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On the non-equivalence of rearranged Walsh and trigonometric systems in L_p

by

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Dedicated to Professor A. Pełczyński
on the occasion of his 70th birthday

Abstract. We consider the question of whether the trigonometric system can be equivalent to some rearrangement of the Walsh system in L_p for some $p \neq 2$. We show that this question is closely related to a combinatorial problem. This enables us to prove non-equivalence for a number of rearrangements. Previously this was known for the Walsh–Paley order only.

1. Introduction. Both the Walsh system and the trigonometric system are systems of characters on a compact abelian group. This explains why many of the results in the theory of those systems are parallel. However, those similarities do not usually extend to the case when the systems are compared directly. So it is known that the Walsh system in the Walsh–Paley order and the trigonometric system are not equivalent in L_p for $p \neq 2$ (see [5]). A "power-type" non-equivalence for those systems was recently shown in [4].

It does not seem natural to fix the order of the systems in this basis equivalence problem. In [4] the conjecture was made that non-equivalence also holds for arbitrary rearrangements of the Walsh system. Nevertheless, the methods used in that paper are very particular to the case of the Walsh–Paley order. The aim of this note is to address the more general equivalence problem.

In a first part, we relate the equivalence question for a fixed ordering to a question of algebraic combinatorial type. In a second part, we apply this approach to prove non-equivalence for a number of orderings. We obtain

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estimates of power type but we do not attempt to find the optimal estimates here.

The method can be generalized to deal with the equivalence of arbitrary systems of characters on compact abelian groups. This will be studied elsewhere.

We consider the trigonometric system $(e_n)_{n\in\mathbb{Z}}$ on $[0,1]$ given by the functions $e_n(t) = \exp(2\pi int)$. Let $r_0, r_1, \ldots$ be the system of Rademacher functions on $[0,1]$. For $n = 0, 1, \ldots$ the binary expansion of n is $n = \sum_{i=0}^{\infty} n_i 2^i$ with $n_i \in \{0,1\}$. Observe that the sum is a finite sum. The nth Walsh function is then given by

$$w_n = \prod_{i=0}^{\infty} r_i^{n_i}.$$

Fix a permutation σ of $\{0,1,\ldots\}$. Given $1 \le p \le \infty$, we say that the trigonometric system and the Walsh system rearranged with σ are *equivalent in L_p* if there exists $c > 0$ such that

$$\frac{1}{c}\Big\|\sum_{k=0}^{\infty}\xi_k e_k\Big\|_p \le \Big\|\sum_{k=0}^{\infty}\xi_k w_{\sigma(k)}\Big\|_p \le c\Big\|\sum_{k=0}^{\infty}\xi_k e_k\Big\|_p$$

for all sequences (ξ_k) of complex numbers with only finitely many non-zero terms. Here $\|\cdot\|_p$ denotes the norm in $L_p[0,1]$. We then write $(e_k)_{k\ge 0} \sim_p (w_{\sigma(k)})_{k\ge 0}$. Observe that we actually only consider one half of the trigonometric system. This is not really essential for what follows but it simplifies the exposition significantly.

We now describe the organization of the paper in some detail. The next section provides a basic duality result which shows that the equivalence questions in L_p and $L_{p'}$ are essentially the same, where as usual p' denotes the conjugate number of p given by $1/p + 1/p' = 1$.

In Section 3 we introduce and study a sequence of functions crucial for our purpose. Norm estimates for these functions provide a tool to prove non-equivalence of Walsh and trigonometric systems. This is a generalization of the method used in [4]. Moreover, it will turn out to be fundamental for our considerations that non-equivalence of Walsh and trigonometric systems in L_p for all $p \ne 2$ can be derived from a non-trivial L_{p_0}-norm estimate of those functions for one fixed $p_0 > 2$.

In Section 4 we show that the L_4-norm of the key functions is determined by the solution of a particular combinatorial problem. This is due to the fact that the fourth power of the L_4-norm is a polynomial function and to the orthogonality of Walsh and trigonometric functions.

Section 5 applies this approach to concrete rearrangements of the Walsh system, in particular to linear and piecewise linear rearrangements and "small" perturbations thereof. This includes all the commonly used orderings of the Walsh functions, i.e. the Walsh–Paley order, the original Walsh order, the Walsh–Kaczmarz order, and the Walsh–Kronecker orders. We give the definitions of these orderings at the appropriate places. More information on Walsh functions and Walsh series can be found in the monographs [1] and [3].

2. Duality. To study the equivalence problem we introduce some notation. By $\mathcal{A}_n$ we denote a general orthonormal system $(a_0, \ldots, a_{n-1})$ in $L_2[0,1]$; usually this will be the system $\mathcal{W}_n$ of the first n Walsh functions $(w_0, \ldots, w_{n-1})$ or the system $\mathcal{E}_n$ of the first n exponential functions $(e_0, \ldots, e_{n-1})$. Given a finite set $\mathbb{F} \subseteq \{0, \ldots, n-1\}$, we denote by $\mathcal{A}(\mathbb{F})$ the system formed by the functions a_k with $k \in \mathbb{F}$. If in particular $\mathbb{F} = [m] := \{0, \ldots, m-1\}$ is the set of the first m members of $\{0, \ldots, n-1\}$ we write again $\mathcal{A}_m$ for $\mathcal{A}([m])$. Given a permutation σ of $\{0, 1, \ldots\}$ we denote by $\mathcal{A}^\sigma(\mathbb{F})$ the system formed by all $a_{\sigma(k)}$ where $k \in \mathbb{F}$. Note that $\mathcal{A}^\sigma(\mathbb{F})$ and $\mathcal{A}(\sigma(\mathbb{F}))$ differ just by their order.

We let $\boldsymbol{\varrho}_p(\mathcal{A}(\mathbb{F}), \mathcal{B}(\mathbb{F}))$ denote the smallest constant c such that

$$\Big\| \sum_{k \in \mathbb{F}} \xi_k a_k \Big\|_p \le c \Big\| \sum_{k \in \mathbb{F}} \xi_k b_k \Big\|_p$$

for all complex numbers ξ_k with $k \in \mathbb{F}$.

We are interested in the quantities

$$\boldsymbol{\varrho}_p(\mathcal{E}_n, \mathcal{W}_n^\sigma), \quad \boldsymbol{\varrho}_p(\mathcal{W}_n, \mathcal{E}_n^{\sigma^{-1}}),$$
$$\boldsymbol{\varrho}_p(\mathcal{W}_n^\sigma, \mathcal{E}_n), \quad \boldsymbol{\varrho}_p(\mathcal{E}_n^{\sigma^{-1}}, \mathcal{W}_n),$$

where σ^{-1} is the inverse permutation of σ.

In order to get some information on duality, we need another quantity, which behaves better under passing from p to p'. The kth *Fourier coefficient* of a function $f \in L_p$ with respect to the system $\mathcal{B}_n$ is given by $\langle f, b_k \rangle = \int_0^1 f(t)\overline{b_k(t)}\, dt$. For simplicity, we will henceforth assume that all orthonormal systems considered consist of bounded functions, so Fourier coefficients exist for all L_p-functions. We let $\boldsymbol{\delta}_p(\mathcal{A}(\mathbb{F}), \mathcal{B}(\mathbb{F}))$ denote the smallest constant c such that

$$\Big\| \sum_{k \in \mathbb{F}} \langle f, b_k \rangle a_k \Big\|_p \le c \|f\|_p$$

for all functions $f \in L_p[0,1]$. Observe that $\boldsymbol{\delta}_p(\mathcal{A}(\mathbb{F}), \mathcal{B}(\mathbb{F}))$ is the norm of the operator $S : L_p[0,1] \to L_p[0,1]$ given by $Sf = \sum_{k \in \mathbb{F}} \langle f, b_k \rangle a_k$.

We have the following facts about the quantities $\boldsymbol{\varrho}_p$ and $\boldsymbol{\delta}_p$, which are either obvious or proved in [2]:

$$\boldsymbol{\varrho}_p(\mathcal{A}(\mathbb{F}), \mathcal{B}(\mathbb{F})) \le \boldsymbol{\delta}_p(\mathcal{A}(\mathbb{F}), \mathcal{B}(\mathbb{F})), \tag{1}$$

$$\boldsymbol{\delta}_p(\mathcal{A}(\mathbb{F}), \mathcal{B}(\mathbb{F})) = \boldsymbol{\delta}_{p'}(\mathcal{B}(\mathbb{F}), \mathcal{A}(\mathbb{F})), \tag{2}$$

$$\boldsymbol{\delta}_p(\mathcal{A}(\mathbb{F}),\mathcal{B}(\mathbb{F})) \le \boldsymbol{\varrho}_p(\mathcal{A}(\mathbb{F}),\mathcal{B}(\mathbb{F}))\boldsymbol{\delta}_p(\mathcal{B}(\mathbb{F}),\mathcal{B}(\mathbb{F})), \tag{3}$$

$$\boldsymbol{\varrho}_p(\mathcal{A}^\sigma(\mathbb{F}),\mathcal{B}^\sigma(\mathbb{F})) = \boldsymbol{\varrho}_p(\mathcal{A}(\sigma(\mathbb{F})),\mathcal{B}(\sigma(\mathbb{F}))), \tag{4}$$

$$\boldsymbol{\delta}_p(\mathcal{A}^\sigma(\mathbb{F}),\mathcal{B}^\sigma(\mathbb{F})) = \boldsymbol{\delta}_p(\mathcal{A}(\sigma(\mathbb{F})),\mathcal{B}(\sigma(\mathbb{F}))). \tag{5}$$

If $\mathbb{F} \subseteq \mathbb{G}$ then also

$$\boldsymbol{\varrho}_p(\mathcal{A}(\mathbb{F}),\mathcal{B}(\mathbb{F})) \le \boldsymbol{\varrho}_p(\mathcal{A}(\mathbb{G}),\mathcal{B}(\mathbb{G})). \tag{6}$$

Moreover, if $\theta \in (0,1)$ is given by $1/p = (1-\theta)/p_0 + \theta/p_1$, complex interpolation shows that

$$\boldsymbol{\delta}_p(\mathcal{A}(\mathbb{F}),\mathcal{B}(\mathbb{F})) \le \boldsymbol{\delta}_{p_0}(\mathcal{A}(\mathbb{F}),\mathcal{B}(\mathbb{F}))^{1-\theta}\boldsymbol{\delta}_{p_1}(\mathcal{A}(\mathbb{F}),\mathcal{B}(\mathbb{F}))^{\theta}. \tag{7}$$

The next fact follows from the boundedness of the Riesz transform in any $L_p[0,1]$ for $1 < p < \infty$ (see [6, Vol. I, p. 67]), and the boundedness of the canonical projection from $L_p[0,1]$ onto the span of the first 2^m Walsh functions (see [3, p. 142]).

Lemma 2.1. *For $1 < p < \infty$, there is a constant c_p such that*

$$\boldsymbol{\delta}_p(\mathcal{E}_n,\mathcal{E}_n) \le c_p \tag{8}$$

for $n = 1, 2, \ldots$ Moreover, for all $m = 1, 2, \ldots$ we have

$$\boldsymbol{\delta}_p(\mathcal{W}_{2^m},\mathcal{W}_{2^m}) = 1. \tag{9}$$

Combining (4) and (6) gives the following lemma.

Lemma 2.2. *If $\sigma[n] \subseteq [N]$ then*

$$\boldsymbol{\varrho}_p(\mathcal{A}_n^\sigma,\mathcal{B}_n) \le \boldsymbol{\varrho}_p(\mathcal{A}_N,\mathcal{B}_N^{\sigma^{-1}}). \tag{10}$$

Proof. We have

$$\begin{aligned}\boldsymbol{\varrho}_p(\mathcal{A}_n^\sigma,\mathcal{B}_n) &= \boldsymbol{\varrho}_p(\mathcal{A}(\sigma[n]),\mathcal{B}^{\sigma^{-1}}(\sigma[n]))\\ &\le \boldsymbol{\varrho}_p(\mathcal{A}([N]),\mathcal{B}^{\sigma^{-1}}([N])) = \boldsymbol{\varrho}_p(\mathcal{A}_N,\mathcal{B}_N^{\sigma^{-1}}). \ \blacksquare\end{aligned}$$

Next we prove a first duality result.

Lemma 2.3. *For any orthonormal system $\mathcal{A}_n$ and $1 < p < \infty$ we have*

$$\boldsymbol{\varrho}_p(\mathcal{E}_n,\mathcal{A}_n) \le c_p\boldsymbol{\varrho}_{p'}(\mathcal{A}_n,\mathcal{E}_n), \tag{11}$$

$$\boldsymbol{\varrho}_p(\mathcal{W}_{2^m},\mathcal{A}_{2^m}) \le \boldsymbol{\varrho}_{p'}(\mathcal{A}_{2^m},\mathcal{W}_{2^m}). \tag{12}$$

Proof. It follows successively from (1), (2), (3) and (8) that

$$\begin{aligned}\boldsymbol{\varrho}_p(\mathcal{E}_n,\mathcal{A}_n) \le \boldsymbol{\delta}_p(\mathcal{E}_n,\mathcal{A}_n) = \boldsymbol{\delta}_{p'}(\mathcal{A}_n,\mathcal{E}_n) &\le \boldsymbol{\varrho}_{p'}(\mathcal{A}_n,\mathcal{E}_n)\boldsymbol{\delta}_{p'}(\mathcal{E}_n,\mathcal{E}_n)\\ &\le c_p\boldsymbol{\varrho}_{p'}(\mathcal{A}_n,\mathcal{E}_n).\end{aligned}$$

The second inequality follows in the same way if we use (9) instead of (8). ■

We can now prove the complete duality result.

PROPOSITION 2.4. *Given a permutation σ and $n \in \{0, 1, \ldots\}$ there exists a number N such that*

$$\varrho_p(\mathcal{E}_n, \mathcal{W}_n^\sigma) \le c_p \varrho_{p'}(\mathcal{W}_N^\sigma, \mathcal{E}_N), \qquad \varrho_p(\mathcal{E}_n^\sigma, \mathcal{W}_n) \le c_p \varrho_{p'}(\mathcal{W}_N, \mathcal{E}_N^\sigma), \tag{13}$$

$$\varrho_p(\mathcal{W}_n, \mathcal{E}_n^\sigma) \le \varrho_{p'}(\mathcal{E}_N^\sigma, \mathcal{W}_N), \qquad \varrho_p(\mathcal{W}_n^\sigma, \mathcal{E}_n) \le \varrho_{p'}(\mathcal{E}_N, \mathcal{W}_N^\sigma), \tag{14}$$

for all $1 < p < \infty$.

Proof. The left hand inequalities are immediate consequences of (11) and (12): use $N = n$ and $N = 2^m > n$ and (6) respectively.

The right hand inequalities follow from (10), the corresponding left hand inequalities and (10) again. ∎

We can now summarize the duality results as follows.

PROPOSITION 2.5. *Let $1 < p < \infty$. Then the systems $(e_k)_{k\ge 0}$ and $(w_{\sigma(k)})_{k\ge 0}$ are equivalent in L_p if and only if they are equivalent in $L_{p'}$.*

Proof. We only have to note that $(e_k)_{k\ge 0} \sim_p (w_{\sigma(k)})_{k\ge 0}$ if and only if the parameters $\varrho_p(\mathcal{E}_n, \mathcal{W}_n^\sigma)$ and $\varrho_p(\mathcal{W}_n^\sigma, \mathcal{E}_n)$ are uniformly bounded. ∎

3. The key functions. To show non-equivalence of the trigonometric and rearranged Walsh systems, norm estimates for the functions

$$F_n^\sigma(s,t) = \sum_{k=0}^{n-1} e_k(s) w_{\sigma(k)}(t)$$

play an essential rôle. This is due to the next observation.

PROPOSITION 3.1. *For each p with $1 < p < \infty$, there exists some constant $c_p > 0$ such that*

$$\varrho_p(\mathcal{E}_n, \mathcal{W}_n^\sigma) \ge c_p n^{1-1/p} \|F_n^\sigma\|_p^{-1} \quad \text{for } n = 1, 2, \ldots,$$

where $\|F_n^\sigma\|_p$ is the norm of F_n^σ in $L_p([0,1]^2)$.

Proof. From the definition of $\varrho_p(\mathcal{E}_n, \mathcal{W}_n^\sigma)$ we find that

$$\int_0^1 \Big| \sum_{k=0}^{n-1} \xi_k e_k(t) \Big|^p dt \le \varrho_p(\mathcal{E}_n, \mathcal{W}_n^\sigma)^p \int_0^1 \Big| \sum_{k=0}^{n-1} \xi_k w_{\sigma(k)}(t) \Big|^p dt$$

for all complex numbers $\xi_0, \ldots, \xi_{n-1}$. Using this for $\xi_k = e_k(s)$, integrating over $s \in [0,1]$ and taking pth roots, we obtain

$$\Big(\int_0^1 \Big| \sum_{k=0}^{n-1} e_k(t) \Big|^p dt \Big)^{1/p} \le \varrho_p(\mathcal{E}_n, \mathcal{W}_n^\sigma) \|F_n^\sigma\|_p.$$

The left hand side is the L_p-norm of the Dirichlet kernel. The well known properties of this kernel imply that

$$c_1 n^{1-1/p} \le \Big(\int_0^1 \Big|\sum_{k=0}^{n-1} e_k(t)\Big|^p dt\Big)^{1/p} \le c_2 n^{1-1/p},$$

where c_1 and c_2 depend only on p (see [6, Vol. I, p. 67]). This completes the proof. ∎

Since by Parseval's equality $\|F_n^\sigma\|_2 = \sqrt{n}$ and since obviously $\|F_n^\sigma\|_\infty = n$, Hölder's inequality yields the upper bound $\|F_n^\sigma\|_p \le n^{1-1/p}$ for any $2 < p < \infty$. If we can show for some $p > 2$ that actually

$$\liminf_{n\to\infty} n^{1/p-1} \|F_n^\sigma\|_p = 0 \tag{15}$$

then Proposition 3.1 implies that $(e_k)_{k\ge 0}$ and $(w_{\sigma(k)})_{k\ge 0}$ cannot be equivalent in L_p. The duality result, Proposition 2.5, shows that this is also true in $L_{p'}$.

We now derive that (15) for one p_0 in $(2,\infty)$ already implies (15) for all p in $(2,\infty)$. Indeed, if $p \in (2,p_0)$, defining $\theta \in (0,1)$ by $1/p = \theta/2 + (1-\theta)/p_0$ and using Hölder's inequality together with $\|F_n^\sigma\|_2 = n^{1/2}$ yields

$$n^{1/p-1}\|F_n^\sigma\|_p \le n^{1/p-1}\|F_n^\sigma\|_2^\theta \|F_n^\sigma\|_{p_0}^{1-\theta} = (n^{1/p_0-1}\|F_n^\sigma\|_{p_0})^{1-\theta}.$$

Similarly, if $p \in (p_0,\infty)$, defining $\theta \in (0,1)$ by $1/p = \theta/p_0$, from $\|F_n^\sigma\|_\infty = n$ we obtain

$$n^{1/p-1}\|F_n^\sigma\|_p \le n^{1/p-1}\|F_n^\sigma\|_{p_0}^\theta \|F_n^\sigma\|_\infty^{1-\theta} = (n^{1/p_0-1}\|F_n^\sigma\|_{p_0})^{\theta}.$$

Altogether, we have proved the following theorem.

THEOREM 3.2. *If there exists $p_0 \in (2,\infty)$ such that*

$$\liminf_{n\to\infty} n^{1/p_0-1} \|F_n^\sigma\|_{p_0} = 0$$

then, for all $p \in (1,\infty)$ with $p \ne 2$, the systems (e_k) and $(w_{\sigma(k)})$ are not equivalent in L_p.

REMARK. In the cases $p = 1$ and $p = \infty$, some additional care has to be taken. Using the well known estimates

$$\boldsymbol{\delta}_1(\mathcal{E}_n,\mathcal{E}_n) = \boldsymbol{\delta}_\infty(\mathcal{E}_n,\mathcal{E}_n) \le c(1+\log n),$$

one deduces from the interpolation formula (7) that if there exists $p_0 \in (2,\infty)$ such that

$$\liminf_{n\to\infty} n^{1/p_0-1}(1+\log n)^{1-2/p_0} \|F_n^\sigma\|_{p_0} = 0$$

then (e_k) and $(w_{\sigma(k)})$ are not equivalent in L_1 and L_∞.

4. The case $p = 4$. By the results of the previous section, we can now concentrate on upper bounds for the L_p-norm of F_n^σ for a convenient value of $p \in (2, \infty)$. We use $p = 4$ here. By Theorem 3.2, to show non-equivalence of (e_k) and $(w_{\sigma(k)})$ in L_p for all $p \in (1, \infty)$ it is enough to verify that

$$\liminf_{n\to\infty} n^{-3/4} \|F_n^\sigma\|_4 = 0.$$

We are going to formulate an equivalent combinatorial condition. To this end, let us introduce some more notation. Given two numbers $m, n \in \mathbb{N}_0$ with binary expansions $m = \sum_{i=0}^\infty m_i 2^i$ and $n = \sum_{i=0}^\infty n_i 2^i$, their *dyadic sum* is given by $m \oplus n = \sum_{i=0}^\infty |m_i - n_i| 2^i$. The set $\mathbb{N}_0$ with dyadic addition is isomorphic to the group of Walsh functions, which is expressed in the equation $w_{m\oplus n} = w_m w_n$ for all $m, n \in \mathbb{N}_0$. Since the Walsh system relates to the powers of two, we will from now on concentrate on the norms of F_{2^n} instead of F_n for all $n \in \mathbb{N}_0$. Let

$$A_n^\sigma = \{(k, l, m) \in [2^n]^3 : k+l-m \in [2^n],\ \sigma(k) \oplus \sigma(l) \oplus \sigma(m) = \sigma(k+l-m)\}.$$

In the next lemma and throughout the paper, the notation $\#A$ means the cardinality of a set A.

Lemma 4.1. $\|F_{2^n}^\sigma\|_4^4 = \#A_n^\sigma$.

Proof. It follows from

$$|F_{2^n}^\sigma(s,t)|^2 = F_{2^n}^\sigma(s,t)\overline{F_{2^n}^\sigma(s,t)} = \Big(\sum_{k=0}^{2^n-1} e_k(s) w_{\sigma(k)}(t)\Big)\Big(\sum_{l=0}^{2^n-1} e_{-l}(s) w_{\sigma(l)}(t)\Big)$$
$$= \sum_{k,l=0}^{2^n-1} e_{k-l}(s) w_{\sigma(k)\oplus\sigma(l)}(t)$$

that

$$|F_{2^n}^\sigma(s,t)|^4 = \sum_{k_1,l_1=0}^{2^n-1} \sum_{k_2,l_2=0}^{2^n-1} e_{k_1-l_1+k_2-l_2}(s) w_{\sigma(k_1)\oplus\sigma(l_1)\oplus\sigma(k_2)\oplus\sigma(l_2)}(t).$$

Since

$$\int_0^1\!\!\int_0^1 e_a(s) w_b(t)\, ds\, dt = \begin{cases} 1 & \text{if } a = b = 0, \\ 0 & \text{otherwise,} \end{cases}$$

we find by integration that $\|F_{2^n}^\sigma\|_4^4 = \#B_n^\sigma$ where

$$B_n^\sigma = \{(k_1, l_1, k_2, l_2) \in [2^n]^4 : k_1 - l_1 + k_2 - l_2 = 0,\ \sigma(k_1) \oplus \sigma(l_1) \oplus \sigma(k_2) \oplus \sigma(l_2) = 0\}.$$

Obviously, B_n^σ has the same cardinality as A_n^σ. ■

Corollary 4.2. *If* $\liminf_{n\to\infty} 8^{-n} \#A_n^\sigma = 0$ *then, for all* $p \in (1, \infty)$ *with* $p \neq 2$, *the systems* (e_k) *and* $(w_{\sigma(k)})$ *are not equivalent in* L_p.

Remark. Using the remark following Theorem 3.2 we also see that

$$\liminf_{n\to\infty} 8^{-n} n^2 \# A_n^\sigma = 0$$

implies that (e_k) and $(w_{\sigma(k)})$ are not equivalent in L_1 and L_∞.

5. Application to concrete rearrangements. In this section, we apply the results of the previous section to the study of the equivalence problem for some specific rearrangements. In particular, we treat the (besides the Walsh–Paley order) most frequently used cases of the original Walsh system, the Walsh–Kaczmarz system and the Walsh–Kronecker systems. For the properties and alternative definitions of the above orderings, we refer the reader to [3].

5.1. *Dyadically linear rearrangements.* The original Walsh system is a particular case of a linear rearrangement of the Walsh–Paley system. A *dyadically linear rearrangement* is represented by a matrix $T = (t_{i,j})_{i,j=0}^\infty$ with entries in $\{0,1\}$ such that the ith coefficient in the binary expansion of $\sigma(n)$ is given as

$$\sigma(n)_i = \sum_{j=0}^\infty t_{i,j} n_j \bmod 2.$$

This is equivalent to the condition that σ is linear with respect to binary addition: $\sigma(m \oplus n) = \sigma(m) \oplus \sigma(n)$. The *original Walsh system* is obtained using the matrix T with entries $t_{i,j} = 1$ if and only if $j = i$ or $j = i+1$.

For linear rearrangements σ the sets A_n^σ behave nicely.

Proposition 5.1. *If σ is dyadically linear and π is an arbitrary permutation, then*

$$A_n^{\sigma\circ\pi} = A_n^\pi$$

and consequently

$$\# A_n^{\sigma\circ\pi} = \# A_n^\pi.$$

Proof. We simply observe that by linearity and injectivity of σ we have

$$\begin{aligned} A_n^{\sigma\circ\pi} &= \{(x,y,z) \in [2^n]^3 : x+y-z \in [2^n], \\ &\qquad \sigma(\pi(x)) \oplus \sigma(\pi(y)) \oplus \sigma(\pi(z)) = \sigma(\pi(x+y-z))\} \\ &= \{(x,y,z) \in [2^n]^3 : x+y-z \in [2^n], \\ &\qquad \sigma(\pi(x) \oplus \pi(y) \oplus \pi(z)) = \sigma(\pi(x+y-z))\} \\ &= \{(x,y,z) \in [2^n]^3 : x+y-z \in [2^n], \\ &\qquad \pi(x) \oplus \pi(y) \oplus \pi(z) = \pi(x+y-z)\} = A_n^\pi. \quad \blacksquare \end{aligned}$$

To use our general combinatorial condition for dyadically linear rearrangements of the Walsh–Paley system, we need the following result, which may also have some interest in its own right.

Theorem 5.2. *Let $\psi : \mathbb{N}_0 \to \mathbb{Z}$ be an arbitrary map. Then for all $n = 0, 1, \ldots$ we have*

$$\#\{(x,y) \in [2^n]^2 : \psi(x \oplus y) = x + y\} \le 3^n.$$

Proof. For $u = 0, 1, \ldots,$ define

$$\begin{aligned} B_n(u) &= \{(x,y) \in [2^n]^2 : x \oplus y = u\}, \\ C_n^{\psi}(u) &= \{(x,y) \in [2^n]^2 : x + y = \psi(u)\}. \end{aligned}$$

Then

$$\{(x,y) \in [2^n]^2 : \psi(x \oplus y) = x + y\} = \bigcup_u B_n(u) \cap C_n^{\psi}(u).$$

So all we have to show is

$$\sum_u \#(B_n(u) \cap C_n^{\psi}(u)) \le 3^n.$$

We use induction over n. The statement for $n = 0$ is trivial. So assume we already know the statement for a certain value of n and all functions ψ. Let us partition $B_{n+1}(u)$ into four disjoint subsets as follows:

$$\begin{aligned} B_{00}(u) &= B_{n+1}(u) \cap ([2^n] \times [2^n]), \\ B_{01}(u) &= B_{n+1}(u) \cap ([2^n] \times (2^n + [2^n])), \\ B_{10}(u) &= B_{n+1}(u) \cap ((2^n + [2^n]) \times [2^n]), \\ B_{11}(u) &= B_{n+1}(u) \cap ((2^n + [2^n]) \times (2^n + [2^n])). \end{aligned}$$

We are going to use the induction hypothesis to show that

$$\sum_u \#(B_{01}(u) \cap C_{n+1}^{\psi}(u)) \le 3^n, \tag{16}$$

$$\sum_u \#(B_{10}(u) \cap C_{n+1}^{\psi}(u)) \le 3^n, \tag{17}$$

$$\sum_u \#((B_{00}(u) \cup B_{11}(u)) \cap C_{n+1}^{\psi}(u)) \le 3^n. \tag{18}$$

This implies $\sum_u \#(B_{n+1}(u) \cap C_{n+1}^{\psi}(u)) \le 3^{n+1}$, completing the induction.

To verify (16), we observe that for $(x,y) \in [2^n] \times (2^n + [2^n])$ we have $y \oplus 2^n = y - 2^n$ and therefore

$$\begin{aligned} (x,y) \in B_{01}(u) \cap C_{n+1}^{\psi}(u) \ &\Leftrightarrow\ x \oplus y = u \text{ and } x + y = \psi(u) \\ &\Leftrightarrow\ x \oplus y \oplus 2^n = u \oplus 2^n \text{ and } x + y - 2^n = \psi(u) - 2^n \\ &\Leftrightarrow\ (x, y - 2^n) \in B_n(u \oplus 2^n) \cap C_{n+1}^{\widetilde{\psi}}(u \oplus 2^n), \end{aligned}$$

where we define $\widetilde{\psi}(\widetilde{u}) = \psi(\widetilde{u} \oplus 2^n) - 2^n$. So

$$\#(B_{01} \cap C_{n+1}^{\psi}(u)) = \#(B_n(u \oplus 2^n) \cap C_n^{\widetilde{\psi}}(u \oplus 2^n)),$$

which implies by induction hypothesis that

$$\sum_u \#(B_{01}(u) \cap C^{\psi}_{n+1}(u)) \le \sum_u \#(B_n(u) \cap C^{\widetilde{\psi}}_n(u)) \le 3^n.$$

Inequality (17) is symmetric to (16).

To prove (18), we observe that

$$\begin{aligned}(x,y) \in B_{00}(u) &\quad \text{implies} \quad x+y < 2^{n+1},\\ (x,y) \in B_{11}(u) &\quad \text{implies} \quad x+y \ge 2^{n+1},\end{aligned}$$

which gives

$$\begin{aligned}&\text{if } \psi(u) \ge 2^{n+1} \text{ then } B_{00}(u) \cap C^{\psi}_{n+1}(u) = \emptyset,\\ &\text{if } \psi(u) < 2^{n+1} \text{ then } B_{11}(u) \cap C^{\psi}_{n+1}(u) = \emptyset.\end{aligned}$$

So

$$\begin{aligned}&\sum_u \#((B_{00}(u) \cup B_{11}(u)) \cap C^{\psi}_{n+1}(u))\\ &\quad = \sum_{\psi(u)<2^{n+1}} \#(B_{00}(u) \cap C^{\psi}_{n+1}(u)) + \sum_{\psi(u)\ge 2^{n+1}} \#(B_{11}(u) \cap C^{\psi}_{n+1}(u)).\end{aligned}$$

Defining $\widetilde{\psi}$ by

$$\widetilde{\psi}(u) = \begin{cases} \psi(u) & \text{if } \psi(u) < 2^{n+1},\\ \psi(u) - 2^{n+1} & \text{if } \psi(u) \ge 2^{n+1},\end{cases}$$

we obtain, for u with $\psi(u) < 2^{n+1}$,

$$(x,y) \in B_{00}(u) \cap C^{\psi}_{n+1}(u) \ \Leftrightarrow\ (x,y) \in B_n(u) \cap C^{\widetilde{\psi}}_n(u),$$

and for u with $\psi(u) \ge 2^{n+1}$

$$(x,y) \in B_{11}(u) \cap C^{\psi}_{n+1}(u) \ \Leftrightarrow\ (x-2^n, y-2^n) \in B_n(u) \cap C^{\widetilde{\psi}}_n(u).$$

So

$$\sum_{\psi(u)<2^{n+1}} \#(B_{00}(u) \cap C^{\psi}_{n+1}(u)) = \sum_{\psi(u)<2^{n+1}} \#(B_n(u) \cap C^{\widetilde{\psi}}_n(u))$$

and

$$\sum_{\psi(u)\ge 2^{n+1}} \#(B_{11}(u) \cap C^{\psi}_{n+1}(u)) = \sum_{\psi(u)\ge 2^{n+1}} \#(B_n(u) \cap C^{\widetilde{\psi}}_n(u)),$$

together with the induction hypothesis, finally imply that

$$\sum_u \#((B_{00}(u) \cup B_{11}(u)) \cap C^{\psi}_{n+1}(u)) \le \sum_u \#(B_n(u) \cap C^{\widetilde{\psi}}_n(u)) \le 3^n. \ ■$$

Denoting by ι the identity $\iota(x) = x$ for all $x \in \mathbb{N}_0$ we can now prove the following result.

Corollary 5.3. $\#A^{\iota}_n \le 6^n$.

Proof. For each $z \in [2^n]$, we consider the set

$$A_n(z) = \{(x, y) \in [2^n]^2 : x + y - z \in [2^n],\ x \oplus y \oplus z = x + y - z\}.$$

Defining $\psi(u) = (u \oplus z) + z$, we obtain

$$A_n(z) \subseteq \{(x, y) \in [2^n]^2 : \psi(x \oplus y) = x + y\},$$

so from Theorem 5.2 we infer that $\#A_n(z) \le 3^n$. Consequently,

$$\#A_n^\iota \le \sum_{z \in [2^n]} \#A_n(z) \le 2^n 3^n = 6^n. \ \blacksquare$$

THEOREM 5.4. *If σ is dyadically linear then $\#A_n^\sigma \le 6^n$ for $n = 0, 1, \ldots$ So the systems (e_k) and $(w_{\sigma(k)})$ are not equivalent in L_p for $p \neq 2$. In particular, the Walsh–Paley system and the original Walsh system are not equivalent to the trigonometric system in L_p for $p \in [1, \infty]$ with $p \neq 2$.*

Proof. The assertion follows immediately from Proposition 5.1 and Corollaries 5.3 and 4.2 and the remark following Corollary 4.2. ∎

REMARK. The *Walsh–Kronecker systems* $W_{2^n}^{\sigma_n}$ are special rearrangements of the first 2^n Walsh functions, different for each n, which are the basis for the fast Walsh–Fourier transform. They can also be obtained from the Walsh matrices. Here σ_n is a dyadically linear map on $[2^n]$ so that our results also apply to this case giving lower estimates for $\varrho_p(\mathcal{E}_{2^n}, \mathcal{W}_{2^n}^{\sigma_n})$.

5.2. *Piecewise linear rearrangements.* Unfortunately, one of the frequently used rearrangements of the Walsh system, the *Walsh–Kaczmarz system*, is not a linear rearrangement. It seems more natural in the equivalence problem than the Walsh–Paley order since it arranges the Walsh functions in the order of increasing number of sign changes. The corresponding permutation σ is given by $\sigma(0) = 0$ and

$$\sigma\Big(2^k + \sum_{i=0}^{k-1} x_i 2^i\Big) = 2^k + \sum_{i=0}^{k-1} x_{k-1-i} 2^i$$

for $k = 0, 1, \ldots$ and $x_0, \ldots, x_{k-1} \in \{0, 1\}$. It is possible to estimate the cardinality of the set A_n^σ from the previous section for this rearrangement directly. Nevertheless, we prefer to sketch an alternative approach which works for all piecewise linear rearrangements.

A permutation σ defines a *piecewise linear rearrangement* if $\sigma(0) = 0$ and

$$\sigma(2^k + m) = 2^k + \sigma_k(m)$$

for $k = 0, 1, \ldots,\ 0 \le m \le 2^k - 1$, and bijections $\sigma_k : [2^k] \to [2^k]$ which are linear with respect to binary addition. In particular, σ leaves the blocks $\{2^k, 2^k + 1, \ldots, 2^{k+1} - 1\}$ invariant. Obviously, the Walsh–Kaczmarz order is a piecewise linear rearrangement.

Instead of using the functions F_n^σ, we now use the functions

$$\widetilde{F}_n^\sigma(s,t) = F_{2n}^\sigma(s,t) - F_n^\sigma(s,t) = \sum_{k=n}^{2n-1} e_k(s) w_{\sigma(k)}(t).$$

As an analogue of Proposition 3.1, we obtain

PROPOSITION 5.5. *For each p with $1 < p < \infty$, there exists some constant $c_p > 0$ such that*

$$\boldsymbol{\varrho}_p(\mathcal{E}_{2n}, \mathcal{W}_{2n}^\sigma) \geq c_p n^{1-1/p} \|\widetilde{F}_n^\sigma\|_p^{-1} \quad \textit{for } n = 1, 2, \ldots$$

Similarly, we obtain analogues of Lemma 4.1 and Corollary 4.2 if we replace the set A_n^σ by the set

$$\widetilde{A}_n^\sigma = \{(k,l,m) \in (2^n + [2^n])^3 : k + l - m \in 2^n + [2^n], \\ \sigma(k) \oplus \sigma(l) \oplus \sigma(m) = \sigma(k+l-m)\}.$$

LEMMA 5.6. $\|\widetilde{F}_{2^n}^\sigma\|_4^4 = \#\widetilde{A}_n^\sigma$.

COROLLARY 5.7. *If $\liminf_{n\to\infty} 8^{-n} \#\widetilde{A}_n^\sigma = 0$ then, for all $p \in (1,\infty)$ with $p \neq 2$, the systems (e_k) and $(w_{\sigma(k)})$ are not equivalent in L_p.*

REMARK. Using the remark following Theorem 3.2 we also deduce that

$$\liminf_{n\to\infty} 8^{-n} n^2 \#\widetilde{A}_n^\sigma = 0$$

implies that (e_k) and $(w_{\sigma(k)})$ are not equivalent in L_1 and L_∞.

We are now in a position to treat the case of piecewise linear rearrangements.

THEOREM 5.8. *If σ is a piecewise linear rearrangement then $\#\widetilde{A}_n^\sigma \leq 6^n$ for $n = 0, 1, \ldots$ So the systems (e_k) and $(w_{\sigma(k)})$ are not equivalent in L_p for $p \neq 2$. In particular, the Walsh–Kaczmarz system is not equivalent to the trigonometric system in L_p for $p \in [1,\infty]$ with $p \neq 2$.*

Proof. For each $z \in 2^n + [2^n]$, we consider the set

$$\widetilde{A}_n^\sigma(z) = \{(x,y) \in (2^n + [2^n])^2 : x + y - z \in 2^n + [2^n], \\ \sigma(x) \oplus \sigma(y) \oplus \sigma(z) = \sigma(x+y-z)\}.$$

Let $\sigma_k : [2^k] \to [2^k]$ denote the linear maps from the definition of piecewise linearity. Then we infer for any $(x,y) \in \widetilde{A}_n^\sigma(z)$ that $x + y - z \in 2^n + [2^n]$ and

$$\sigma_n(\widetilde{x} \oplus \widetilde{y} \oplus \widetilde{z}) = \sigma_n(\widetilde{x} + \widetilde{y} - \widetilde{z}),$$

where $\widetilde{x} = x - 2^n$, $\widetilde{y} = y - 2^n$, $\widetilde{z} = z - 2^n$. Since σ_n is a permutation this implies $\widetilde{x} \oplus \widetilde{y} \oplus \widetilde{z} = \widetilde{x} + \widetilde{y} - \widetilde{z}$. So

$$\#\widetilde{A}_n^\sigma(z) \leq \#\{(\widetilde{x},\widetilde{y}) \in [2^n]^2 : \widetilde{x} \oplus \widetilde{y} \oplus \widetilde{z} = \widetilde{x} + \widetilde{y} - \widetilde{z}\}.$$

This can be estimated by 3^n as in the proof of Corollary 5.3 and gives $\#\widetilde{A}_n^\sigma \le \sum_{z\in 2^n+[2^n]} \#\widetilde{A}_n^\sigma(z) \le 6^n$. The claim now follows from Corollary 5.7 if $p \in (1,\infty)$ and the remark after that corollary if $p = 1,\infty$. ∎

5.3. *Small perturbations.* Besides (piecewise) linear rearrangements, we can treat a further class of rearrangements, namely small perturbations of rearrangements of the Walsh system, which are known to be non-equivalent.

To this end, for $v \in \mathbb{Z}$, we also consider the sets

$$A_n^\sigma(v) := \{(x,y,z) \in [2^n]^3 : x+y-z \in [2^n],\\ \sigma(x)\oplus\sigma(y)\oplus\sigma(z)\oplus\sigma(v) = \sigma(x+y-z)\}.$$

So instead of asking for $\sigma(x)\oplus\sigma(y)\oplus\sigma(z)\oplus\sigma(x+y-z) = 0$ we require that the left hand side of this equality equals a fixed number $\sigma(v)$. Note that $A_n^\sigma(0) = A_n^\sigma$.

As in the proof of Theorem 5.4, we can control the size of $A_n^\sigma(v)$ for dyadically linear rearrangements σ and all v.

PROPOSITION 5.9. *If σ is dyadically linear then $\#A_n^\sigma(v) \le 6^n$ for $n = 0,1,\ldots$*

Proof. By linearity and injectivity of σ we need only consider the case $\sigma = \iota$. Using $\psi(u) = (u\oplus z\oplus v)+z$ we derive the result as in the proof of Corollary 5.3. ∎

Given two permutations π and σ let

$$f(u) := \pi(u)\oplus\sigma(u) \quad \text{and} \quad f_n^* := \max_{u\in[2^n]} f(u).$$

The function f measures, in some sense, how much π deviates from σ. In particular $|\pi(u)-\sigma(u)| \le f(u)$. We say that π *dyadically differs from σ by f.*

PROPOSITION 5.10. *We have*

$$A_n^\sigma \subseteq \bigcup_{\pi(v)\le 4f_n^*} A_n^\pi(v).$$

In particular

$$\#A_n^\sigma \le (4f_n^*+1)\,\#A_n^\pi.$$

Proof. Note that for all x,y,z we have

$$\sigma(x)\oplus\sigma(y)\oplus\sigma(z)\oplus\sigma(x+y-z)\oplus\pi(x)\oplus\pi(y)\oplus\pi(z)\oplus\pi(x+y-z)\\ = f(x)\oplus f(y)\oplus f(z)\oplus f(x+y-z). \tag{19}$$

Also, for any $x,y\ge 0$ the dyadic addition satisfies

$$x\oplus y \le x+y.$$

Therefore if $(x, y, z) \in A_n^\sigma$ then $x, y, z, x + y - z \in [2^n]$ and hence

$$f(x) \oplus f(y) \oplus f(z) \oplus f(x + y - z) \leq 4f_n^*.$$

Defining v by

$$\pi(v) = \pi(x) \oplus \pi(y) \oplus \pi(z) \oplus \pi(x + y - z),$$

we obtain $(x, y, z) \in A_n^\pi(v)$. It now follows from

$$\sigma(x) \oplus \sigma(y) \oplus \sigma(z) \oplus \sigma(x + y - z) = 0$$

and (19) that $\pi(v) \leq 4f_n^*$. This completes the proof. ∎

This proposition immediately implies

THEOREM 5.11. *If π dyadically differs from σ by f, and f and $\#A_n^\pi$ satisfy*

$$\liminf_{n\to\infty} 8^{-n} \# A_n^\pi f_n^* = 0,$$

then, for all $p \in (1, \infty)$ with $p \neq 2$, the systems (e_k) and $(w_{\sigma(k)})$ are not equivalent in L_p. This is in particular the case if π is dyadically linear and f satisfies

$$f_n^* = o\left(\frac{4^n}{3^n}\right).$$

REMARK. In the cases $p = 1, \infty$, we again have to adjust the condition to

$$\liminf_{n\to\infty} 8^{-n} n^2 \# A_n^\pi f_n^* = 0.$$

We now develop a dual version of the last results. We will mostly leave the proofs to the reader, since they are completely analogous to the previous ones. For $v \in \mathbb{Z}$, we define the sets

$$\widehat{A}_n^\sigma(v) := \{(x, y, z) \in [2^n]^3 : \sigma(x) + \sigma(y) - \sigma(z) + v = \sigma(x \oplus y \oplus z)\}$$

and we let $\widehat{A}_n^\sigma = \widehat{A}_n^\sigma(0)$. As before, we can show that for $p \in (1, \infty)$ with $p \neq 2$ the systems $(e_{\sigma(k)})$ and (w_k) are not equivalent in L_p if

$$\liminf_{n\to\infty} 8^{-n} \# \widehat{A}_n^\sigma = 0.$$

Again, we can control the size of $\widehat{A}_n^\sigma(v)$ for dyadically linear rearrangements σ and all v.

PROPOSITION 5.12. *If σ is dyadically linear then $\#\widehat{A}_n^\sigma(v) \leq 6^n$ for $n =$* $0, 1, \ldots$

Given two permutations π and σ let

$$\widehat{f}(u) := |\pi(u) - \sigma(u)| \quad \text{and} \quad \widehat{f}_n^* := \max_{u\in[2^n]} \widehat{f}(u).$$

The function $\widehat{f}$ measures how much π deviates from σ. We say that π *differs from σ by $\widehat{f}$.*

PROPOSITION 5.13. *We have*

$$\widehat{A}_n^\sigma \subseteq \bigcup_{|v|\le 4\widehat{f}_n^*} \widehat{A}_n^\pi(v).$$

In particular

$$\#\widehat{A}_n^\sigma \le (8\widehat{f}_n^* + 1)\,\#\widehat{A}_n^\pi.$$

Proof. Note that for all x, y, z we have

$$\begin{aligned}(20)\qquad &|\sigma(x\oplus y\oplus z)-\sigma(x)-\sigma(y)+\sigma(z)-\pi(x\oplus y\oplus z)+\pi(x)+\pi(y)-\pi(z)|\\ &\qquad\le \widehat{f}(x\oplus y\oplus z)+\widehat{f}(x)+\widehat{f}(y)+\widehat{f}(z).\end{aligned}$$

Therefore if $(x, y, z) \in A_n^\sigma$ then $x, y, z, x \oplus y \oplus z \in [2^n]$ and hence

$$\widehat{f}(x\oplus y\oplus z)+\widehat{f}(x)+\widehat{f}(y)+\widehat{f}(z) \le 4\widehat{f}_n^*.$$

Defining v by

$$v = \pi(x\oplus y\oplus z)-\pi(x)-\pi(y)+\pi(z),$$

we obtain $(x, y, z) \in \widehat{A}_n^\pi(v)$. It now follows from

$$\sigma(x\oplus y\oplus z)-\sigma(x)-\sigma(y)+\sigma(z) = 0$$

and (20) that $|v| \le 4\widehat{f}_n^*$. This completes the proof. ∎

Again we immediately obtain

THEOREM 5.14. *If π differs from σ by $\widehat{f}$, and $\widehat{f}$ and $\#\widehat{A}_n^\pi$ satisfy*

$$\liminf_{n\to\infty} 8^{-n}\#\widehat{A}_n^\pi \widehat{f}_n^* = 0,$$

then, for all $p \in (1,\infty)$ with $p \ne 2$, the systems $(e_{\sigma(k)})$ and (w_k) are not equivalent in L_p. This is in particular the case if π is dyadically linear and $\widehat{f}$ satisfies

$$\widehat{f}_n^* = o\left(\frac{4^n}{3^n}\right).$$

REMARK. In the cases $p = 1, \infty$, we again have to adjust the condition to

$$\liminf_{n\to\infty} 8^{-n} n^2 \#\widehat{A}_n^\pi \widehat{f}_n^* = 0.$$

To illustrate the power of this perturbation method, we add another example.

EXAMPLE. Let $\mathbb{F}$ be a subset of $\mathbb{N}_0$ such that

$$\liminf_{n\to\infty} \frac{\#(\mathbb{F}\cap[n])}{n} < 2 - \log_2 3 = 0.415037\ldots$$

Let σ be such that for $x = \sum_{i=0}^\infty x_i 2^i$ we have

$$\sigma(x) = \sum_{i\in\mathbb{F}} \widetilde{x}_i 2^i \oplus \sum_{i\notin\mathbb{F}} x_i 2^i,$$

where $\widetilde{x}_i \in \{0,1\}$ are such that σ is a permutation and otherwise arbitrary. In other words, σ acts arbitrarily on the binary coefficients in $\mathbb{F}$ and as the identity on the remaining binary coefficients. Then the systems (e_k) and $(w_{\sigma(k)})$ are not equivalent in L_p with $p \neq 2$.

Proof. Fix $n \in \mathbb{N}_0$. Let $m = \#(\mathbb{F} \cap [n])$ and write $\mathbb{F} \cap [n] = \{k_0, \ldots, k_{m-1}\}$ and $[n] \setminus \mathbb{F} = \{k_m, \ldots, k_{n-1}\}$. Define a permutation π_n by

$$\pi_n\Big(\sum_{i=0}^{n-1} x_i 2^i\Big) = \sum_{i=0}^{n-1} x_{k_i} 2^i.$$

Then π_n is a dyadically linear permutation on $[2^n]$ so by Proposition 5.1 we have $\#A_n^{\pi_n} = \#A_n^{\pi_n \circ \sigma}$. Moreover

$$\pi_n \circ \sigma(u) \oplus \pi_n(u) = \sum_{i=0}^{m-1} \widetilde{u}_{k_i} 2^i \oplus \sum_{i=0}^{m-1} u_{k_i} 2^i < 2^m.$$

This implies by Proposition 5.10 that

$$\#A_n^{\pi_n \circ \sigma} \le 4 \cdot 2^m \# A_n^{\pi_n},$$

or by the linearity of π_n and Corollary 5.3,

$$\#A_n^{\sigma} \le 4 \cdot 2^m 6^n.$$

The growth condition on $\#(\mathbb{F} \cap [n])$ ensures that

$$\liminf_{n \to \infty} 8^{-n} \# A_n^{\sigma} = 0.$$

The claim now follows from Corollary 4.2. ∎

Final remarks. 1. The estimates for the non-equivalence quantities obtained by our methods have power type behavior. Nevertheless, since they do not give optimal exponents except possibly in the case $p = 4$, we did not state those estimates explicitly. In the case $2 < p \le 4$, our estimates for the Walsh–Paley system are the same as the lower bounds obtained in [4]. In the cases $p > 4$ and $1 \le p < 2$, the estimates for the special case of the Walsh–Paley order in [4] are better than ours. It would be interesting to find the optimal estimates at least in the cases of the usual orderings.

2. Although we were not able to give general estimates for the cardinalities of the sets A_n^{σ}, we conjecture that the identical permutation already gives the maximal possible cardinality. A similar question, very natural from the combinatorial point of view, is to find good upper bounds for the cardinalities of the sets

$$B_n^{\sigma} = \{(k,l) \in [2^n]^2 : k + l \in [2^n],\ \sigma(k) \oplus \sigma(l) = \sigma(k+l)\}.$$

For linear and piecewise linear rearrangements one can obtain $\#B_n^{\sigma} \le 3^n$ and for the identity $\#B_n^{\iota} = 3^n$. Again we conjecture that $\#B_n^{\sigma} \le 3^n$ for any permutation σ. Basically, this is a question about how big the set of pairs (k, l) can be for which σ behaves like a homomorphism between the integers and the Cantor group. We checked this claim for $n \leq 4$ and for all permutations σ of $[2^n]$ by computer. The running time for the case $n = 4$ on a PC was about four days.

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