

Continuous Reinhardt domains from a Jordan viewpoint

by

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Abstract. As a natural extension of bounded complete Reinhardt domains in \mathbb{C}^N to spaces of continuous functions, continuous Reinhardt domains (CRD) are bounded open connected solid sets in commutative C^* -algebras with respect to the natural ordering. We give a complete parametric description for the structure of holomorphic isomorphisms between CRDs and characterize the partial Jordan triple structures which can be associated with some CRDs. On the basis of these results, we test two conjectures concerning the Jordan structure of bounded circular domains. It turns out that both the problems of bidualization and unique extension of inner derivations have positive solution in the setting of CRDs.

1. Introduction. A classical complete Reinhardt domain is an open connected subset in the space \mathbb{C}^n of all complex n -tuples, which is invariant under all coordinate multiplications $(z_1, \dots, z_n) \mapsto (\lambda_1 z_1, \dots, \lambda_n z_n)$ with $|\lambda_1|, \dots, |\lambda_n| \leq 1$. If we regard \mathbb{C}^n as the complex ordered space of functions $z : \{1, \dots, n\} \rightarrow \mathbb{C}$, this property can be stated as

$$(CR) \quad f \in D \text{ and } |g| \leq |f| \Rightarrow g \in D.$$

Postulating (CR) for the order absolute value, we can speak of bounded complete Reinhardt domains in complex Banach lattices in a natural manner.

1974 Sunada [18] obtained a rather thorough description of classical bounded Reinhardt domains containing the origin from the viewpoint of holomorphic equivalence. Later on several authors investigated holomorphic equivalence of generalized Reinhardt domains in atomic Banach lattices [1, 2, 12]. Motivated by an interesting work of Vigué [19] on the possible lack of symmetry of continuous products of discs with different radii, in [17] we introduced the concept of *continuous Reinhardt domains* (CRD for short).

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By definition, a CRD is a bounded complete Reinhardt domain in the C^* -algebra of all bounded continuous functions over some topological space, or more generally, in a commutative C^* -algebra. In [17] we have shown that a symmetric CRD is a continuous mixture of finite-dimensional Euclidean balls, essentially more involved than direct sums of topological products of balls. In [7] we found matrix representations for linear isomorphisms between two symmetric CRDs. To achieve these results we intensively used the Jordan theory of the bidual embedding of symmetric domains. However, the main points of both Sunada's and Vigué's papers concern the non-symmetric case. Recently, based upon the Lie theory of Hermitian operators in the dual space, in [16] we managed to extend the matrix representations of [7] to a Banach–Stone type theorem on isomorphisms of general Banach lattice normed commutative C^* -algebras. This result includes implicitly the description of all possible linear isomorphisms between CRDs because the convex hull of any CRD can be regarded as the unit ball of some lattice norm in a commutative C^* -algebra and linear isomorphisms preserve convex hulls.

The aim of this paper is a description of all possible holomorphic equivalences of CRDs. According to a classical result of [4], every bounded circular domain and hence even a non-symmetric CRD admits a natural partial Jordan $*$ -triple structure, a so-called partial JB^* -triple, which gives rise to the description of its complete holomorphic vector fields—a crucial piece of information about its holomorphic geometry. In particular, every holomorphic isomorphism between two bounded circular domains is the composition of a linear isomorphism with the exponential of a suitable complete holomorphic vector field over one of the two domains.

In Section 2 we review the basic material [4, 13, 14] concerning partial JB^* -triples. The bidual embedding arguments used in [17] to treat the Jordan structure in the symmetric case are not available for general CRDs. Although Dineen [5] and Barton–Timoney [3] established a satisfactory bidual Jordan theory for all bounded convex circular domains already in 1986, it is still one of the fundamental open questions in geometric Jordan theory, with no interesting partial results so far, whether the canonical partial triple product associated with any *non-convex* bounded circular domain extends in a weak*-continuous manner to the canonical partial triple product of some bounded circular domain in the bidual. Instead, in Section 3 we develop an alternative approach for determining the partial JB^* -triple product associated with a CRD. The conclusion, Theorem 3.5, is an integral representation of this triple product. In classical finite-dimensional complex analysis, Reinhardt domains are popular test objects for conjectures. In the second half of the paper we use this integral representation as a starting point to solve the special case of two open problems on bounded circular domains in the setting of CRDs.

The first problem we treat has its origin in a work of Panou [10] where it is shown that every inner derivation of the Jordan triple associated with the symmetric part of a finite-dimensional bounded circular domain admits a unique extension to an inner derivation of the partial Jordan triple associated with the whole domain. Though it is natural to expect that the analog holds in general Banach spaces, the only known infinite-dimensional results concern domains with nearly atomic symmetric part [15]. On the basis of Theorem 3.5 along with the fine structure description of the Jordan triple product associated with a symmetric CRD [7], in Section 4 we can establish immediately that the partial Jordan triple of a CRD has the unique extension property for inner derivations.

The second question we solve for CRDs is the above mentioned open problem of the Jordan structure of the second dual of a partial JB^* -triple. First, in Section 5 we refine Theorem 3.5 to a natural extension of the results for symmetric CRDs given in [7] whose proofs relied upon some bidual considerations in [17]. By proceeding the opposite way, in Section 6 we apply the fine structure description obtained in Section 5 along with function representations of $C_0(\Omega)''$ spaces to establish that the Jordan triple product associated with a CRD admits a separately weak*-continuous bidual extension which can be regarded as the canonical Jordan triple of some not necessarily unique CRD.

In the course of Section 5 we apply a Riesz type representation theorem for positive multilinear functionals on products of C_0 -spaces which seems to have never been stated explicitly in the literature. Actually the result we need is contained implicitly in a recent work of Villanueva [20]. We close the paper with an Appendix including a short direct proof.

2. Preliminaries on partial JB^* -triples. Recall [14] that given a complex Banach space E (with norm $\|\cdot\|$), the tuple $(E, E_0, \{\dots\})$ is called a *partial Jordan *-triple* if E_0 is a closed complex subspace of E , and $\{\dots\}$ is a continuous operation $E \times E_0 \times E \rightarrow E$ with the following properties:

(J1) $\{xay\}$ is symmetric bilinear in the variables x, y ($\in E$), conjugate linear in a ($\in E_0$) and $\{E_0E_0E_0\} \subset E_0$;

(J2) the *Jordan identity* holds, i.e. for all $a, b, c \in E_0$ and $x, y \in V$,

$$\{ab\{xyc\}\} = \{\{abx\}cy\} - \{x\{bac\}y\} + \{xc\{abz\}\};$$

(J3) we have the weak associativity

$$\{\{xax\}bx\} = \{xa\{xbx\}\}, \quad a, b \in E_0, x \in E.$$

Notice that in the case of *full* Jordan *-triples, i.e. if $E_0 = E$, axiom (J3) is a consequence of (J2) (see e.g. [6, Ch. 10]). The geometric importance of partial Jordan *-triples relies upon the fact established first implicitly in

[4, 9, 6] that given any bounded circular domain D in a Banach space E , there is a necessarily unique partial Jordan $*$ -triple $(E, E_D, \{\dots\}_D)$ called the *canonical partial Jordan $*$ -triple of D* such that $D \cap E_D$ consists of the centers of holomorphic symmetries of D and $\lim_{t \downarrow 0} S_{ta} S_0(x) = a - \{xax\}_D$ for all $a \in E_D$ and $x \in E$ where S_c denotes the holomorphic symmetry of D with center $c \in D \cap E_D$. We say that $(E, E_0, \{\dots\})$ is a *partial JB^* -triple* if it is a subtriple of some canonical partial Jordan $*$ -triple $(E, E_D, \{\dots\}_D)$. In other words, this means that all the vector fields $[a - \{xax\}]\partial/\partial x$, $a \in E_0$, are complete in a suitable bounded circular domain $D (\subset E)$. This terminology is in accordance with the customary use of the term JB^* -triple for full Jordan triples. Indeed, by Kaup's Riemann mapping theorem [8], in the case $E = E_D$ the domain D is necessarily convex and hence the carrier space E can be renormed so that D becomes the unit ball and the usual C^* and hermitian positivity axioms are satisfied. By the results of [13, 14], we have a complete axiomatic description of partial JB^* -triples. A partial Jordan $*$ -triple $(E, E_0, \{\dots\})$ is a partial JB^* -triple if and only if

(J4) the operators $L(a) : x \mapsto \{aax\}$, $a \in E_0$, have spectrum ≥ 0 with

$$\inf_{\|a\|=1} \|L(a)a\| \neq 0;$$

(J5) $L(a) \in \text{Her}(B)$, $a \in E_0$, for some bounded circular domain B .

It is well-known that the domain B in (J5) can be chosen to be convex and such that its gauge function $\|\cdot\|_B$ should satisfy the C^* -axiom

(J4') $\text{Sp}(L(a)) \geq 0$ with $\|L(a)a\|_B = \|a\|_B^3$ for all $a \in E_0$.

Given any bounded circular domain B satisfying (J5), there exists $\varepsilon > 0$ such that for any $\delta \in (0, \varepsilon)$, $(E, E_0, \{\dots\})$ is a subtriple of $(E, E_{D_\delta}, \{\dots\}_{D_\delta})$ with the bounded circular domain

$$(2.1) \quad D_\delta := \bigcup_{a \in E_0} [\exp((a - \{xax\})\partial/\partial x)](\delta B).$$

Our next aim will be to describe the canonical partial JB^* -triples of CRDs. Recall [4] that the group $\text{Aut}(E, E_0, \{\dots\}) := \{L \in \mathcal{L}(E) : LE_0 \subset E_0, L\{xay\} = \{(Lx)(La)(Ly)\} \text{ for } a \in E_0, x, y \in E\}$ of all automorphisms of the triple $\mathbf{E} = (E, E_0, \{\dots\})$ coincides with the set of all injective linear transformations $L : E \rightarrow E$ such that $LD = D$ whenever \mathbf{E} is the canonical JB^* -triple of a bounded circular domain D . In particular, if $D \subset C_0(\Omega)$ is a CRD, all multiplications with continuous functions of absolute value one belong to $\text{Aut}(E, E_D, \{\dots\}_D)$. So first we consider the effect of linear automorphisms on the construction (2.1).

2.2. LEMMA. Let $(E, E_0, \{\dots\})$ be a partial JB^* -triple and Ψ a bounded subgroup of $\text{Aut}(E, E_0, \{\dots\})$. Then there exists a Ψ -invariant bounded circular domain $D \subset E$ such that $(E, E_0, \{\dots\})$ is a subtriple of $(E, E_D, \{\dots\}_D)$.

Proof. Choose a bounded circular domain B in E satisfying axiom (J5). Define $B_1 := \bigcup_{\psi \in \Psi} \psi B$. Since Φ is a bounded group of linear mappings, B_1 is a bounded Ψ -invariant circular domain in E . Given any $a \in E_0$ and $\psi \in \Psi$, since $\psi \in \text{Aut}(E, E_0, \{\dots\})$, we have $L(\psi^{-1}a) = \psi^{-1}L(a)\psi$. Since $\psi^{-1}a \in E_0$, by axiom (J5) it follows that $\exp(itL(\psi^{-1}a))\psi B = \psi B$, $t \in \mathbb{R}$. Since $\psi^{-1}a$ can be any element in E_0 , we also get $\exp(itL(a))\psi B = \psi B$ for all $a \in E_0$, $t \in \mathbb{R}$ and $\psi \in \Psi$. That is, $L(a) \in \text{Her}(\psi B)$, $\psi \in \Psi$, and hence $L(a) \in \text{Her}(\bigcup_{\psi \in \Psi} \psi B) = \text{Her}(B_1)$ for all $a \in E_0$. Thus the domain B_1 satisfies axiom (J5) and we can use it in the construction (2.1) instead of B with some $\delta > 0$. Given any $\psi \in \Psi$, it only remains to prove that

$$\psi \bigcup_{a \in E_0} [\exp((a - \{axx\})\partial/\partial x)](\delta B_1) = \bigcup_{a \in E_0} [\exp((a - \{axx\})\partial/\partial x)](\delta B_1).$$

However, this is again a direct consequence of the facts $\psi(\delta B_1) = \delta B_1$ and

$$\psi \exp((a - \{axx\})\partial/\partial x) = [\exp((\psi a - \{x(\psi a)x\})\partial/\partial x)]\psi.$$

The latter identity follows from the relation $\psi \in \text{Aut}(E, E_0, \{\dots\})$. ■

3. Integral formula for the canonical partial triple product for a CRD. Let $(E, E_0, \{\dots\})$ denote a fixed partial JB^* -triple over $E := \mathcal{C}_0(\Omega)$ with a locally compact topological Hausdorff space Ω . Throughout this section we assume the Reinhardt property

$$(R) \quad \Psi \subset \text{Aut}(E, E_0, \{\dots\}) \quad \text{where} \quad \Psi := \{\psi \cdot : \psi \in \mathcal{C}(\Omega), |\psi| = 1\}$$

and $\psi \cdot$ denotes the multiplication operator $\mathcal{C}_0(\Omega) \ni f \mapsto \psi f$. As we mentioned, the canonical partial JB^* -triple $(E, E_D, \{\dots\}_D)$ of any CRD D has property (R). Moreover, from Lemma 2.2, $(E, E_0, \{\dots\})$ can be regarded as a subtriple of the canonical JB^* -triple of some CRD in $\mathcal{C}_0(\Omega)$.

As a first consequence of (R), we have $e^{it\phi}E_0 \subset E_0$ and

$$e^{it\phi}\{xay\} = \{(e^{it\phi}x)(e^{it\phi}a)(e^{it\phi}y)\}, \quad t \in \mathbb{R},$$

for any bounded continuous function $\phi : \Omega \rightarrow \mathbb{R}$ (with $a \in E_0$, $x, y \in E$). Hence derivation with respect to the variable t yields

$$(3.1) \quad \psi E_0 \subset E_0, \quad \psi\{xay\} = \{(\psi x)ay\} - \{x(\overline{\psi}a)y\} + \{xa(\psi y)\}$$

for all bounded continuous functions $\psi : \Omega \rightarrow \mathbb{C}$. In particular E_0 is a closed ideal in $\mathcal{C}_0(\Omega)$ regarded as a commutative C^* -algebra with the pointwise product of functions. Therefore necessarily

$$E_0 = \mathcal{C}_0(\Omega_0) := \{f \in \mathcal{C}_0(\Omega) : f(\Omega \setminus \Omega_0) = 0\}$$

with the open set $\Omega_0 := \{\omega \in \Omega : \exists a \in E_0 \ a(\omega) \neq 0\}$.

3.2. LEMMA. $\{xay\}(\omega) = 0$ whenever $x(\omega) = y(\omega) = 0$.

Proof. By the symmetry (J1), it suffices to prove the statement for $x = y$. Furthermore, by the continuity of the triple product and since continuous functions vanishing at ω ($\in \Omega$) can be uniformly approximated by continuous functions vanishing on some neighborhood of ω , it suffices to see that $\{xax\}(\omega) = 0$ if $x(U) = 0$ for some neighborhood $U \subset \Omega$ of ω .

Assume $\omega \in U$ where U is open in Ω , $x \in E$, and $x(U) = 0$. Choose a compact neighborhood V of ω within U and let $\phi : \Omega \rightarrow [0, 1]$ be a continuous function such that $\phi(\omega) = 1$ and $\phi(\Omega \setminus V) = 0$. Observe that if $c \in E_0$ is a function with $c(V) = 0$ then $\phi x = \phi c = 0$ and, by (5.1),

$$\{xcx\}(\omega) = \phi\{xcx\}(\omega) = 2\{(\phi x)cx\}(\omega) - \{x(\phi c)x\}(\omega) = 0.$$

Consider any $a \in E_0$. Choose a continuous function $\psi : \Omega \rightarrow [0, 1]$ with $\psi(V) = 0$ and $\psi(\Omega \setminus U) = 1$. Since $\psi x = x$, applying the function $c := \psi a$ vanishing on V we get

$$\begin{aligned} 0 &= \psi\{xax\}(\omega) = 2\{(\psi x)ax\}(\omega) - \{x(\psi a)x\}(\omega) \\ &= 2\{xax\} - \{xcx\} = 2\{xax\}. \blacksquare \end{aligned}$$

3.3. COROLLARY. *We have*

$$\{xay\} = \frac{1}{2}x\{z(\bar{y}a)z\} + \frac{1}{2}y\{z(\bar{x}a)z\} \quad \text{if } z \in E \text{ with } xz = x \text{ and } yz = y.$$

Proof. Suppose $xz = x$ and $yz = y$. For any $\omega \in \Omega$ apply Lemma 3.2 to the functions $x_\omega := x - x(\omega)z$ and $y_\omega := y - y(\omega)z$ satisfying $x_\omega(\omega) = y_\omega(\omega) = 0$ to get

$$\begin{aligned} 0 &= \{x_\omega a y_\omega\}(\omega) = \{[x - x(\omega)z]a[y - y(\omega)z]\}(\omega) \\ &= \{xay\}(\omega) + x(\omega)y(\omega)\{zaz\}(\omega) - x(\omega)\{zay\}(\omega) - y(\omega)\{zax\}(\omega). \end{aligned}$$

Thus, everywhere on Ω ,

$$\{xay\} = -xy\{zaz\} + x\{zay\} + y\{zax\}.$$

Observe that, by (3.1) and since $yz = y$, here we have

$$xy\{zaz\} = x[2\{yaz\} - \{z(\bar{y}a)z\}]$$

and similarly $yx\{zaz\} = y[2\{xaz\} - \{z(\bar{x}a)z\}]$. Therefore

$$\begin{aligned} xy\{zaz\} &= x\{zay\} + y\{zax\} - \frac{1}{2}\{z(\bar{x}a)z\} - \frac{1}{2}\{z(\bar{y}a)z\}, \\ \{xay\} &= -xy\{zaz\} + x\{zay\} + y\{zax\} = \frac{1}{2}x\{z(\bar{y}a)z\} + \frac{1}{2}y\{z(\bar{x}a)z\}. \blacksquare \end{aligned}$$

3.4. LEMMA. *The triple product $\{\dots\}$ is positive in the sense that*

$$x, a, y \geq 0 \Rightarrow \{xay\} \geq 0.$$

Proof. Fix $0 \leq x, y \in E$ and $0 \leq a \in E_0$. Since the functions with compact support are dense in \mathcal{C}_0 -spaces, we may assume

$$\text{supp}(x), \text{supp}(y) \text{ compact} \subset \Omega, \quad \text{supp}(a) \text{ compact} \subset \Omega_0.$$

Then we can choose $0 \leq x_0, x_1, y_0, y_1, z \in \mathcal{C}_0(\Omega)$ with compact support such that

$$\begin{aligned} x &= x_0 + x_1, & y &= y_0 + y_1, \\ \text{supp}(x_0), \text{supp}(y_0) &\subset \Omega_0, \\ \text{supp}(x_1) \cap \text{supp}(a) &= \text{supp}(y_1) \cap \text{supp}(a) = \emptyset, \\ \text{supp}(x) \cup \text{supp}(y) \cup \text{supp}(a) &\subset \{\zeta \in \Omega : z(\zeta) = 1\}. \end{aligned}$$

By (J1) we have $\{xay\} = \sum_{k,l=0}^1 \{x_k a y_l\}$. Here $\{x_0 a y_0\} \geq 0$ for the following reasons. The subtriple $(E_0, E_0, \{\dots\} | E_0^3)$ is a JB^* -triple with $\Psi \subset \text{Aut}(E_0, E_0, \{\dots\} | E_0^3)$. Therefore it is necessarily the canonical JB^* -triple of a bounded symmetric continuous Reinhardt domain in $E_0 = \mathcal{C}_0(\Omega_0)$. However, by [17, Theorem 2] the triple product is non-negative for non-negative functions for symmetric CRDs. Thus indeed $\{x_0 a y_0\} \geq 0$ since $a, x_0, y_0 \in E_0$. On the other hand, by Corollary 3.3,

$$\{x_1 a y_1\} = \frac{1}{2} x_1 \{z(\bar{y}_1 a)z\} + \frac{1}{2} y_1 \{z(\bar{x}_1 a)z\} = 0$$

because $\bar{x}_1 a = \bar{y}_1 a = 0$. It only remains to see that $\{x_0 a y_1\} \geq 0$ (since the proof of $\{x_1 a y_0\} = \{y_0 a x_1\} \geq 0$ is analogous). Define

$$c := \sqrt{x_0 a}.$$

Since $\text{supp}(c) \subset \text{supp}(a) \subset \Omega_0$, we have $c \in E_0$ and $cz = c$. By Corollary 3.3 (applied with y_1 instead of y and first with c instead both of a and x and then with x_0 instead of x),

$$\begin{aligned} \{c c y_1\} &= \frac{1}{2} c \{z(y_1 c)z\} + \frac{1}{2} y_1 \{z(c^2)z\} = \frac{1}{2} y_1 \{z(c^2)z\}, \\ \{x_0 a y_1\} &= \frac{1}{2} x_0 \{z(y_1 a)z\} + \frac{1}{2} y_1 \{z(x_0 a)z\} = \frac{1}{2} y_1 \{z(x_0 a)z\} \end{aligned}$$

because $y_1 a = y_1 c = 0$. That is,

$$\{x_0 a y_1\} = \{c c y_1\} = \frac{1}{2} y_1 \{z(x_0 a)z\}.$$

According to (J4), $\text{Sp}(L(c)) \geq 0$. However, it is a basic fact about the spectra of multipliers in commutative Banach algebras that

$$u(X) \subset \text{Sp}[\mathcal{C}_0(X) \ni f \mapsto uf] \quad \text{if } X \text{ open} \subset \Omega \text{ and } u \in \mathcal{C}_0(\Omega).$$

In particular, by taking $X := \Omega \setminus \text{supp}(a)$ and $u := \{z(x_0 a)z\}$ we have

$$\{z(x_0 a)z\}(\Omega \setminus \text{supp}(a)) \subset \text{Sp } L(c) \subset [0, \infty).$$

Thus $\{z(x_0 a)z\} \geq 0$ on $\text{supp}(y_1)$ and hence $2\{x_0 a y_1\} = y_1 \{z(x_0 a)z\} \geq 0$. ■

We can summarize the results of this section in the following theorem.

3.5. THEOREM. Let Ω be a locally compact space, $E := C_0(\Omega)$ and suppose $(E, E_0, \{.. \})$ is a partial JB^* -triple with the Reinhardt property (R). Then there exists an open subset Ω_0 in Ω such that $E_0 = \{f \in E : f(\Omega \setminus \Omega_0) = 0\}$. Given any $\omega \in \Omega$, there is a (unique) positive Radon measure μ_ω on Ω_0 with total mass $\leq M := \sup_{0 \leq x, a, y \leq 1} \max\{xay\}$ and

$$(3.6) \quad \{xay\}(\omega) = \frac{1}{2}x(\omega) \int \bar{a}y \, d\mu_\omega + \frac{1}{2}y(\omega) \int \bar{a}x \, d\mu_\omega, \quad x, y \in E, a \in E_0.$$

Proof. We have already established the relation $E_0 = C_0(\Omega_0)$ and the positivity of the triple product in the sense of Lemma 3.4. Fix $\omega \in \Omega$. According to [20] ⁽¹⁾, the positivity of the bounded 3-linear functional $(x, a, y) \mapsto \{x\bar{a}y\}(\omega)$ implies the existence of a positive Radon measure ν_ω of finite total variation on $\Omega \times \Omega_0 \times \Omega$ such that

$$\{x\bar{a}y\}(\omega) = \int x \otimes a \otimes y \, d\nu_\omega, \quad x, y \in E, a \in E_0,$$

where $x \otimes a \otimes y$ denotes the function $(\xi, \alpha, \eta) \mapsto x(\xi)a(\alpha)y(\eta)$ on $\Omega \times \Omega_0 \times \Omega$. It is well-known that $\nu_\omega(\Omega \times \Omega_0 \times \Omega) = \sup\{\int x \otimes a \otimes y \, d\nu_\omega : a \in C_0(\Omega_0), x, y \in C_0(\Omega), 0 \leq a, x \leq 1\} = M$. By the inner compact regularity of Radon measures, given any functions $x, y \in E$ with compact support and $a \in E_0$, we can choose an increasing sequence $K_1 \subset K_2 \subset \dots \subset \Omega$ of compact sets such that $\text{supp}(x) \cup \text{supp}(y) \subset K_1$ and $\lim_{n \rightarrow \infty} \nu(\Omega \setminus K_n) = 0$. Also we can choose a sequence $(z_n) \subset C_0(\Omega) = E$ such that $0 \leq z_1 \leq z_2 \leq \dots \leq 1$ and $z_n(K_n) = 1$ ($n = 1, 2, \dots$). Then, by Corollary 3.3, we have

$$\begin{aligned} \{xay\}(\omega) &= \frac{1}{2}x(\omega)\{z_n(\bar{y}a)z_n\} + \frac{1}{2}y(\omega)\{z_n(\bar{x}a)z_n\} \\ &= \frac{1}{2}x(\omega) \int z_n \otimes (\bar{a}y) \otimes z_n \, d\nu_\omega + \frac{1}{2}y(\omega) \int z_n \otimes (\bar{a}x) \otimes z_n \, d\nu_\omega \\ &\rightarrow \frac{1}{2}x(\omega) \int 1_\Omega \otimes (\bar{a}y) \otimes 1_\Omega \, d\nu_\omega + \frac{1}{2}y(\omega) \int 1_\Omega \otimes (\bar{a}x) \otimes 1_\Omega \, d\nu_\omega \end{aligned}$$

where 1_Ω denotes the function identically 1 on Ω . Thus with the measure

$$\mu_\omega(X) := \nu_\omega(\Omega \times X \times \Omega) \quad (X \text{ Borel } \subset \Omega)$$

we have the stated relation for $x, y \in E$ with compact support. The statement follows by the uniform density in E of functions with compact support. ■

3.7. REMARK. It would be tempting to conjecture that every partial JB^* -triple satisfying the hypothesis of Theorem 3.5 is the canonical JB^* -triple of some CRD. However, there is a counterexample even in two dimensions.

Let $\Omega := \{1, 2\}$, $\Omega_0 := \{1\}$ and

$$\{xay\} := [1 \mapsto x(1)\overline{a(1)}y(1), 2 \mapsto x(1)\overline{a(1)}y(2)/2 + x(2)\overline{a(1)}y(1)/2].$$

⁽¹⁾ This fact is implicit in [20]. For the sake of completeness, we include a short direct proof in the Appendix.

Then any CRD D over Ω such that the vector fields $[a - \{xax\}]\partial/\partial x$ are complete in D must be an ellipsoid of the form $D = \{x : |x(1)|^2 + \lambda|x(1)|^2 < 1\}$ for some $\lambda > 0$ with $E_D = E \neq E_0$.

4. Extension from the symmetric part. Next we are going to study partial Jordan $*$ -triples with triple product of the form obtained in Theorem 3.5 for canonical JB^* -triples of CRDs. With the aid of bidual embedding, a technique not yet available for general partial JB^* -triples, in [17] we have achieved a finer analog of Theorem 3.5 in the special case of symmetric CRDs. Applying [17, Theorem 2] to the restriction of the triple product to the symmetric part E_0 in Theorem 3.5, we see that the measures μ_ω , $\omega \in \Omega_0$, have finite support. Moreover, there exists a partition $\{\Omega_i : i \in I\}$ of Ω_0 consisting of finite sets along with a function $m : \Omega_0 \rightarrow \mathbb{R}$ such that, by writing $i(\omega)$ for the unique $i \in I$ with $\omega \in \Omega_i$, we have

$$(4.1) \quad \mu_\omega = \sum_{\eta \in \Omega_{i(\omega)}} m(\eta)\delta_\eta, \quad \omega \in \Omega_0, \quad 0 < \inf m \leq \sup_{i \in I} \sum_{\eta \in \Omega_i} m(\eta) < \infty.$$

4.2. THEOREM. *Let $(E, E_0, \{\dots\})$ be a partial Jordan $*$ -triple where $E := C_0(\Omega)$ with a locally compact topological space Ω , $E_0 := \{f \in E : f(\Omega \setminus \Omega_0) = 0\}$ with a non-empty open subset $\Omega_0 \subset \Omega$, and let the triple product $\{\dots\}$ have the form (3.6). If the measures μ_ω are all positive and (4.1) holds then $(E, E_0, \{\dots\})$ is a subtriple of the canonical partial JB^* -triple of some CRD.*

Proof. By assumption, the triple product $\{\dots\}$ satisfies axioms (J1)–(J3) in Section 2. Thus to see that $(E, E_0, \{\dots\})$ is a partial JB^* -triple, we have to verify axioms (J4) and (J5). According to [17, Theorem 2], the restriction of $\{\dots\}$ to the symmetric part $E_0 \times E_0 \times E_0$ is the canonical JB^* -triple product of the symmetric CRD $D_0 := \{f \in E_0 : \sup_{i \in I} \sum_{\eta \in \Omega_i} m(\eta)|f(\eta)|^2 < 1\}$ in E_0 . Hence $\inf_{\|a\|_\infty=1} \|L(a)a\|_\infty > 0$. Consider the set

$$B := \left\{ f \in E : [\inf m] \max |f|^2 < 1, \sup_{i \in I} \sum_{\eta \in \Omega_i} m(\eta)|f(\eta)|^2 < 1 \right\}.$$

This is clearly a bounded convex CRD in E . We have

$$B = \bigcap_{\omega \in \Omega \setminus \Omega_0} B_\omega \cap \bigcap_{i \in I} B^{(i)}$$

with the (unbounded) CRDs $B_\omega := \{f \in E : |f(\omega)| < 1/\inf m\}$ and $B^{(i)} := \{f \in E : \langle [f|\Omega_i] | [f|\Omega_i] \rangle \leq 1\}$ where $\langle \cdot | \cdot \rangle_i$ denotes the scalar product $\langle \varphi | \psi \rangle_i := \sum_{\eta \in \Omega_i} m(\eta)\varphi(\eta)\overline{\psi(\eta)}$. Given any function $a \in E_0$, we have

$$L(a)x(\omega) = \{aax\}(\omega) = \frac{1}{2}x(\omega) \int_{\eta \in \Omega_0} |a(\eta)|^2 d\mu_\omega(\eta), \quad \omega \in \Omega \setminus \Omega_0;$$

$$L(a)x|\Omega_i = \frac{1}{2}\langle [a|\Omega_i] | [a|\Omega_i] \rangle_i [x|\Omega_i] + \frac{1}{2}\langle [x|\Omega_i] | [a|\Omega_i] \rangle_i [a|\Omega_i], \quad i \in I.$$

For any fixed $\omega \in \Omega \setminus \Omega_0$ and $\tau \in \mathbb{C}$,

$$\exp(i\tau L(a))x(\omega) = e^{i\tau \int |a|^2 d\mu_\omega} x(\omega).$$

As a consequence, $\exp(i\tau L(a))B_\omega \subset B_\omega$ whenever $\text{Im } \tau \geq 0$. Similarly, $\exp(i\tau L(a))B^{(i)} \subset B^{(i)}$ for $\text{Im } \tau \geq 0$ and $i \in I$ because any mapping $\varphi \mapsto \frac{1}{2}\langle \alpha | \alpha \rangle_i \varphi + \frac{1}{2}\langle \varphi | \alpha \rangle_i \alpha$ is a positive linear operator with respect to the scalar product $\langle \cdot | \cdot \rangle_i$. Therefore $\exp(i\tau L(a))B \subset B$ if $\text{Im } \tau \geq 0$. Hence axioms (J4) and (J5) are immediate. Thus $(E, E_0, \{ \dots \})$ is a partial JB^* -triple.

Due to the form (3.6) of the triple product, the group Ψ of multiplications with functions of modulus 1 consists of automorphisms of $(E, E_0, \{ \dots \})$. In view of Lemma 2.2 we conclude that $(E, E_0, \{ \dots \})$ is a subtriple of the canonical triple of some bounded domain which is invariant under multiplications with continuous functions of modulus 1, that is, a CRD. ■

We can also apply the structural descriptions of Section 3 together with Theorem 4.2 to test if all inner derivations of the canonical partial JB^* -triple of a CRD can be extended in a uniformly continuous manner from the symmetric part to the whole space. As we shall see, this category does not give any counterexample.

4.3. THEOREM. *On the space $E := \mathcal{C}_0(\Omega)$, let $(E, E_0, \{ \dots \})$ be a partial JB^* -triple such that $\{ \psi \cdot : \psi \in \mathcal{C}(\Omega), |\psi| = 1 \} \subset \text{Aut}(E, E_0, \{ \dots \})$. Then there exists a finite constant M such that $\|\Delta\| \leq M\|\Delta|E_0\|$ for all inner derivations Δ of $(E, E_0, \{ \dots \})$.*

Proof. We know there exists an open subset Ω_0 of Ω with $E_0 = \{ f \in \mathcal{C}_0(\Omega_0) : f(\Omega \setminus \Omega_0) = 0 \}$. Furthermore, for any $\omega \in \Omega$, there is a positive Radon measure μ_ω on Ω_0 such that

$$\{xax\}(\omega) = x(\omega) \int_{\Omega_0} x\bar{a} d\mu_\omega, \quad x \in E, a \in E_0, \omega \in \Omega.$$

Since $\mathcal{C}_0(\Omega_0) = \{x\bar{a}|_{\Omega_0} : x \in E, a \in E_0\}$, the function $\omega \mapsto \int_{\Omega_0} f d\mu_\omega$ is necessarily continuous for all fixed $f \in \mathcal{C}_0(\Omega_0)$. Finally, we may assume the measures $\mu_\omega, \omega \in \Omega_0$, to be of the form (4.1). Thus, by writing $S(\omega) := \Omega_{i(\omega)}$ for short, we have

$$\int_{\Omega_0} f d\mu_\omega = \sum_{\eta \in S(\omega)} m(\eta)f(\eta), \quad \omega \in \Omega_0, f \in \mathcal{C}_0(\Omega_0).$$

Notice also that $\omega \in S(\omega)$ for all $\omega \in \Omega_0$ and $0 < \inf m \leq \sup m < \infty$ and $\sup_{\omega \in \Omega_0} \#S(\omega) < \infty$. Consider an inner derivation Δ of $(E, E_0, \{ \dots \})$.

That is,

$$\Delta x = \sum_{k=1}^N \{a_k b_k x\}, \quad x \in E,$$

for some finite sequence $a_1, b_1, \dots, a_N, b_N \in E_0$. In particular, given any function $x \in E = \mathcal{C}_0(\Omega)$,

$$2\Delta x(\omega) = \sum_{\eta \in S(\omega)} m(\eta) \sum_{k=1}^N [a_k(\eta) \overline{b_k(\eta)} x(\omega) + x(\eta) \overline{b_k(\eta)} a_k(\omega)] \quad \text{for } \omega \in \Omega_0,$$

$$\Delta x(\omega) = \int_{\Omega_0} \sum_{k=1}^N a_k(\zeta) \overline{b_k(\zeta)} d\mu_\omega(\zeta) x(\omega) \quad \text{for } \omega \in \Omega \setminus \Omega_0.$$

The continuity $\|\{xay\}\| \leq K\|x\| \|a\| \|y\|$ of the partial triple product implies that $\sup_{\omega \in \Omega} \mu_\omega(\Omega_0) \leq K < \infty$. Hence it suffices to see that

$$(4.4) \quad \sup_{\zeta \in \Omega_0} \left| \sum_{k=1}^N a_k(\zeta) \overline{b_k(\zeta)} \right| \leq \frac{4\|\Delta|E_0\|}{\inf m}.$$

To prove this inequality, fix $\zeta \in \Omega_0$. Since the set $S(\zeta)$ is finite, for each $\omega \in S(\zeta)$ we can find $e_\omega \in E_0 \equiv \mathcal{C}_0(\Omega_0)$ such that $1 = e_\omega(\zeta) = \sup |e_\omega(\cdot)|$ but $e_\omega(\eta) = 0$ for $\zeta \neq \eta \in S(\zeta)$. Then

$$\begin{aligned} 2\Delta e_\omega &= \left[\sum_{k=1}^N \sum_{\eta \in S(\zeta)} m(\eta) a_k(\eta) \overline{b_k(\eta)} \right] e_\omega + \sum_{k=1}^N m(\omega) \overline{b_k(\omega)} a_k, \\ 2 \sum_{\omega \in S(\zeta)} [\Delta e_\omega](\omega) &= \sum_{k=1}^N \#S(\zeta) \sum_{\omega \in S(\zeta)} m(\omega) a_k(\omega) \overline{b_k(\omega)} \\ &\quad + \sum_{k=1}^N \sum_{\omega \in S(\zeta)} m(\omega) a_k(\omega) \overline{b_k(\omega)} \\ &= [\#S(\zeta) + 1] \sum_{\omega \in S(\zeta)} \sum_{k=1}^N m(\omega) a_k(\omega) \overline{b_k(\omega)}. \end{aligned}$$

It follows that

$$\begin{aligned} m(\zeta) \sum_{k=1}^N \overline{b_k(\zeta)} a_k(\zeta) &= 2[\Delta e_\zeta](\zeta) - \sum_{\omega \in S(\zeta)} m(\omega) \sum_{k=1}^N a_k(\omega) \overline{b_k(\omega)}, \\ \sum_{\omega \in S(\zeta)} m(\omega) \sum_{k=1}^N a_k(\omega) \overline{b_k(\omega)} &= \frac{2}{\#S(\zeta) + 1} \sum_{\omega \in S(\zeta)} [\Delta e_\omega](\omega). \end{aligned}$$

Notice that $\|e_\omega\| = \max |e_\omega(\cdot)| = 1$ and hence $|[\Delta e_\omega](\omega)| \leq \|\Delta|E_0\|$ for all $\omega \in \Omega$. Therefore

$$m(\omega) \left| \sum_{k=1}^N a_k(\omega) \overline{b_k(\omega)} \right| \leq 2\|\Delta|E_0\| + \frac{2}{\#S(\zeta) + 1} \sum_{\omega \in S(\zeta)} \|\Delta|E_0\| \leq 4\|\Delta|E_0\|.$$

This completes the proof of (4.4) and hence the proof of the theorem. ■

5. The fine structure of the canonical partial JB^* -triple of a CRD. Throughout this section, Ω denotes a locally compact Hausdorff space, $\Omega_0 \neq \emptyset$ is a fixed open subset of Ω , and we write $E := \mathcal{C}_0(\Omega)$ and $E_0 := \{f \in E : f(\Omega \setminus \Omega_0) = 0\}$. Also we reserve the notations $[\mu_\omega : \omega \in \Omega]$, $\{\Omega_i : i \in I\}$ and m , respectively, for a given measure-valued map $\Omega \rightarrow \mathcal{M}(\Omega_0)_+$, a partition of Ω_0 into finite non-empty sets, and a function $m : \Omega_0 \rightarrow \mathbb{R}_+$ such that (4.1) holds. We know from Theorems 3.5 and 4.2 that the canonical triple product of a CRD has necessarily the form (3.6) in terms of these objects.

Our purpose will be to find a description in terms of the topological properties of the partition $\{\Omega_i : i \in I\}$ when a triple product of the form (3.6) is the canonical triple product of some CRD. It is clear that there are plenty of mappings $\omega \mapsto \mu_\omega$, even satisfying (4.1), for which the operation (3.6) is not a partial JB^* -triple product. Indeed, the following observation is an immediate but fundamental consequence of Theorem 3.5 and its proof. *Given a bounded Reinhardt domain D in E , the canonical triple product $\{\dots\} := \{\dots\}_D$ has the form*

$$(5.1) \quad \{xay\} = \frac{1}{2}xA(\bar{a}y) + \frac{1}{2}yA(\bar{a}x), \quad a \in E_0, x, y \in E,$$

with some positive linear map $A : E_0 \rightarrow \mathcal{C}_b(\Omega) := \{\text{bounded continuous functions } \Omega \rightarrow \mathbb{C}\}$. It is well-known [11] that the positivity of A entails its boundedness automatically. Notice also that, by the Riesz–Kakutani representation theorem, any positive linear mapping $A : \mathcal{C}_0(\Omega_0) \rightarrow \mathcal{C}_b(\Omega)$ has the form $Af(\omega) = \int_{\Omega_0} f d\mu_\omega$ with a uniquely determined mapping $\Omega \ni \omega \mapsto \mu_\omega \in \mathcal{M}(\Omega_0)_+$.

5.2. LEMMA. *Suppose $A : E_0 \rightarrow \mathcal{C}_b(\Omega)$ is a positive linear mapping. Then the structure $(E, E_0, \{\dots\})$ with the operation (5.1) is a partial Jordan * -triple if and only if*

$$(5.3) \quad \begin{aligned} A(fA(g)) &= A(gA(f)), \\ A(fA(g))|_{\Omega_0} &= A(f)A(g)|_{\Omega_0}, \quad f, g \in E_0. \end{aligned}$$

Proof. Since E_0 and E are closed ideals in $\mathcal{C}_b(\Omega)$ with respect to the pointwise product of functions, the operation (5.1) is a well-defined positive

continuous sesquitrilinear map $E \times E_0 \times E \rightarrow E$. It satisfies the identities

$$\begin{aligned} \{xa\{xbx\}\} &= \frac{1}{2}xA(\bar{a}xA(\bar{b}x)) + \frac{1}{2}xA(\bar{b}x)A(\bar{b}x), \\ \{xa\{xbx\}\} - \{xb\{xax\}\} &= \frac{1}{2}xA[\bar{a}xA(\bar{b}x)] - \bar{b}xA(\bar{a}x). \end{aligned}$$

Hence, by taking $f := \bar{a}x$ and $g := \bar{b}x$, we see that (5.3) implies axiom (J3). Conversely, assume (J3) holds. Then $xA[\bar{a}xA(\bar{b}x)] - \bar{b}xA(\bar{a}x) = 0$ for $a, b \in E_0$ and $x \in E$. Consider any functions $f, g \in E_0$ with compact support. Then, given any point $\omega \in \Omega$, we can choose a function $x_\omega \in E$ with compact support such that the interior of $\text{supp}(x_\omega)$ contains $\{\omega\} \cup \text{supp}(f) \cup \text{supp}(g)$. Then we can write $f = \bar{a}_\omega x_\omega$ and $g = \bar{b}_\omega x_\omega$ with some $a_\omega, b_\omega \in E_0$ and hence (J3) implies $0 = A[fA(g) - gA(f)](\omega)$. Thus, since the functions with compact support are dense in E_0 , axiom (J3) is equivalent to the identity $A(fA(g)) = A(gA(f))$ in (5.3).

Let us now proceed to the axiom (J2) of the Jordan identity. By polarization, (J2) is equivalent to its special case

$$(J2') \quad \{aa\{xbx\}\} = 2\{\{aax\}bx\} - \{x\{aab\}x\}, \quad a, b \in E_0, x \in E.$$

In terms of the operation A , this identity (multiplied by 2) can be stated as

$$\begin{aligned} aA(\bar{a}xA(\bar{b}x)) + xA(\bar{b}x)A(|a|^2) \\ = [aA(\bar{a}x) + xA(|a|^2)]A(\bar{b}x) + xA(\bar{b}[aA(\bar{a}x) + xA(|a|^2)]) \\ - xA([aA(\bar{a}b) + bA(|a|^2)]^- x). \end{aligned}$$

By the positivity of A , we can write the last term above as $-xA([\bar{a}A(a\bar{b}) + \bar{b}A(|a|^2)]x)$. Thus, by the linearity of A , axiom (J2') is equivalent to

$$\begin{aligned} aA(\bar{a}xA(\bar{b}x)) + xA(\bar{b}x)A(|a|^2) \\ = aA(\bar{a}x)A(\bar{b}x) + xA(|a|^2)A(\bar{b}x) + xA(\bar{b}aA(\bar{a}x)) + xA(\bar{b}xA(|a|^2)) \\ - xA(\bar{a}xA(a\bar{b})) - xA(\bar{b}xA(|a|^2)). \end{aligned}$$

Here the terms $xA(\bar{b}x)A(|a|^2)$ and $xA(\bar{b}xA(|a|^2))$ cancel each other, whence we get

$$(J2'') \quad aA(\bar{a}xA(\bar{b}x)) = aA(\bar{a}x)A(\bar{b}x) + xA(\bar{b}aA(\bar{a}x)) - xA(\bar{a}xA(a\bar{b})).$$

Observe that (5.3) implies (J2'') immediately. To finish the proof, assume (J2)+(J3). As we have shown, this is nothing else than the identity $A(fA(g)) = A(gA(f))$ along with (J2''). By substituting $f := \bar{a}x$ and $g := a\bar{b}$ in (J2''), we see that two terms cancel each other and the remaining identity $aA(\bar{a}xA(\bar{b}x)) = aA(\bar{a}x)A(\bar{b}x)$ is equivalent to its polarized form

$$a_1A(\bar{a}_2xA(\bar{b}x)) = a_1A(\bar{a}_2x)A(\bar{b}x), \quad a_1, a_2, b \in E_0, x \in E.$$

Since each $a_1 \in E_0$ vanishes outside Ω_0 , but for any $\omega \in \Omega_0$ there is a func-

tion $a_{1,\omega} \in E_0$ with $a_{1,\omega}(\omega) \neq 0$, the polarized identity is further equivalent to

$$A(\bar{a}xA(\bar{b}x)) \mid \Omega_0 = A(\bar{a}x)A(\bar{b}x) \mid \Omega_0, \quad a, b \in E_0, x \in E.$$

As we have seen, any functions $f, g \in E_0$ with compact support can be written in the form $f = \bar{a}x, g = \bar{b}x$ for suitable $a, b \in E_0$ and $x \in E$ with compact support. This implies the second identity in (5.3) for functions with compact support, and the statement follows by a standard density argument. ■

5.4. REMARK. An application of the results in [7] concerning symmetric CRDs to the symmetric part of the canonical partial JB^* -triple of a CRD yields the following observation. If $(E, E_0, \{ \dots \})$ is a partial JB^* -triple with a triple product of the form (3.6) and having property (4.1), then the set-valued function $\omega \mapsto \Omega_{i(\omega)} \cup \{ \infty \}$ (where $i(\omega)$ denotes the unique $i \in I$ with $\omega \in \Omega_i$) is continuous with respect to the Hausdorff topology on the set of non-empty compact subsets of $\Omega_0 \cup \{ \infty \}$. As a consequence, *given a relatively closed subset F of Ω_0 and a point $\omega \in F$ such that $\#[F \cap \Omega_{i(\omega)}] = N_F := \max_{\eta \in F} \#[F \cap \Omega_{i(\eta)}]$, there are disjoint open sets $U_1, \dots, U_{N_F} \subset \Omega_0$ such that $\omega \in U_1$ and $\#[U_k \cap F \cap \Omega_{i(\eta)}] = 1$ for any $\eta \in U_1 \cup \dots \cup U_{N_F}$ and $k = 1, \dots, N_F$.*

5.5. LEMMA. *Assume the mapping $\omega \mapsto \Omega_{i(\omega)} \cup \{ \infty \}$ is Hausdorff continuous in the sense of 5.4. Then given any $\omega \in \Omega$, there exists a finite family of disjoint Borel subsets $G_1, \dots, G_N \subset \Omega_0$ such that $\mu_\omega(\Omega_0 \setminus \bigcup_{k=1}^N G_k) = 0$ and $\#[\Omega_i \cap G_k] \leq 1$ for all $i \in I$ and $k = 1, \dots, N$.*

Proof. We use countable transfinite exhaustion to construct the sets G_1, \dots, G_N . To start, let $N := N_{\Omega_0}$, $F^{(0)} := \Omega_0$ and $U_1^{(0)}, \dots, U_N^{(0)} := \emptyset$. For any countable ordinal $r \succ 0$, if until each set $U_k^{(s)}$ with $s \prec r$ and $1 \leq k \leq N$ is open and $\mu_\omega(\bigcup_{s \prec r} \bigcup_{k=1}^N U_k^{(s)}) < \mu_\omega(\Omega_0)$, define $F^{(r)} := \Omega_0 \setminus \bigcup_{s \prec r} \bigcup_{k=1}^N U_k^{(s)}$, $N_r := \max_{\eta \in F^{(r)}} \#[F^{(r)} \cap \Omega_{i(\eta)}]$. We also choose $\omega_r \in F^{(r)}$ with $\#[F^{(r)} \cap \Omega_{i(\omega_r)}] = N_r$ along with a finite disjoint family $U_1^{(r)}, \dots, U_{N_r}^{(r)}$ such that $\mu_\omega(U_1^{(r)}) > 0$ and $\#[U_k^{(r)} \cap \Omega_{i(\eta)}] = 1$ for all $\eta \in U_1^{(r)} \cup \dots \cup U_{N_r}^{(r)}$ and $k = 1, \dots, N_r$. Finally, we set $U_k^r := \emptyset$ for $N_r < k \leq N$. This can be done in view of Remark 5.4 and the fact that trivially $N_r \leq N$. Since the measure μ_ω is finite, in this manner, for some countable ordinal r^* , we get a family $\{U_k^{(r)} : r \prec r^*, k = 1, \dots, N\}$ of open subsets of Ω_0 such that $\mu_\omega(\Omega_0 \setminus \bigcup_{s \prec r^*} \bigcup_{k=1}^N U_k^{(s)}) = 0$ and $\#[U_k^{(r)} \cap F^{(r)} \cap \Omega_{i(\eta)}] \leq 1, 1 \leq k \leq N$, but $\Omega_{i(\eta)} \subset \bigcup_{k=1}^N [U_k^{(r)} \cap F^{(r)}]$ for all $\eta \in \bigcup_{k=1}^N [U_k^{(r)} \cap F^{(r)}]$ for all ordinals $r \prec r^*$. Therefore the choice $G_k := \bigcup_{s \prec r^*} [U_k^{(s)} \cap F^{(s)}], k = 1, \dots, N$, suits our requirements. ■

5.6. COROLLARY. Let $\mathcal{K} := \{K \subset I : \bigcup_{i \in K} \Omega_i \text{ is Borel measurable}\}$ and define $\tilde{\mu}_\omega(K) := \mu_\omega(\bigcup_{i \in K} \Omega_i)$, $K \in \mathcal{K}$. Then (under the hypothesis of Lemma 5.5) there is a Borel function $p_\omega : \Omega_0 \rightarrow [0, 1]$ such that $\sum_{\eta \in \Omega_i} p_\omega(\eta) = 1$, $i \in I$, and for all bounded Borel functions $f : \Omega_0 \rightarrow \mathbb{C}$ we have

$$\int_{\Omega_0} f d\mu_\omega = \int \sum_{i \in I} f(\eta) p_\omega(\eta) d\tilde{\mu}_\omega(i).$$

Proof. As already noted, the sets

$$G_k^{(r)} := U_k^{(r)} \cap F^{(r)} = U_k^{(r)} \setminus \bigcup_{s < r} \bigcup_{k=1}^{N_r} U_k^{(s)}, \quad r < r^*, 1 \leq k \leq N_r,$$

form a disjoint covering of Ω_0 up to a set of μ_ω -measure 0. Let $\tilde{p}_k^{(r)}$ denote the Radon–Nikodým derivative $d\tilde{\mu}_{\omega,k}^{(r)}/d\tilde{\mu}_\omega$ with the measure $\tilde{\mu}_{\omega,k}^{(r)}(K) := \mu_\omega(G_k^{(r)} \cap \bigcup_{i \in K} \Omega_i)$, $K \in \mathcal{K}$. These are functions $I \rightarrow \mathbb{R}$ defined up to a set of $\tilde{\mu}_\omega$ -measure 0, and we can choose Borel measurable representatives with $0 \leq \tilde{p}_k^{(r)} \leq 1$ and $\sum_{k=1}^{N_r} \tilde{p}_k^{(r)} = 1$ on $I^{(r)} := \{i \in I : \Omega_i \subset \bigcup_{k=1}^{N_r} G_k^{(r)}\}$ and vanishing outside $I^{(r)}$. This can be done because every Ω_i meets any set $G_k^{(r)}$ in at most one point and for the sets $G^{(r)} := \bigcup_{k=1}^{N_r} G_k^{(r)}$ we have either $\Omega_i \subset G^{(r)}$ or $\Omega_i \cap G^{(r)} = \emptyset$. Hence the statement holds with the function $p(\eta) := \sum_{r < r^*} \sum_{k=1}^{N_r} \tilde{p}_k^{(r)}(i(\eta))$, $\eta \in \Omega_0$. ■

5.7. COROLLARY. Suppose we have (4.1) with a weight function $m > 0$ and let $A : \mathcal{C}_0(\Omega_0) \rightarrow \mathcal{C}_b(\Omega)$ have the form $Af(\omega) = \int_{\eta \in \Omega_0} f d\mu_\omega$ with suitable Radon measures μ_ω , $\omega \in \Omega$. Then the identity $A(fA(g)) = A(gA(f))$ is equivalent to the fact that

$$(5.8) \quad \mu_\omega(X) = \int \sum_{i \in I} m(\eta) d\kappa_\omega(i), \quad X \subset \Omega_0,$$

with suitable measures $\kappa_\omega : \{K \subset I : \bigcup_{i \in K} \Omega_i \text{ is Borel measurable}\} \rightarrow \mathbb{R}_+$, $\omega \in \Omega$.

Proof. Using Corollary 5.6, we can write

$$\begin{aligned} [A(fA(g))](\omega) &= \int \sum_{i \in I} f(\eta) [A(g)](\eta) p_\omega(\eta) d\tilde{\mu}_\omega(i) \\ &= \int \sum_{i \in I} f(\eta) \sum_{\zeta \in \Omega_{i(\eta)}} g(\zeta) m(\zeta) p_\omega(\eta) d\tilde{\mu}_\omega(i) \\ &= \int \sum_{i \in I} f(\eta) g(\zeta) m(\zeta) p_\omega(\eta) d\tilde{\mu}_\omega(i) \end{aligned}$$

because $i(\eta) = i$ for $\eta \in \Omega_i$. Thus the identity $A(fA(g)) = A(gA(f))$ is

equivalent to

$$(5.9) \quad 0 = \int \sum_{i \in I, \zeta, \eta \in \Omega_i} f(\eta)g(\zeta)[m(\zeta)p_\omega(\eta) - m(\eta)p_\omega(\zeta)] d\tilde{\mu}_\omega(i)$$

for all $f, g \in \mathcal{C}_0(\Omega_0)$ and $\omega \in \Omega$. By passing to limits of monotone sequences, we see that (5.9) holds for all $f, g \in \mathcal{C}_0(\Omega_0)$ if and only if it holds for all bounded Borel measurable functions $f, g : \Omega_0 \rightarrow \mathbb{C}$. Consider (5.9) for the partition $\Omega_0 = \bigcup_{r \prec r^*} \bigcup_{k=1}^{N_r} G_k^{(r)} \cup [\mu_\omega\text{-zero-set}]$ constructed in the proof of Corollary 5.6. By writing $\zeta_{i,k}^{(r)}$ for the unique element of $\Omega_i \cap G_k^{(r)}$, we get

$$(5.9') \quad 0 = \sum_{r \prec r^*} \sum_{k,l=1}^{N_r} \int_{i \in I^{(r)}} f(\zeta_{i,k}^{(r)})g(\zeta_{i,l}^{(r)})[m(\zeta_{i,k}^{(r)})p_\omega(\zeta_{i,l}^{(r)}) - m(\zeta_{i,l}^{(r)})p_\omega(\zeta_{i,k}^{(r)})] d\tilde{\mu}_\omega(i).$$

This holds for all bounded Borel functions $f, g : \Omega_0 \rightarrow \mathbb{C}$ if and only if, given any $r \prec r^*$, for $\tilde{\mu}_\omega$ -almost every $i \in I^{(r)}$ we have

$$m(\zeta_{i,k}^{(r)})p_\omega(\zeta_{i,l}^{(r)}) - m(\zeta_{i,l}^{(r)})p_\omega(\zeta_{i,k}^{(r)}) = 0, \quad 1 \leq k, l \leq N_r.$$

Indeed, if we just consider functions f, g vanishing outside $G_k^{(r)}$ respectively $G_l^{(r)}$ (with fixed $r \prec r^*$ and $1 \leq k, l \leq N_r$), we obtain (5.9') without the summations $\sum_{r \prec r^*}$ and $\sum_{k,l=1}^{N_r}$, whence the statement is immediate. Thus, since $\sum_{\zeta \in \Omega_i} p_\omega(\zeta) = 1$ for $\tilde{\mu}_\omega$ -almost every $i \in I$, (5.9') holds for all bounded Borel functions if and only if

$$p_\omega(\eta) = m(\eta) \left[\sum_{\zeta \in \Omega_i} m(\zeta) \right]^{-1} \quad \text{for } \tilde{\mu}\text{-almost every } i \in I \text{ and } \eta \in \Omega_i.$$

This observation establishes the statement of 5.7 with the measures $\kappa_\omega(K) := \int_{i \in K} [\sum_{\zeta \in \Omega_i} m(\zeta)]^{-1} d\tilde{\mu}_\omega(i)$, $K \in \mathcal{K}$. ■

6. Bidual of the canonical JB^* -triple of a CRD. On the basis of the previous section, first we give an exhaustive parametric description of the canonical JB^* -triples of continuous Reinhardt domains. Also we answer in the affirmative the question if the bidual of the canonical JB^* -triple of a continuous Reinhardt domain can be regarded as the canonical JB^* -triple of a continuous Reinhardt domain in the bidual commutative C^* -algebra.

As in the previous sections, Ω denotes a fixed locally compact Hausdorff space, Ω_0 is a non-empty open subset of Ω , m is a function $\Omega_0 \rightarrow \mathbb{R}$ and $\Pi = \{\Omega_i : i \in I\}$ is a partition of Ω_0 . We shall write Ω_0/Π for the index set I of Π equipped with the topology inherited from the Hausdorff topology of $\tilde{\Omega}_0 := \{\Omega_i \cup \{\infty\} : i \in I\}$. That is, a set $J \subset I$ is open if $\{\Omega_i \cup \{\infty\} : i \in J\}$ is an open subset of $\tilde{\Omega}_0$ with respect to the Hausdorff topology of the compact subsets on $\Omega_0 \cup \{\infty\}$ restricted to $\tilde{\Omega}_0$.

6.1. DEFINITION (cf. [7, 1.1-2]). We say that the couple (m, Π) is *admissible* if $\sup_{i \in I} \#\Omega_i < \infty$, $0 < \inf m \leq \sup m < \infty$ and all the functions $\Omega_0 \ni \omega \mapsto \sum_{\eta \in \Omega_{i(\omega)}} m(\eta) f(\eta)$, $f \in \mathcal{C}_0(\Omega_0)$, are continuous.

According to [7, 1.2], the couple (m, Π) is admissible if and only if the function space $\mathcal{C}_0(\Omega_0)$ endowed with the triple product polarized from $\{xax\}(\omega) := \sum_{\zeta \in \Omega_{i(\omega)}} m(\zeta) x(\zeta) \overline{a(\zeta)} x(\omega)$ (where $i(\omega)$ denotes the (unique) index with $\omega \in \Omega_{i(\omega)}$) is the canonical triple of some symmetric Reinhardt domain in $\mathcal{C}_0(\Omega_0)$. Furthermore, as a consequence of [7, 1.3(iii)], given an admissible couple (m, Π) , the topological space Ω_0/Π is locally compact and Hausdorff.

6.2. LEMMA. *Let (m, Π) be an admissible couple.*

- (1) *A function $\phi : I \rightarrow \mathbb{C}$ belongs to $\mathcal{C}_0(\Omega_0/\Pi)$ if and only if the function $f_\phi := [\omega \mapsto \phi(i(\omega))]$ is bounded, continuous on Ω_0 , constant on the sets Ω_i , $i \in I$, and such that for any $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset \Omega_0$ with $|f_\phi(\Omega_i)| < \varepsilon$ whenever $\Omega_i \cap K_\varepsilon = \emptyset$.*
- (2) *The range of the operator \tilde{A}_0 on $\mathcal{C}_0(\Omega_0)$ defined by*

$$(6.3) \quad \tilde{A}_0 f(i) := \sum_{\zeta \in \Omega_{i(\omega)}} m(\zeta) f(\zeta), \quad i \in I, f \in \mathcal{C}_0(\Omega_0),$$

is a uniformly dense multiplicative ideal in $\mathcal{C}_0(\Omega_0/\Pi)$.

Proof. (1) Let $\phi \in \mathcal{C}_0(\Omega_0/\Pi)$. By construction, f_ϕ is constant on each Ω_i . Also the ranges of ϕ and f_ϕ coincide, thus f_ϕ is necessarily bounded. Consider a convergent net $\omega_j \rightarrow \omega_0$ in Ω_0 . According to [7, 1.2(iv)], we have $\Omega_{i(\omega_j)} \cup \{\infty\} \rightarrow \Omega_{i(\omega_0)} \cup \{\infty\}$ with respect to the Hausdorff topology. Therefore $f_\phi(\omega_j) \rightarrow f_\phi(\omega_0)$, showing that $f_\phi \in \mathcal{C}_b(\Omega_0)$. The stated vanishing property of f_ϕ at infinity is straightforward. Conversely, assume that $\phi : I \rightarrow \mathbb{C}$ is such that $f_\phi \in \mathcal{C}_b(\Omega_0)$ with the behavior at infinity as in (1). Then ϕ vanishes at infinity in the sense of the locally compact inherited Hausdorff topology of Ω_0/Π . We show the continuity of ϕ as follows. Let $[i_j : j \in J]$ be a net in I such that $\Omega_{i_j} \cup \{\infty\} \rightarrow \Omega_{i_0} \cup \{\infty\}$ in the Hausdorff sense. By [7, 1.3(i)] we can find a convergent net $\omega_j \rightarrow \omega_0$ in Ω_0 with $\omega_j \in \Omega_{i_j}$, $j \in J$ and $\omega_0 \in \Omega_{i_0}$. Hence $\phi(i_j) = f_\phi(\Omega_{i_j}) = f_\phi(\omega_j) \rightarrow f_\phi(\omega_0) = f_\phi(\Omega_{i_0}) = \phi(i_0)$.

(2) As already noted, for each $f \in \mathcal{C}_0(\Omega_0)$ the function $A_0 f := [\Omega_0 \ni \omega \mapsto \sum_{\zeta \in \Omega_{i(\omega)}} m(\zeta) f(\zeta)]$ is continuous. Obviously, $A_0 f$ is constant on each Ω_i . Given a net $[i_j : j \in J]$ such that $\Omega_{i_j} \rightarrow \{\infty\}$ in the Hausdorff sense (i.e. for every K compact $\subset \Omega_0$ there exists $j_K \in J$ with $K \cap \Omega_{i_j} = \emptyset$ for $j \geq j_K$), we have $A_0 f(\Omega_{i_j}) \rightarrow 0$ because

$$|A_0 f(\Omega_{i_j})| \leq \sup_{\omega} m(\omega) \max_i \#\Omega_i \max_{\zeta \in V} |f(\zeta)|$$

and $\max_{\zeta \in V} |f(\zeta)| \rightarrow 0$. By (1), $A_0 f = f_\phi$ for some $\phi \in \mathcal{C}_0(\Omega_0/\Pi)$. Thus $\text{ran } \tilde{A}_0 \subset \mathcal{C}_0(\Omega_0/\Pi)$. Observe that, for any $\psi \in \mathcal{C}_0(\Omega_0/\Pi)$ we have $\psi[\tilde{A}_0 f] = \tilde{A}_0(\psi f)$. Thus $\text{ran } \tilde{A}_0$ is an ideal in $\mathcal{C}_0(\Omega_0/\Pi)$. For any $i \in I$, there exists $\phi \in \text{ran } \tilde{A}_0$ with $\phi(i) \neq 0$. Indeed, by choosing any element $\omega \in \Omega_i$, there exists a function $f \in \mathcal{C}_0(\Omega_0)$ with $f(\omega) = 1$ and $f(\zeta) = 0$ for $\zeta \in \Omega_i \setminus \{\omega\}$ and $\tilde{A}_0 f(i) = \sum_{\zeta \in \Omega_i} m(\zeta) f(\zeta) = m(\omega) f(\omega) > 0$. Hence, by the Stone–Weierstrass theorem, the ideal $\text{ran } \tilde{A}_0$ is uniformly dense in $\mathcal{C}_0(\Omega_0/\Pi)$. ■

6.4. DEFINITION. Given an admissible couple (m, Π) and a non-negative measure-valued mapping $\omega \mapsto \kappa_\omega$ from Ω to $\mathcal{M}(\Omega/\Pi)$, write

$$\mathbf{E}(\Omega, \Omega_0, m, \Pi, \kappa)$$

for the structure $(\mathcal{C}_0(\Omega), \{f \in \mathcal{C}_0(\Omega) : f(\Omega \setminus \Omega_0) = 0\}, \{\dots\})$ where the triple product $\{\dots\}$ is the polarized form of

$$(6.5) \quad \{xax\}(\omega) = x(\omega) \int \sum_{i \in I} \sum_{\zeta \in \Omega_i} m(\zeta) \overline{a(\zeta)} x(\zeta) d\kappa_\omega(i),$$

$\omega \in \Omega, a \in E_0, x \in E.$

We say that the tuple $(\Omega, \Omega_0, m, \Pi, \kappa)$ is *admissible* if (m, Π) is an admissible couple and the measure-valued mapping $\omega \mapsto \kappa_\omega$ is weakly continuous ⁽²⁾ and such that $\kappa_\omega = \delta_{i(\omega)}$ whenever $\omega \in \Omega_0$.

6.6. THEOREM. *Let Ω be a locally compact Hausdorff space and $\emptyset \neq \Omega_0 \subset \Omega$ an open subset. By setting $E := \mathcal{C}_0(\Omega)$, $E_0 := \{f \in E : f(\Omega \setminus \Omega_0) = 0\}$ the triple $(E, E_0, \{\dots\})$ is a subtriple of the canonical JB^* -triple of some Reinhardt domain in E if and only if it is of the form $\mathbf{E}(\Omega, \Omega_0, m, \Pi, \kappa)$ with an admissible tuple $(\Omega, \Omega_0, m, \Pi, \kappa)$.*

The canonical JB^ -triple of any Reinhardt domain in $\mathcal{C}_0(\Omega)$ with non-zero symmetric part has the form $\mathbf{E}(\Omega, \Omega_0, m, \Pi, \kappa)$ with a suitable admissible tuple $(\Omega, \Omega_0, m, \Pi, \kappa)$.*

Proof. We already know the following facts from Theorem 3.5 and Corollary 5.7. The canonical JB^* -triple of any Reinhardt domain with non-zero symmetric part in $E := \mathcal{C}_0(\Omega)$ coincides with $\mathbf{E}(\Omega, \Omega_0, m, \Pi, \kappa)$ for some open $\emptyset \neq \Omega_0 \subset \Omega$ and an admissible couple (Π, m) . Moreover any partial Jordan * -triple $(E, E_0, \{\dots\})$ with $E_0 = \{f \in E : f(\Omega \setminus \Omega_0) = 0\}$ for some $\emptyset \neq \Omega_0 \subset \Omega$ and such that all multiplications with continuous functions $\Omega \rightarrow \mathbb{T}$ ($= \{z \in \mathbb{C} : |z| = 1\}$) belong to $\text{Aut}(E, E_0, \{\dots\})$ must have the form $\mathbf{E}(\Omega, \Omega_0, m, \Pi, \kappa)$ with suitable open $\emptyset \neq \Omega_0 \subset \Omega$ and an admissible couple (Π, m) . Finally, by Lemma 2.2 and Corollary 5.7, each $\mathbf{E}(\Omega, \Omega_0, m, \Pi, \kappa)$ is a subtriple of the canonical JB^* -triple of some Reinhardt domain in E if and only if the triple product maps $E \times E_0 \times E$ to E .

⁽²⁾ That is, $\omega \mapsto \int_{i \in I} \phi(i) d\kappa_\omega(i)$ is continuous for every $\phi \in \mathcal{C}_0(\Omega/\pi)$.

Thus it remains to prove that, in a structure of the form $\mathbf{E}(\Omega, \Omega_0, m, \Pi, \kappa)$, the triple product maps $E \times E_0 \times E$ into E (where $E := \mathcal{C}_0(\Omega)$ and $E_0 := \{f \in E : f(\Omega \setminus \Omega_0) = 0\}$) if and only if the mapping $\omega \mapsto \kappa_\omega$ is weakly continuous. The sufficiency of the weak continuity of κ for $\{EE_0E\} \subset E$ is immediate. Conversely, suppose (m, Π) is an admissible couple and the triple product (6.5) is continuous and satisfies $\{EE_0E\} \subset E$. Then, by Corollary 5.7, $(E, E_0, \{\dots\}) = \mathbf{E}(\Omega, \Omega_0, m, \Pi, \kappa)$ is a partial JB^* -triple and, in particular, the operation

$$Af(\omega) = \int_{\eta \in \Omega_0} \sum_{\zeta \in \Omega_{i(\eta)}} m(\zeta)f(\zeta) d\mu_\omega(\eta) = \int \tilde{A}_0 f d\kappa_\omega, \quad \omega \in \Omega, f \in E_0,$$

has values in the space $\mathcal{C}_b(\Omega)$ of all bounded continuous functions $\Omega \rightarrow \mathbb{C}$. Therefore also the operation $T_0g := [\Omega \ni \omega \mapsto \int \psi d\kappa_\omega]$, $\psi \in \text{ran } \tilde{A}_0$, has values in $\mathcal{C}_b(\Omega)$. We know that the measures μ_ω , $\omega \in \Omega$, have total mass bounded by the norm $M := \sup_{a \in E_0, x, y \in E} \|\{xay\}\|$ of the triple product. It follows that $\kappa_\omega(I) \leq M$, $\omega \in \Omega$, and hence T_0 is bounded with norm $\leq M$ (i.e. $\sup_{\omega \in \Omega} |T_0\psi(\omega)| \leq M \sup_{i \in I} |\psi(i)|$, $\psi \in \text{ran } \tilde{A}_0$). Therefore T_0 admits a continuous extension $T : [\text{ran } \tilde{A}_0]^- \rightarrow \mathcal{C}_b(\Omega)$ to the closure of the range of \tilde{A}_0 with $T\phi(\omega) = \int \phi d\kappa_\omega$, $\phi \in [\text{ran } \tilde{A}_0]^-$. By Lemma 6.2, we have $\mathcal{C}_0(\Omega_0/\Pi) = [\text{ran } \tilde{A}_0]^-$. This implies the weak continuity of $\omega \mapsto \kappa_\omega$. ■

Next we proceed to the bidualization of the partial triple $(E, E_0, \{\dots\}) := \mathbf{E}(\Omega, \Omega_0, m, \Pi, \kappa)$. As usual, we shall regard the commutative C^* -algebra $E := \mathcal{C}_0(\Omega)$ with the spectral norm as a weak*-dense subspace of the bidual $\mathbf{E} := E^{**} \equiv \mathcal{C}(\mathbf{\Omega})$ where $\mathbf{\Omega}$ is the hyperstonian compact topological space of all norm-continuous multiplicative functionals with respect to the jointly weak*-continuous extension of the product in E equipped with the weak* topology inherited from E^{***} . That is, we identify any element $\mathbf{a} \in \mathbf{E}$ canonically with the evaluation function $\omega \mapsto \omega(\mathbf{a})$ on $\mathbf{\Omega}$.

6.7. THEOREM. *Let D be a bounded Reinhardt domain in $E := \mathcal{C}_0(\Omega)$. Then there exists a bounded Reinhardt domain \mathbf{D} in $\mathbf{E} := E^{**} \equiv \mathcal{C}(\mathbf{\Omega})$ such that the canonical JB^* -triple $(E, E_D, \{\dots\}_D)$ is a subtriple of $(\mathbf{E}, \mathbf{E}_D, \{\dots\}_D)$ and \mathbf{E}_D is the weak* closure of E in \mathbf{E} and the triple product $\{\dots\}_D$ is the jointly weak*-continuous extension of $\{\dots\}_D$.*

Proof. According to Lemma 5.2, there is a positive and hence norm-continuous mapping $A : E_D \rightarrow F$ satisfying the identities (5.3) such that $2\{xay\}_D = xA(\bar{a}y) + yA(\bar{a}x)$, $a \in E_D$, $x, y \in E$. To study the bidual continuation of A , let us regard the commutative C^* -algebra $F := \mathcal{C}_b(\Omega)$ as a weak*-dense subspace of the bidual $\mathbf{F} := F^{**} \equiv \mathcal{C}(\widehat{\Omega})$ where $\widehat{\Omega}$ is a suitable compact hyperstonian topological space. Since $E = \mathcal{C}_0(\Omega)$ is a closed multiplicative ideal in F and E_D is a closed multiplicative ideal in E , also the

weak* closures $\mathbf{E} = \overline{E}^{w^*}$ and $\mathbf{E}_0 := \overline{E_0}^{w^*}$ are weak*-closed M-ideals in \mathbf{F} . Hence we may assume without loss of generality that

$$\mathbf{E} = \{\mathbf{f} \in \mathbf{F} : \mathbf{f}(\widehat{\Omega} \setminus \Omega) = 0\}, \quad \mathbf{E}_0 = \{\mathbf{f} \in \mathbf{F} : \mathbf{f}(\widehat{\Omega} \setminus \Omega_0) = 0\}$$

for some open-closed subsets $\Omega_0 \subset \Omega \subset \widehat{\Omega}$, and the biadjoint A^{**} maps \mathbf{E}_0 into \mathbf{F} . Consider the operation

$$(6.8) \quad \{\mathbf{xay}\}_{**} := \frac{1}{2}[A^{**}(\mathbf{x}\bar{\mathbf{a}})]\mathbf{y} + \frac{1}{2}[A^{**}(\mathbf{y}\bar{\mathbf{a}})]\mathbf{x}, \quad \mathbf{a} \in \mathbf{E}_0, \mathbf{x}, \mathbf{y} \in \mathbf{E}.$$

Since the biadjoint of any positive linear operator (between Banach lattices) is weak*-continuous and positive, and since the product in \mathbf{F} is separately weak*-continuous, the product (6.8) is a separately weak*-continuous extension of the triple product $\{\dots\}_D$. From (5.3) it also follows that $A^{**}(\mathbf{f}A^{**}(\mathbf{g})) = A^{**}(\mathbf{g}A^{**}(\mathbf{f}))$ and $\mathbf{a}A^{**}(\mathbf{f}A^{**}(\mathbf{g})) = \mathbf{a}A^{**}(\mathbf{f})A^{**}(\mathbf{g})$ for all $\mathbf{a}, \mathbf{f}, \mathbf{g} \in \mathbf{E}_0$. Thus $A^{**} : \mathbf{E}_0 \rightarrow \mathbf{F}$ is a positive linear operator with the property (5.3) and, by Lemma 5.2, the operation $\{\dots\}_{**}$ is a partial Jordan *-triple product.

To complete the proof, it remains to verify axioms (J4), (J5) for the product $\{\dots\}_{**}$ with some bounded circular domain $\mathbf{B} \subset \mathbf{E}$. The weak*-closure of the domain D seems a tempting but technically unsuitable choice for \mathbf{B} in our setting. Instead we proceed as follows. Let $\Omega_1 := \Omega \setminus \Omega_0$ and regard \mathbf{E} as the ℓ^∞ -direct sum of the weak*-closed ideals \mathbf{E}_0 and $\mathbf{E}_1 := \{\mathbf{f} \in \mathbf{F} : \mathbf{f}(\widehat{\Omega} \setminus \Omega_1) = 0\}$. Define

$$\mathbf{B} := \mathbf{B}_0 + \mathbf{B}_1$$

where

$$\mathbf{B}_0 := \text{int}_{\mathbf{E}_0} \overline{D \cap E_D}^{w^*}, \quad \mathbf{B}_1 := \{\mathbf{x} \in \mathbf{E}_1 : \max|\mathbf{x}| < 1\}.$$

Recall [5, 3] that the bidual of a (full) JB^* -triple is a JB^* -triple with the separately weak*-continuous extension of the triple product. Hence, since the set $B_0 := D \cap E_D$ is the open unit ball of the canonical norm $\|a\|_{\{\dots\}_D} := [\max \text{Sp}[E_D \ni c \mapsto \{aac\}_D]]^{1/2}$ on E_D , its weak*-closure \mathbf{B}_0 is the norm closure of the open unit ball of the norm $\|\cdot\|_{\{\dots\}_{**}}$ on \mathbf{E}_0 . We show that actually \mathbf{B}_0 is a (bounded symmetric) complete Reinhardt domain in the function space $\mathbf{E}_0 \simeq \mathcal{C}(\Omega_0)$. Indeed, by Lemma 5.2 we have $A(\mathbf{f}A(\mathbf{g}))|_{\Omega_0} = A(\mathbf{f})A(\mathbf{g})|_{\Omega_0}$ for $\mathbf{f}, \mathbf{g} \in E_D$. Hence $A^{**}(\mathbf{f}A^{**}(\mathbf{g}))|_{\Omega_0} = A^{**}(\mathbf{f})A^{**}(\mathbf{g})|_{\Omega_0}$ for $\mathbf{f}, \mathbf{g} \in \mathbf{E}_0$. Since \mathbf{B}_0 is the canonical unit ball of the triple product $\{\dots\}_{**}$ restricted to \mathbf{E}_0^3 , Lemma 5.2 implies the Reinhardt property of \mathbf{B}_0 . On the other hand, \mathbf{B}_1 is trivially a (bounded symmetric) complete Reinhardt domain in $\mathbf{E}_1 \simeq \mathcal{C}(\Omega_1)$. Since $(\mathbf{E}_0, \mathbf{E}_0, \{\dots\}_{**}|_{\mathbf{E}_0^3})$ is a (full) JB^* -triple, for each $\mathbf{a} \in \mathbf{E}_0$, the operator $L(\mathbf{a})\mathbf{x} := \{\mathbf{aax}\}_{**}$, $\mathbf{x} \in \mathbf{E}$, is \mathbf{B}_0 -hermitian. On the other hand, the positivity of A^{**} (in the sense that it preserves the cone of all non-negative functions) entails the positivity of all $L(\mathbf{a})$, $\mathbf{a} \in \mathbf{E}_0$. Hence (J4) is immediate for the partial Jordan *-triple $(\mathbf{E}, \mathbf{E}_0, \{\dots\}_{**})$ with \mathbf{B} in the role of B there. To establish (J5), we only have to see that given any $\mathbf{a} \in \mathbf{E}_0$, the operator $L(\mathbf{a})$ is \mathbf{B} -hermitian. We have $L(\mathbf{a}) = \frac{1}{2}L_0(\mathbf{a}) + \frac{1}{2}L_1(\mathbf{a})$

where $L_0(\mathbf{a})\mathbf{x} := A^{**}(|\mathbf{a}|^2)\mathbf{x}$ and $L_1(\mathbf{a})\mathbf{x} := A^{**}(\mathbf{x}\bar{\mathbf{a}})\mathbf{a}$. The operator $L_1(\mathbf{a})$ is multiplication with a non-negative function in \mathbf{E} and hence necessarily both \mathbf{B}_0 - and \mathbf{B}_1 -hermitian. For $L_0(\mathbf{a})$ we have $L_0(\mathbf{a})\mathbf{E} \subset \mathbf{F}\mathbf{a} \subset \mathbf{F}\mathbf{E}_0 = \mathbf{E}_0$ and $L_0(\mathbf{a})\mathbf{E}_1 = A^{**}(\mathbf{E}_1\bar{\mathbf{a}})\mathbf{a} = A^{**}(0)\mathbf{a} = 0$. Thus the complementary ideals \mathbf{E}_0 and \mathbf{E}_1 are invariant subspaces of the operator $L(\mathbf{a})$ which acts on \mathbf{E}_k as a \mathbf{B}_k -hermitian operator for both $k = 0, 1$. Therefore $L(\mathbf{a})$ is $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ -hermitian. ■

7. Appendix

7.1. THEOREM. *Let Ω be a locally compact Hausdorff space and $\phi : \mathcal{C}_0(\Omega)^N \rightarrow \mathbb{R}$ a continuous positive N -linear form (that is, $\Phi(f_1, \dots, f_N) \geq 0$ for $f_1, \dots, f_N \geq 0$). Then with $f_1 \otimes \dots \otimes f_N : (\omega_1, \dots, \omega_N) \mapsto \prod_{k=1}^N f_k(\omega_k)$ we have*

$$\Phi(f_1, \dots, f_N) = \int f_1 \otimes \dots \otimes f_N d\mu, \quad f_1, \dots, f_N \in \mathcal{C}_0(\Omega),$$

for some bounded Radon measure μ on Ω^N .

Proof. Consider the family \mathbf{U} of all finite minimal open coverings of Ω including at most one non-precompact member. That is, each $\mathcal{U} \in \mathbf{U}$ can be written in the form $\mathcal{U} = \{U_1, \dots, U_m\}$ where $\Omega = \bigcup_{k=1}^m U_k$ with open sets U_k such that U_1, \dots, U_{m-1} have compact closure in Ω and $\bigcup_{i \in I} U_i \neq \Omega$ whenever I is a proper subset of $\{1, \dots, m\}$. The latter property means that the \mathcal{U} is minimal. This minimality property guarantees that for any covering $\mathcal{U} \in \mathbf{U}$ we can fix a system $\{\mathcal{U}_{\omega_U} : U \in \mathcal{U}\}$ of points such that

$$\mathcal{U}_{\omega_U} \in U \setminus \bigcup_{U \neq V \in \mathcal{U}} V, \quad U \in \mathcal{U}.$$

Since locally compact spaces are paracompact, we can also choose a partition of unity $\{\mathcal{U}_{\varphi_U} : U \in \mathcal{U}\}$ subordinated to the covering \mathcal{U} . That is, $\sum_{U \in \mathcal{U}} \mathcal{U}_{\varphi_U} = 1$ where $0 \leq \mathcal{U}_{\varphi_U} \in \mathcal{C}(\Omega)$ with $\mathcal{U}_{\varphi_U}(\Omega \setminus U) = 0$. Notice that necessarily $\mathcal{U}_{\varphi_U}(\mathcal{U}_{\omega_V}) = \delta_{UV}$ ($= 1$ if $U = V$, 0 otherwise). Hence the linear operator

$$P_{\mathcal{U}}f := \sum_{\substack{U \in \mathcal{U} \\ \text{precompact}}} f(\mathcal{U}_{\omega_U})\mathcal{U}_{\varphi_U}, \quad f \in \mathcal{C}_0(\Omega),$$

is a projection of $\mathcal{C}_0(\Omega)$ onto its finite-dimensional subspace with linear basis $\{\mathcal{U}_{\varphi_U} : U \in \mathcal{U} \text{ precompact}\}$.

The class \mathbf{U} has the natural net ordering $\mathcal{U} \prec \mathcal{V}$ of being finer. That is, $\mathcal{U} \prec \mathcal{V}$ if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ with $V \subset U$. It is well-known that, given any $f \in \mathcal{C}_0(\Omega)$ and $\varepsilon > 0$, there exists $\mathcal{U} \in \mathbf{U}$ such that $\sup_{\omega_1, \omega_2 \in U} |f(\omega_1) - f(\omega_2)| \leq \varepsilon$ for all $U \in \mathcal{U}$. This means that

$$\lim_{\mathcal{U} \in \mathbf{U}} \|P_{\mathcal{U}}f - f\| = 0, \quad f \in \mathcal{C}_0(\Omega).$$

Consider the linear functionals

$$\widehat{\Phi}_U \widehat{f} := \sum_{\substack{U_1, \dots, U_N \in \mathcal{U} \\ \text{precompact}}} \widehat{f}(\omega_{U_1}, \dots, \omega_{U_N}) \Phi(u_{\varphi_{U_1}}, \dots, u_{\varphi_{U_N}})$$

on $\mathcal{C}_0(\Omega^N)$. Observe that, for $f_1, \dots, f_N \in \mathcal{C}_0(\Omega)$,

$$\Phi(P_U f_1, \dots, P_U f_N) = \widehat{\Phi}_U(f_1 \otimes \dots \otimes f_N).$$

Since the form Φ is assumed to be positive, if $-1 \leq \widehat{f} \leq 1$ we have

$$\sum_{\substack{U_1, \dots, U_N \in \mathcal{U} \\ \text{precompact}}} (-1) \Phi(u_{\varphi_{U_1}}, \dots, u_{\varphi_{U_N}}) \leq \widehat{\Phi}_U \widehat{f} \leq \sum_{\substack{U_1, \dots, U_N \in \mathcal{U} \\ \text{precompact}}} \Phi(u_{\varphi_{U_1}}, \dots, u_{\varphi_{U_N}}),$$

which shows that

$$\|\widehat{\Phi}_U\| \leq \sum_{\substack{U_1, \dots, U_N \in \mathcal{U} \\ \text{precompact}}} \Phi(u_{\varphi_{U_1}}, \dots, u_{\varphi_{U_N}}), \quad U \in \mathbf{U}.$$

On the other hand, the functions $Uf := \sum_{U \in \mathcal{U} \text{ precompact}} u_{\varphi_U}$ satisfy

$$\begin{aligned} 0 &\leq Uf \leq 1, \\ 0 &\leq \Phi(Uf, \dots, Uf) = \sum_{\substack{U_1, \dots, U_N \in \mathcal{U} \\ \text{precompact}}} \Phi(u_{\varphi_{U_1}}, \dots, u_{\varphi_{U_N}}), \quad U \in \mathbf{U}. \end{aligned}$$

Hence

$$\begin{aligned} \|\widehat{\Phi}_U\| &= \sum_{\substack{U_1, \dots, U_N \in \mathcal{U} \\ \text{precompact}}} \Phi(u_{\varphi_{U_1}}, \dots, u_{\varphi_{U_N}}) \\ &\leq \|\phi\| \quad (:= \sup_{\|f_1\| = \dots = \|f_N\| = 1} |\Phi(f_1, \dots, f_N)|). \end{aligned}$$

By the continuity of Φ we have $\|\Phi\| < \infty$. According to the Alaoglu–Bourbaki theorem, the bounded net $(\widehat{\Phi}_U)_{U \in \mathbf{U}}$ admits cluster points in the dual of $\mathcal{C}_0(\Omega)$ in the weak* sense. (Actually one could even prove its weak* convergence but we do not need this finer argument). By taking any cluster point $\widehat{\Phi}$ of $(\widehat{\Phi}_U)_{U \in \mathbf{U}}$, for all $f_1, \dots, f_N \in \mathcal{C}_0(\Omega)$ we have

$$\begin{aligned} \Phi(f_1, \dots, f_N) &= \lim_{U \in \mathbf{U}} \Phi(P_U f_1, \dots, P_U f_N) = \lim_{U \in \mathbf{U}} \widehat{\Phi}_U(f_1 \otimes \dots \otimes f_N) \\ &= \widehat{\Phi}(f_1 \otimes \dots \otimes f_N). \end{aligned}$$

The proof is complete. ■

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