha, A)^* = Q \tag{2}$$

for all $\alpha > 0$. We conclude that Q is a projection, $QX^* \subset \operatorname{Ker}((AR(\alpha, A))^*)$ and $(AR(\alpha, A))^* X^* \subset (I - Q)X^*$. Finally, the inclusion $\operatorname{Ker}((AR(\alpha, A))^*) \subset QX^*$ holds since $\operatorname{Ker}((AR(\alpha, A))^*) = \operatorname{Ker}(W_n^* - I)$ for all $n \in \mathbb{N}$. ∎

Corollary 1. *Let A be an operator in the class* (UBR). *Then $Y = \mathrm{Ker}(A) \oplus \overline{A(D(A))}$ is a closed subspace of X.*

Proof. If $\mathrm{Ker}(A) = \{0\}$, then Y is obviously closed. Assume that $u \in \mathrm{Ker}(A)$, $\|u\| = 1$. Let $f \in X^*$ be such that $\|f\| = f(u) = 1$. Then $(Qf)(u) = \mathrm{Lim}\, f(W_n u) = 1$ and $(Qf)(\overline{A(D(A))}) = \{0\}$. Hence, whenever $v \in \overline{A(D(A))}$, $\|u\| = 1 = (Qf)(u+v) \le \|Q\|\,\|u+v\|$. The inequalities

$$\|u\| \le \|Q\|\,\|u+v\| \quad \text{and} \quad \|v\| \le \|v+u\| + \|u\| \le (1+\|Q\|)\|u+v\| \tag{3}$$

then hold for all $u \in \mathrm{Ker}(A)$ and $v \in \overline{A(D(A))}$. We conclude that the linear mapping $\varphi(u,v) = u+v$ from $\mathrm{Ker}(A) \times \overline{A(D(A))}$, which is a closed subspace of $X \times X$, to Y is a continuous bijection and by (3) the inverse of φ is continuous. Hence Y is closed. ∎

Remark 1. The assumptions (i) and (iii) in the theorem below are in particular satisfied if X is a dual space and A is the adjoint of an operator which on the predual space generates a (UBR) and has dense domain. If P is the canonical projection of the predual, then the operators $R(\alpha, A)^{**}$ commute with P.

We can assume that $A \neq 0$. Then $Q \neq I$ and $\|I - Q\| \ge 1$. We also assume that the canonical image $\hat{X}$ of X is complemented in the second dual space of X with a projection P onto $\hat{X}$.

Theorem 1. *Let X be a Banach space and A an operator in the class* (UBR). *Assume that*

(i) $X^{**} = \widehat{X} \oplus Z$,
(ii) $\|I - 2P\|\,\|I - Q\| < 2$,
(iii) *the resolvent operators satisfy $R(\alpha, A)^{**} Z \subset Z$ for all $\alpha > 0$.*

Then

$$X = \mathrm{Ker}(A) \oplus \overline{A(D(A))}.$$

Proof. Let $Y = \mathrm{Ker}(A) \oplus \overline{A(D(A))}$. Assume that $X \neq \overline{Y}$. Then there exists a linear functional $h \in X^*$ such that $\|h\| = 1$ and $h(Y) = \{0\}$. Let $\gamma = \|I - 2P\|\,\|I - Q\| < 2$. Let x_0 be a point in X such that $\|x_0\| = 1$ and $h(x_0) > \gamma/2$. Since $h((AR(\alpha, A))X) = \{0\}$ for all $\alpha > 0$, $h = Qh$. Let $Q^*\widehat{x_0} = \hat{v} + F$ where $v \in X$ and $F \in Z$. Let α be some positive number. Then by (2), $\alpha R(\alpha, A)^{**} F = \alpha R(\alpha, A)^{**} Q^* \widehat{x_0} - \alpha \widehat{R(\alpha, A)} v = Q^* \widehat{x_0} - \alpha \widehat{R(\alpha, A)} v = \widehat{v} - \alpha \widehat{R(\alpha, A)} v + F$. Since $R(\alpha, A)^{**} Z \subset Z$ it follows that that $v = \alpha R(\alpha, A) v$. This implies $v \in D(A)$ and $Av = 0$. Since, by assumption, $h(\mathrm{Ker}(A)) = \{0\}$, we have $h(v) = 0$ and therefore $F(h) = Qh(x_0) - h(v) = h(x_0)$. Now

$$\begin{aligned}\|\widehat{x_0 - v} + F\| &= \|(I-2P)(\widehat{x_0 - v} - F)\| = \|(I-2P)((I-Q^*)\widehat{x_0})\| \\ &\le \|I-2P\|\,\|I-Q\| = \gamma\end{aligned}$$

and $|(\widehat{x_0 - v} + F)(h)| = 2|h(x_0)| \le \gamma$. Hence $|h(x_0)| \le \gamma/2$. This contradicts the assumption that $h(x_0) > \gamma/2$. Therefore $X = \overline{Y}$ and since Y is closed by Corollary 1, $X = Y$. ∎

In Section 3 we will give an example of an unbounded operator A satisfying the conditions in Theorem 1. The following theorem is an extension of [12, Corollary VIII.4.1] where ergodicity was proved for reflexive spaces.

Theorem 2. *Let A be in the class* (UBR). *If* $\mathrm{Ker}((AR(\alpha,A))^{**}) \subset \widehat{X}$ *for some $\alpha > 0$* (*and then for all $\alpha > 0$*), *in particular if $R(\alpha,A)$ is weakly compact or if X is reflexive, then*

$$X = \mathrm{Ker}(A) \oplus \overline{A(D(A))}.$$

Proof. Let Q be as in Lemma 1. By (2), $Q^*X^{**} \subset \mathrm{Ker}((AR(\alpha,A))^{**})$. Hence $Q^*X^{**} \subset \widehat{X}$. Let $Y = \mathrm{Ker}(A) \oplus \overline{A(D(A))}$. Assume that $X \ne \overline{Y}$. Then there exists $h \in X^*$ such that $\|h\| = 1$ and $h(Y) = \{0\}$ and there is an element $x_0 \in X$ so that $\|x_0\| = 1$ and $h(x_0) \ne 0$. Now $Q^*\widehat{x_0} \in \widehat{X}$. Let $Q^*\widehat{x_0} = \hat{w}$. Since $(AR(\alpha,A))^{**}Q^*\widehat{x_0} = 0$, it is obvious that $AR(\alpha,A)w = 0$. Hence $w \in D(A)$ and $Aw = 0$. Thus $h(w) = 0$. Since $h(w) = (Q^*\widehat{x_0})(h) = (Qh)(x_0) = h(x_0)$, it follows that $h(x_0) = 0$ contrary to the choice of x_0. ∎

In the case that the projection P is an L-projection we have the following result. A bounded linear projection P is an *L-projection* if $\|x\| = \|Px\| + \|(I-P)x\|$ for all $x \in X$. For an L-projection $\|I-2P\| = 1$.

Theorem 3. *Let X be a Banach space such that there is an L-projection on X^{**} with range $\hat{X}$ and kernel Z. Assume that A is in the class* (UBR) *with $K = 1$ and $\|I-Q\| < 2$. Then*

$$X = \mathrm{Ker}(A) \oplus \overline{A(D(A))}.$$

Proof. Let h be as in the proof of Theorem 1 and $x_0 \in X$ be such that $\|x_0\| = 1$ and $h(x_0) > \gamma/2$, where $\gamma = \|I-Q\|$. Since X^{**} is the direct sum of $\hat{X}$ and Z, there are elements $v \in X$ and $F \in Z$ such that $Q^*\widehat{x_0} = \hat{v} + F$. Given $\alpha > 0$, we have $\alpha R(\alpha,A)^{**}F = \alpha R(\alpha,A)^{**}Q^*\widehat{x_0} - \alpha\widehat{R(\alpha,A)}v = Q^*\widehat{x_0} - \alpha\widehat{R(\alpha,A)}v = \widehat{v} - \alpha\widehat{R(\alpha,A)}v + F$. Let $z_\alpha = (I - \alpha R(\alpha,A))v$. By assumption, $\|\alpha R(\alpha,A)^{**}F\| \le \|F\|$. Hence $\|\widehat{z_\alpha} + F\| = \|z_\alpha\| + \|F\| \le \|F\|$. We conclude that $z_\alpha = 0$. This implies that $v \in D(A)$ and $Av = 0$. Since, by the choice of h, $h(\mathrm{Ker}(A)) = \{0\}$, we have $h(v) = 0$ and therefore $F(h) = (Qh)(x_0) - h(v) = h(x_0)$. Now

$$\|\widehat{x_0 - v} + F\| = \|\widehat{x_0 - v} - F\| = \|(I-Q^*)\widehat{x_0}\| \le \|I-Q^*\| = \gamma$$

and $|(\widehat{x_0 - v + F})(h)| = 2|h(x_0)| \leq \gamma$. Hence $|h(x_0)| \leq \gamma/2$ contrary to the assumption $h(x_0) > \gamma/2$. This completes the proof. ∎

The canonical projection of c_0 is an L-projection (see [6]) and $c_0^* = l^1$. Theorems 1 and 3 are not necessarily true if $\|I - Q\| \geq 2$.

Example 3. Let $X = l^1 = l^1(\mathbb{N})$ and let T be the shift operator $Te_k = e_{k+1}$ for $k \in \mathbb{N}$, where e_k is the kth unit vector $(0, \ldots, 0, 1, 0, \ldots)$. Let $A = T - I$. It is well known that A is injective and A^* is not injective on the dual space l^∞. Hence A is not ergodic. Since A is dissipative, $\|\alpha R(\alpha, A)\| \leq 1$ for all positive numbers α. It follows from Theorem 3 that $\|I - Q\| = 2$.

3. Strongly continuous semigroups. If a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X satisfies the inequality

$$\text{(T)} \qquad \|T(t) - I\| \leq \gamma < 2 \quad \text{for all } t > 0$$

then, for any $\alpha > 0$, $\|AR(\alpha, A)\| \leq \gamma$ where A is the generator of the semigroup. This follows from the equality

$$\text{(4)} \qquad AR(\alpha, A)x = \alpha \int_0^\infty e^{-\alpha s}(T(s) - I)x\, ds.$$

If $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous group with an unbounded generator, then $\limsup_{t\downarrow 0} \|T(t) - I\| \geq 2$ by Williams [11, Theorem 1] (see also [8, Corollary 2.4.13]). Hence for groups, (T) implies boundedness of the generator. For more results on strongly continuous groups of operators see [4].

Remark 2. (i) If (T) holds for small t, i.e., if there is a positive number δ such that

$$\text{(T}'\text{)} \qquad \|T(t) - I\| \leq \gamma < 2 \quad \text{for } 0 < t < \delta,$$

then the semigroup $(T(t))_{t\geq 0}$ has an analytic extension (see [7]).

(ii) Condition (T′) is obviously weaker than (T) since all semigroups with bounded generator satisfy (T′) for some $\delta > 0$. For example the bounded operator $Af(s) = isf(s)$ on $C[0,1]$ generates a group of invertible isometries. Then, for large values of t, $i\pi t^{-1} \in \sigma(A)$. Hence $-1 \in \sigma(T(t))$ and $\|T(t) - I\| = 2$.

The following example was given in [11]. In this example $(T(t))_{t\geq 0}$ is a strongly continuous semigroup with an unbounded generator. The assumptions of Theorems 1 and 3 are satisfied. Therefore the generator is ergodic.

Example 4. For the following semigroup, $\|T(t)-I\| = 1$ and $\|T(t)\| = 1$ for all $t > 0$. Therefore, by (4), $\|I - Q\| = 1$. Let $X = l^1$ and let

$$\begin{cases} (T(t)x)_{2n-1} = \frac{1}{2}(1 + e^{-nt})x_{2n-1} + \frac{1}{2}(1 - e^{-nt})x_{2n}, \\ (T(t)x)_{2n} = \frac{1}{2}(1 - e^{-nt})x_{2n-1} + \frac{1}{2}(1 + e^{-nt})x_{2n} \end{cases}$$

whenever $x = \{x_n\}_{n=1}^{\infty} \in l^1$ and $n \in \mathbb{N}$. It is easy to see that $\lim_{t\to\infty} T(t)x = Px$ for each $x \in X$, where P is the projection

$$(Px)_{2n-1} = (Px)_{2n} = \tfrac{1}{2}(x_{2n-1} + x_{2n})$$

for $n \in \mathbb{N}$. The convergence is even uniform. Since $\|T(t) - P\| \le e^{-t}$, $\lim_{t\to\infty} \|T(t) - P\| = 0$. Further $PX = \mathrm{Ker}(A)$ and $\mathrm{Ker}(P) = A(D(A))$ (see [3, Theorem V.4.10]). The semigroup $T(t)$ is obviously adjoint of a strongly continuous semigroup on c_0.

Whenever a strongly continuous semigroup satisfies (T), then

$$\mathrm{Ker}(A) \oplus \overline{A(D(A))} = \mathrm{Ker}(T(t) - I) \oplus \overline{(T(t) - I)X}$$

for every $t > 0$. This follows from the Theorem of Sato in [9]. There the semigroup was assumed to satisfy the inequality $\|T(t) - I\| < 2$ for every $0 < t < \delta$ for some $\delta > 0$. According to Mathematical Reviews (MR0627692) this condition can be replaced by the assumption that $-1 \in \rho(T(t))$ for $0 < t < \delta$ for some positive δ. We only need to assume that, for some $\delta > 0$, -1 is not an eigenvalue of $T(t)$ or $T(t)^*$ for all $0 < t < \delta$. We use the method of Shaw [10, Lemma 3.2] in the proof of the equalities below.

PROPOSITION 1. *Let $(T(t))_{t\ge 0}$ be a strongly continuous semigroup on X. Suppose that there is a positive number δ such that for all $0 < t < \delta$, -1 is neither an eigenvalue of $T(t)$ nor of $T(t)^*$. Then*

$$\mathrm{Ker}(T(t) - I) = \bigcap_{s>0} \mathrm{Ker}(T(s) - I) \quad \text{and} \quad \overline{(T(t) - I)X} = \overline{\bigcup_{s>0} (T(s) - I)X}$$

whenever $0 < t < \delta$.

Proof. Let $0 < t_0 < \delta$ and let $x \in \mathrm{Ker}(T(t_0) - I)$. Then

$$(T(t_0/2) + I)(T(t_0/2) - I)x = 0.$$

Since $T(t_0/2)+I$ is injective, $(T(t_0/2)-I)x = 0$. In this way a sequence $\{t_n\}$ can be found such that $T(t_n)x = x$ for each $n = 0, 1, \ldots$ and $\lim_{n\to\infty} t_n = 0$. Let $E = \{t > 0 : T(t)x = x\}$. Then $E \subset (0, \infty)$ and E is a subsemigroup of $(0, \infty)$. Further, E is closed in $(0, \infty)$, and if $t \in E$, then $t + t_n \in E$ and $\lim_{n\to\infty}(t+t_n) = t$. It follows that $E = (0, \infty)$. We conclude that $T(t)x = x$ for all $t > 0$. Hence $\mathrm{Ker}(T(t_0) - I) \subset \mathrm{Ker}(T(t) - I)$ for every $t > 0$. This proves the first equality.

To prove the second equality it is enough to show $\bigcup_{s>0}(T(s) - I)X \subset \overline{(T(t_0) - I)X}$ where $0 < t_0 < \delta$. Assume that f is an element of X^* such that $T(t_0)^* f = f$. The set $\{t > 0 : T(t)^* f = f\}$ is closed in $(0, \infty)$ since, by the strong continuity of $(T(t))_{t\ge 0}$, the mapping $t \mapsto (T(t)^* f)(x)$ is continuous for every $x \in X$. As $T(t)^* + I$ is injective for all $0 < t < \delta$, we conclude as in the first part of the proof that $T(t)^* f = f$ for all $t > 0$. ∎

Remark 3. (i) The subspaces in Proposition 1 are in terms of the generator, $\mathrm{Ker}(A)$ and $\overline{A(D(A))}$. By the definition of a generator and by [3, Lemma II.1.3],

$$\mathrm{Ker}(A) = \bigcap_{s>0} \mathrm{Ker}(T(s) - I) \quad \text{and} \quad \overline{A(D(A))} = \overline{\bigcup_{s>0} (T(s) - I)X}.$$

(ii) If a strongly continuous semigroup $(T(t))_{t\geq 0}$ is bounded, then the subspace $\mathrm{Ker}(A) \oplus \overline{A(D(A))}$ is a direct sum and a closed subspace of X (see for example [10] or Corollary 1). The same is true for the subspace $\mathrm{Ker}(T(t) - I) \oplus \overline{(T(t) - I)X}$. Further, for all $t > 0$,

$$\mathrm{Ker}(T(t) - I) \oplus \overline{(T(t) - I)X} \subset \mathrm{Ker}(A) \oplus \overline{A(D(A))}$$

(see [10, Theorem 3.4]).

We will now consider strongly continuous semigroups which are *eventually norm continuous*, i.e. $\lim_{t\downarrow 0} \|(T(t) - I)T(s)\| = 0$ for $s > t_0$ for some $t_0 \geq 0$. For example strongly continuous analytic semigroups and eventually compact semigroups are in this class. In particular semigroups with a bounded generator are norm continuous, i.e. the map $t \mapsto T(t)$ is continuous for all $t \geq 0$ (for definitions we refer to [3]). Eventually norm continuous semigroups satisfy the spectral mapping theorem $e^{t\sigma(A)} = \sigma(T(t))\backslash\{0\}$ for $t \geq 0$ ([3, Theorem IV.3.10]).

Corollary 2. *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ which is eventually norm continuous. Then there is a positive number δ such that*

$$\mathrm{Ker}(T(t) - I) = \mathrm{Ker}(A) \quad \textit{and} \quad \overline{(T(t) - I)X} = \overline{A(D(A))} \tag{5}$$

for all $t \in (0, \delta)$. If the semigroup is also bounded and A is ergodic, then $T(t) - I$ is ergodic for all $t \in (0, \delta)$.

Proof. By [3, Theorem IV.3.10], the spectral mapping theorem $e^{t\sigma(A)} = \sigma(T(t))\backslash\{0\}$ for $t \geq 0$ holds. Further, by [3, Theorem II.4.18], $\sigma(A) \cap i\mathbb{R}$ is bounded. Let $s > 0$ be such that $|\beta| \leq s$ whenever $i\beta \in \sigma(A)$ where $\beta \in \mathbb{R}$. Then, for $0 < t < \pi s^{-1}$, $e^{it\beta} \neq -1$ and therefore $-1 \in \rho(T(t))$. Now the result follows from Proposition 1. ∎

Remark 4. If A is the generator of a bounded analytic semigroup $(T(t))_{t\geq 0}$, then the equalities (5) hold for all $t > 0$. The proof is similar to the proof of Corollary 2. In this case $-1 \in \rho(T(t))$ for all $t > 0$ since ([3, Corollary II.4.6]) $\sigma(A) \cap i\mathbb{R} \subset \{0\}$.

4. The canonical projection of the predual of the space J. We consider the space of James as defined in [1] and [2]. Let J be the space of all complex sequences $x = \{x_n\}_{n=1}^{\infty}$ with the norm

$$\|x\| = \sup\Big\{\Big(\sum_{j=1}^{k}\Big|\sum_{n\in I_j} x_n\Big|^2\Big)^{1/2} : I_1,\ldots,I_k \text{ are disjoint finite intervals of positive integers, } k\ge 1\Big\}.$$

The Banach space J has a (unique isometric) predual ([1]), which we call Y. The canonical projection of Y on $Y^{***} = X^{**}$ corresponding to the decomposition $Y^{***} = \widehat{Y^*} \oplus \hat{Y}^{\perp}$ will be called P_Y. The kernel of P_Y is a one-dimensional subspace $\mathbb{C}F_2$, where $F_2 \in J^{**}$, $\|F_2\| = 1$ and F_2 restricted to $\hat{Y}$ is equal to 0. Hence $J^{**} = \hat{J} \oplus \mathbb{C}F_2$. We will show that the projection P_Y has the property $\|I - 2P_Y\| < 2$.

The following definition was given by Brown and Ito in [2, p. 266]. A Banach space Y is in the class (L) if for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that whenever $x \in Y^*$, $G \in (\hat{Y})^{\perp}$, $\|\hat{x} + G\| = 1$ and $\|x\| > 1 - \delta$ then $\|G\| < \epsilon$.

We cite the result in [2, Proposition 7].

Proposition 2 ([2]). *The predual Y of J is in the class (L).*

Theorem 4. *Let P_Y be the canonical projection of the Banach space Y. Then $\|I - P_Y\| = 1$ and $1 < \|I - 2P_Y\| < 2$.*

Proof. First we show that $\|I - 2P_Y\| > 1$. If $\|I - 2P_Y\| = 1$, then the operator $T = 2P - I$ is an invertible isometry since $T^2 = I$. By [5, Proposition 10], $\mathrm{Ker}(T - I) = \hat{J}$ is weakly sequentially complete, which is not possible since J does not contain a subspace isomorphic to l^1 and J is not reflexive.

We now make use of the results of Brown and Ito [1], [2]. For all complex numbers c and all elements $x \in J$,

$$\|\hat{x} + cF_2\| \ge |c|, \tag{6}$$

which follows from [1, Lemma 2] where it is proved that $\|F_3\| = 1$; here F_3 is an element of J^{***} with $F_3(\hat{J}) = \{0\}$, $F_3(F_2) = 1$ and $J^{***} = \hat{J}^* \oplus \mathbb{C}F_3$. From the properties of F_2 it follows that

$$\|\hat{x} + cF_2\| \ge \|x\| \quad \text{for every } x \in J \text{ and all complex numbers } c. \tag{7}$$

For any $F \in J^{**}$ with $\|F\| = 1$ we deduce from inequalities (6) and (7) that $\|(I - P_Y)F\| \le \|F\| = 1$ and $\|P_Y F\| \le 1$. Therefore $\|I - P_Y\| = 1$ and $\|P_Y\| = 1$.

Finally, we prove that $\|I - 2P_Y\| < 2$. Let $0 < \epsilon < 1$. Since $Y^* = J$, it follows from Proposition 2 that there exists $\delta = \delta(\epsilon) > 0$ such that whenever $F \in J^{**}$, $\|F\| = 1$ and $\|P_Y F\| > 1 - \delta$ then $\|(I - P_Y)F\| < \epsilon$. Now for all

$F \in J^{**}$ with $\|F\| = 1$,

$$\|(I - 2P_Y)F\| = \|(I - P_Y)F - P_Y F\| \le \|(I - P_Y)F\| + \|P_Y F\|.$$

If $\|P_Y F\| > 1 - \delta$, then $\|(I - P_Y)F\| < \epsilon$. Hence $\|(I - 2P_Y)F\| < \epsilon + 1$. If $\|P_Y F\| \le 1 - \delta$, then $\|(I - 2P_Y)F\| \le 2 - \delta$. Therefore

$$\|I - 2P_Y\| = \sup\{\|(I - 2P_Y)F\| : \|F\| = 1\} \le \max\{\epsilon + 1, 2 - \delta\} < 2. \blacksquare$$

Remark 5. Theorem 1 can be applied to operators in the class (UBR) on J if $\|I - Q\| < K$ where $K = 2/\|I - 2P_Y\| > 1$.

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Kirsti Mattila
Department of Mathematics
Royal Institute of Technology
SE-10044 Stockholm, Sweden
E-mail: kirsti@math.kth.se

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