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Hyers–Ulam constants of Hilbert spaces

by

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Abstract. The best constant in the Hyers–Ulam theorem on isometric approximation in Hilbert spaces is equal to the Jung constant of the space.

1. Introduction

1.1. *Nearisometries.* We first introduce some notation. Throughout the paper, E will be a real Banach space (possibly a Hilbert space) with $\dim E \geq 1$. The norm of a vector $x \in E$ is written as $|x|$. We let $\overline{B}(x,r)$ denote the closed ball with center x and radius r, and we set $\overline{B}(r) = \overline{B}(0,r)$. The diameter of a set $A \subset E$ is $d(A)$. For maps $f, g : E \to E$ we set

$$d(f,g) = \sup_{x \in E} |fx - gx|,$$

with the possibility $d(f,g) = \infty$.

We say that a map $f : E \to E$ is a *nearisometry* if there is a number $\varepsilon \geq 0$ such that

$$|x-y| - \varepsilon \leq |fx - fy| \leq |x-y| + \varepsilon$$

for all $x, y \in E$. More precisely, such a map is an ε-*nearisometry*. We do not assume that f is continuous.

1.2. *Background.* D. H. Hyers and S. M. Ulam [HU] proved in 1945 that every surjective ε-nearisometry $f : E \to E$ of a Hilbert space E can be approximated by a surjective isometry $T : E \to E$ such that

$$d(T,f) \leq c\varepsilon \tag{1.3}$$

with $c = 10$. The result was extended to all Banach spaces with $c = 5$ by J. Gevirtz [Ge] in 1983; an important step towards this proof was made by P. M. Gruber [Gr] in 1978. M. Omladič and P. Šemrl [OŠ] proved in 1995 that the result is true with $c = 2$. Moreover, if $f(0) = 0$, the isometry T can be chosen so that $T(0) = 0$, that is, T is linear. They also proved that the

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constant 2 is best possible for such T. In 1998 Šemrl [Še, 1.1] proved that even without the condition $T(0) = 0$, the bound 2ε is best possible in the class of all Banach spaces. However, he also proved [Še, 1.2] that for a given Banach space E, the result holds for all c greater than the Jung constant $J(E)$ of E.

1.4. DEFINITIONS. The *Jung constant* $J(E)$ of E is the infimum of all $r > 0$ such that every set $A \subset E$ with $d(A) \le 2$ is contained in a ball of radius r. We say that $J(E)$ is *attained* if this infimum is a minimum, that is, A is contained in some ball $\overline{B}(x, J(E))$.

For all spaces we have $1 \le J(E) \le 2$, and $J(\mathbb{R}^n) = \sqrt{2n/(n+1)} < \sqrt{2}$ by the classical result proved by H. W. E. Jung [Ju] in 1901. Furthermore, $J(E) = \sqrt{2}$ for infinite-dimensional Hilbert spaces; see [Ro, Th. 1] for separable spaces, and [Da, Th. 2] or [Še, p. 704] for arbitrary spaces. Moreover, the Jung constant of a Hilbert space is always attained; see [Ro, Th. 9] and [Da, Th. 1].

The *Hyers–Ulam constant* $H(E)$ of E is the infimum of all $c > 0$ such that for each surjective ε-nearisometry $f : E \to E$ there is a surjective isometry $T : E \to E$ satisfying (1.3). This constant is *attained* if T can always be chosen so that $d(T, f) = H(E)\varepsilon$.

We let $H_c(E)$ denote the constant defined as $H(E)$ but considering only *continuous* surjective nearisometries $f : E \to E$. Clearly $H_c(E) \le H(E)$.

We summarize the results given above as follows.

1.5. THEOREM (Hyers–Ulam–Gruber–Gevirtz–Omladič–Šemrl). *Let E be a Banach space with $\dim E \ge 1$ and let $f : E \to E$ be a surjective ε-nearisometry with $f(0) = 0$. Then there is a surjective linear isometry $T : E \to E$ such that $d(T, f) \le 2\varepsilon$. Moreover, $H(E) \le J(E)$. If $J(E)$ is attained, there is a surjective isometry $T : E \to E$ with $d(T, f) \le J(E)\varepsilon$.*

The proof of the first part is given in [OŠ] and (slightly simplified) in [BL, Th. 15.2]. The second part follows rather easily from the first part, as shown in [Še, 1.2].

The purpose of this paper is to prove that $H_c(E) = H(E) = J(E)$ for all Hilbert spaces. Some remarks on Banach spaces are given in Section 3.

1.6. REMARK. More generally, Theorem 1.5 holds for nearisometries $f : E \to F$ onto another Banach space F. Since it follows that E and F are isometrically isomorphic, the restriction to the case $E = F$ is no loss of generality from the point of view of the present paper.

1.7. *The role of surjectivity.* The surjectivity condition in 1.5 can sometimes be weakened (see [Di, Th. 2]), and it can be entirely omitted if $\dim E < \infty$; see [BŠ, Th. 1] and [Di, Th. 1]. For our purposes it is convenient to consider ε-nearisometries $f : E \to E$ for which $E \setminus fE$ is a bounded set. We let $H_0(E)$ denote the infimum of all $c > 0$ such that for each such f there is a surjective isometry $T : E \to E$ with $d(T, f) \le c\varepsilon$. Trivially $H(E) \le H_0(E)$.

We give the following variation of 1.5.

1.8. THEOREM. *In Theorem* 1.5 *one can replace the surjectivity of f by the condition that $E \setminus fE$ be bounded. In particular, $H_0(E) \le J(E)$.*

Proof. Suppose that $f : E \to E$ is an ε-nearisometry with $f(0) = 0$ and that $E \setminus fE \subset \overline{B}(2R)$ with $R > \varepsilon$. Define a map $g : E \to E$ by $gx = 2x$ for $|x| \le R$ and by $gx = fx$ for $|x| > R$. Then g is a surjective nearisometry with $d(g, f) < \infty$. By 1.5 there is a surjective linear isometry $T : E \to E$ with $d(T, f) < \infty$.

Replacing f by $T^{-1}f$ we may assume that $T = \mathrm{id}$. We can now proceed as in the lower half of p. 362 of [BL], since the points z_n are defined for large n. Note, however, a misprint in the estimate for $\|z_n\|$. It should read

$$\|z_n\| \le \|f(x + z_n) - f(x)\| + \varepsilon = \|x + (n + a)y - (ay + x)\| + \varepsilon = n + \varepsilon.$$

We obtain $|fx - x| \le 2\varepsilon$ for all $x \in E$. The second part of the theorem follows as in [Še, 1.2]. ■

1.9. REMARKS. 1. The condition of 1.8 can still be relaxed as follows:

(1) The function $x \mapsto d(x, fE)$ is bounded in E.

(2) The set $R \cap fE$ is unbounded for each ray $R \subset E$.

Indeed, the proof of [Di, Th. 2] gives a surjective nearisometry $g : E \to E$ with $d(g, f) < \infty$, and we can proceed essentially as in 1.8.

2. We always have

$$1 \le H_c(E) \le H(E) \le H_0(E) \le J(E) \le 2. \tag{1.10}$$

It suffices to prove the first inequality, since the other inequalities either follow from 1.8 or are trivial.

Let $q > 0$ and define a function $\varphi : [0, \infty[\to \mathbb{R}$ by

$$\varphi(t) = \begin{cases} -t & \text{for } 0 \le t \le 1, \\ -1 + (2 + q)(t - 1)/q & \text{for } 1 \le t \le 1 + q, \\ t & \text{for } t \ge 1 + q. \end{cases}$$

The map $f : E \to E$, defined by $f(0) = 0$ and by $f(x) = \varphi(|x|)x/|x|$ for $x \ne 0$, is surjective and continuous with $d(f, \mathrm{id}) < \infty$. If $T : E \to E$ is an isometry with $d(T, f) = c < \infty$, the unit ball is contained in a ball of radius $c/2$ by Lemma 1.13 below, and thus $c \ge 2$. Since $f = \mathrm{id}$ outside the ball $B(1 + q)$ and since f maps this ball onto itself, we easily see that f is a $(2 + 2q)$-nearisometry. Hence $H_c(E) \ge 1/(1 + q)$, which gives the first inequality as $q \to 0$.

We are ready to state the main result of the paper.

1.11. Theorem. *For every Hilbert space E with* $\dim E \geq 1$ *we have* $H_{\mathrm{c}}(E) = H(E) = H_0(E) = J(E)$. *Moreover, the constants* $H_{\mathrm{c}}(E)$, $H(E)$ *and* $H_0(E)$ *are attained. Thus* $H(\mathbb{R}^n) = \sqrt{2n/(n+1)}$, *and* $H(E) = \sqrt{2}$ *if* $\dim E = \infty$.

By (1.10) it suffices to show that $J(E) \leq H_{\mathrm{c}}(E)$. This will be done in 2.12. However, we shall first give a proof for the weaker inequality $J(E) \leq H_0(E)$ in 2.11, because it is more straightforward.

Since the Jung constant of a Hilbert space is attained, the constants $H_{\mathrm{c}}(E)$, $H(E)$ and $H_0(E)$ are attained by 1.5.

The following observations are used several times in the paper.

1.12. Lemma. *Let* $T : E \to E$ *be a surjective isometry of a Banach space* E *with* $d(T, \mathrm{id}) < \infty$. *Then* T *is of the form* $Tx = x + w$ *for some* $w \in E$.

Proof. Replacing Tx by $Tx - T(0)$ we may assume that $T(0) = 0$. Then T is linear by the Mazur–Ulam theorem. Hence $T - \mathrm{id}$ is a linear map with bounded image, and thus $T = \mathrm{id}$. ∎

1.13. Lemma. *Suppose that* $A \subset E$ *and that* $f : E \to E$ *is a map such that* $d(f, \mathrm{id}) < \infty$ *and* $fx = -x$ *for all* $x \in A$. *Suppose also that* $T : E \to E$ *is an isometry with* $d(T, f) \leq q < \infty$. *Then* A *is contained in a ball of radius* $q/2$.

Proof. By Lemma 1.12, T is of the form $Tx = x + w$ for some $w \in E$. For $x \in A$ we have $q \geq |Tx - fx| = |2x + w|$. Hence $A \subset \overline{B}(-w/2, q/2)$. ∎

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2. Proofs

2.1. *Notation.* In this section we assume that E is a real Hilbert space with $\dim E \geq 1$. The inner product of vectors $x, y \in E$ is written as $x \cdot y$. The unit sphere of E is $S(1) = \{x \in E : |x| = 1\}$. For each $u \in S(1)$ we let $p_u : E \to \mathbb{R}$ denote the projection $p_u x = x \cdot u$. The line spanned by u is written as L_u, and $d(x, L_u)$ denotes the distance between a point $x \in E$ and L_u.

2.2. Lemma. *Suppose that* $u \in S(1)$, *that* $s, t \in \mathbb{R}$, *and that* $a, b \in E$ *with* $d(a, L_u) = d(b, L_u)$. *Then*

$$\big||su - a| - |tu - b|\big| \leq \big||s - p_u a| - |t - p_u b|\big|.$$

Proof. By performing auxiliary isometries of the triples $\{a, (p_u a)u, su\}$ and $\{b, (p_u b)u, tu\}$ we may assume that $a = b,\ p_u a = p_u b = 0,\ s \geq 0,\ t \geq 0$, and the lemma follows from the triangle inequality. ∎

2.3. *The basic construction.* In the rest of the section we assume that $A \subset E$ is a set with $0 \in A \subset \overline{B}(3/4)$ and $d(A) \le 1$. We associate to A a map $f : E \to E$ as follows.

For each $u \in S(1)$ we set

$$\alpha_u = \inf p_u A, \qquad \beta_u = \sup p_u A.$$

Then

$$\alpha_u \le 0 \le \beta_u \le \alpha_u + 1. \tag{2.4}$$

Define a function $h : S(1) \to \mathbb{R}$ by

$$h(u) = \begin{cases} 2\beta_u - 1 & \text{if } \beta_u \ge 1/2, \\ 1 + 2\alpha_u & \text{if } \alpha_u \le -1/2, \\ 0 & \text{if } \alpha_u \ge -1/2 \text{ and } \beta_u \le 1/2. \end{cases} \tag{2.5}$$

Since $\beta_u \le \alpha_u + 1$ by (2.4), the function h is well defined. Furthermore,

$$\alpha_{-u} = -\beta_u, \qquad \beta_{-u} = -\alpha_u, \qquad h(-u) = -h(u), \qquad |h(u)| \le 1/2 \tag{2.6}$$

for all $u \in S(1)$. Clearly h is continuous.

The map $f : E \to E$ is now defined by

$$f(x) = \begin{cases} -x & \text{for } x \in A, \\ x - h(u)u & \text{for } x \in L_u \setminus A,\ u \in S(1). \end{cases} \tag{2.7}$$

Since $0 \in A$ and since $h(-u) = -h(u)$ by (2.6), the map f is well defined. Furthermore,

$$d(f, \mathrm{id}) = 2 \sup\{|x| : x \in A\} \le 3/2. \tag{2.8}$$

2.9. LEMMA. *The map $f : E \to E$ defined by* (2.7) *is a* 1*-nearisometry.*

Proof. Let $x, y \in E$. We must show that

$$\delta = \big||fx - fy| - |x - y|\big| \le 1. \tag{2.10}$$

If $x, y \in A$, then $\delta = 0$. Assume that $x, y \in E \setminus A$. Choose $u \in S(1)$ with $x \in L_u$. Then $|fx - x| = |h(u)| \le 1/2$ by (2.6). Similarly $|fy - y| \le 1/2$, and (2.10) follows.

In the rest of the proof we assume that $x \in A$ and $y \in E \setminus A$. Write $y = su$ with $u \in S(1)$, $s \in \mathbb{R}$. We may assume that $h(u) \ge 0$ replacing u by $-u$ if necessary. To simplify notation we write

$$p = p_u, \qquad \alpha = \alpha_u, \qquad \beta = \beta_u, \qquad h = h(u).$$

By Lemma 2.2 we get

$$\delta \le \big||s - h + px| - |s - px|\big| \le |2px - h|.$$

If $h = 0$, then $|px| \le \max\{|\alpha|, \beta\} \le 1/2$ and $\delta \le 1$. Assume that $h > 0$. Then $h = 2\beta - 1$ and $|\alpha| < 1/2 < \beta$. Since $\alpha \le px \le \beta$ and $\beta \le \alpha + 1$, we obtain

$$2px - h \le 2\beta - 2\beta + 1 = 1, \qquad h - 2px \le 2\beta - 1 - 2\alpha \le 1.$$

Thus $\delta \le 1$. ■

We turn to the proof of Theorem 1.11. It will follow from Proposition 2.12 below, but we first prove a weaker result.

2.11. PROPOSITION. *$H_0(E) \geq J(E)$.*

Proof. Let $0 < \varepsilon \leq 3/4 - \sqrt{2}/2$ and set $r = J(E)/2 + \varepsilon$. By the definition 1.4 of the Jung constant, there is a set $A \subset E$ such that $d(A) = 1$, $A \subset \overline{B}(r)$ and A is not contained in any ball of radius $r - 2\varepsilon$. Since $A \subset \overline{B}(3/4) \subset \overline{B}(1)$, the set $A \cup \{0\}$ has the same properties. Hence we may assume that $0 \in A$. Since $J(E) \leq \sqrt{2}$, we have $r \leq 3/4$. We can thus apply the construction of 2.3 and obtain the map $f : E \to E$ defined by (2.7). Since $|h(u)| \leq 1/2$ for all $u \in S(1)$ by (2.6), we have $E \setminus fE \subset \overline{B}(5/4)$. Since f is a 1-nearisometry by 2.9, there is a surjective isometry $T : E \to E$ with $d(T, f) \leq H_0(E) + \varepsilon = q$. By 1.13, there is a ball of radius $q/2$ containing A. Hence $r - 2\varepsilon < q/2$, which yields $J(E) < H_0(E) + 3\varepsilon$. As $\varepsilon \to 0$, this proves the proposition. ∎

2.12. PROPOSITION. *$H_c(E) \geq J(E)$.*

Proof. Assume first that $\dim E = n < \infty$. Set $K = J(E)/2 = \sqrt{n/(2(n+1))}$. Choose a regular n-simplex Δ centered at the origin with $d(\Delta) = 1$. Let $v_0, \ldots, v_n$ be the vertices of Δ. Then $|v_j| = K$ for all $0 \leq j \leq n$. Moreover, Δ is not contained in any ball of radius less than K.

Setting $A = \{0, v_0, \ldots, v_n\}$ we have $d(A) = 1$ and $A \subset \overline{B}(K) \subset \overline{B}(3/4)$. Hence we may apply the construction of 2.3 and obtain a function $h : S(1) \to \mathbb{R}$ and a map $f : E \to E$, defined by (2.5) and (2.7).The map f is a 1-nearisometry by 2.9, and f is continuous in $E \setminus A$. We next modify f in a neighborhood of A to get a continuous nearisometry.

Choose a small number $r > 0$ such that the balls $\overline{B}(a, r)$ are disjoint for $a \in A$. For each $a \in A$ we define $g_r : \overline{B}(a, r) \to E$ by

$$g_r((1-t)a + tz) = (1-t)fa + tfz,$$

where $|z - a| = r$. Setting $g_r x = fx$ for $x \notin A + \overline{B}(r)$ we obtain a continuous map $g_r : E \to E$ such that $g_r x = fx = -x$ for $x \in A$ and such that $d(g_r, \mathrm{id}) < \infty$.

FACT 1. *The map g_r is a λ-nearisometry where $\lambda = \lambda(r) \to 1$ as $r \to 0$.*

Let $x, y \in E$. To prove Fact 1 we must find an estimate

$$\big||g_r x - g_r y| - |x - y|\big| \leq \lambda, \tag{2.13}$$

where $\lambda = \lambda(r) \to 1$ as $r \to 0$.

To simplify the proof we only consider the limiting case $r = 0$. Then g_0 is not a genuine map but a one-to-many relation at the points of A. More precisely, the image of a point v_j is the line segment $I(v_j) = [-v_j, -v_j + u_j]$ where $u_j = v_j/|v_j|$. The image of 0 is

$$I(0) = \bigcup\{[0, -h(u)u] : u \in S(1)\}.$$

Moreover, $g_0x = fx$ for $x \in E \setminus A$. It clearly suffices to prove that

$$\delta = \big||g_0x - g_0y| - |x - y|\big| \le 1, \tag{2.14}$$

where now g_0x [or g_0y] can be an arbitrary point of $I(x)$ [or $I(y)$] if x [or y] is in A. For example, rough estimates show that if $a \in A$ and $|x-a| \le r \le K/4$, then $d(g_rx, I(a)) \le 3r$. Hence (2.14) implies (2.13) with $\lambda = 1 + 8r$ for $r \le K/4$.

We consider six cases.

Case 1: $x, y \notin A$. Since f is a 1-nearisometry, (2.14) is true.

Case 2: $x, y \in A \setminus \{0\}$. Now $|x - y| \le 1$. It is easy to see that $d(I(x) \cup I(y)) = 1$. Hence $|g_0x - g_0y| \le 1$, and (2.14) follows.

Case 3: $x = 0$ and $y \in A \setminus \{0\}$. Since $|h(u)| \le 1/2$ for all $u \in S(1)$ by (2.6), we have $I(0) \subset \overline{B}(1/2)$ and hence $|g_0x - g_0y| \le |g_0x| + |g_0y| \le 1/2 + K$. Since $|x - y| = K < 1$, (2.14) follows.

Case 4: $x = y = 0$. Now $\delta = |g_0x - g_0y| \le d(I(0)) \le 1$.

Case 5: $x = 0$ and $y \notin A$. Since $|g_0x| \le 1/2$ and $|y - g_0y| = |y - fy| \le 1/2$, (2.14) is true.

Case 6: $x \in A \setminus \{0\}$ and $y \notin A$, say $x = v_0$. Then $g_0y = fy$ and $g_0x \in I(v_0) = [-v_0, -v_0 + u_0]$. We must show that

$$|z - fy| \le |x - y| + 1, \tag{2.15}$$

$$|z - fy| \ge |x - y| - 1 \tag{2.16}$$

for all $z \in I(v_0)$.

Since $-v_0 = fx$ and $-v_0 + u_0 = v_0 - h(u_0)u_0 = \lim_{x' \to x} fx'$ and since f is a 1-nearisometry, the inequalities (2.15) and (2.16) are true if z is an endpoint of $I(v_0)$. This implies (2.15) for all $z \in I(v_0)$, and it remains to prove (2.16). Define $p : E \to \mathbb{R}$ by $pw = w \cdot u_0$.

Fact 2. (2.16) *is true whenever* $pfy \notin\,]-K, -K + 1[$.

To prove this let $z \in I(v_0)$. Then $|z - fy| \ge |z_0 - fy|$ where z_0 is one of the endpoints of $I(v_0)$. Since (2.16) holds at the endpoints, Fact 2 follows.

Choose $u \in S(1)$ such that $y = su$ for some $s \in \mathbb{R}$ and $h(u) \ge 0$. Set $h = h(u)$ and define functions $F, G : \mathbb{R} \to \mathbb{R}$ by

$$F(s) = |su - x|, \qquad G(s) = d(su - hu, I(v_0)).$$

Since $f(su) = su - hu$ whenever $su \in E \setminus A$, it suffices to show that

$$F(s) - G(s) \le 1 \tag{2.17}$$

for all $s \in \mathbb{R}$.

Assume first that $u \cdot x \ne 0$. Then there are real numbers $a < b$ such that $p(su - hu) \in [-K, -K + 1]$ if and only if $s \in [a, b]$. For $s \notin\,]a, b[$, (2.17) follows from Fact 2. Suppose that $s \in [a, b]$. Clearly $h \in [a, b]$. The function

G is affine on the intervals $[a,h]$ and $[h,b]$. Hence $F-G$ is convex on these intervals. Consequently, it suffices to prove (2.17) at the points $s=a,h,b$. The cases $s=a,b$ were already considered above. Since $G(h)=0$, it suffices to show that

$$|hu-x|\le 1. \tag{2.18}$$

In the case $u\cdot x=0$, the function $F-G$ is convex on the intervals $]-\infty,h]$ and $[h,\infty[$. Moreover, $F(s)-G(s)\to\pm h$ as $s\to\pm\infty$. Since $|h|\le 1/2<1$, it again suffices to verify (2.18).

The case $h=0$ is clear and we assume that $h>0$. Then $h=2\beta_u-1=2v\cdot u-1$ for some $v\in A$. Thus

$$|hu-x|^2=h(h-2u\cdot x)+|x|^2=h[2u\cdot(v-x)-1]+K^2.$$

Here $u\cdot(v-x)\le|v-x|\le 1$, $h\le 2K-1$ and $K<\sqrt{2}/2$. Hence $|hu-x|^2\le\sqrt{2}-1/2<1$, and (2.18) follows. This completes the proof of Fact 1.

Fact 3. *$d(T,g_r)\ge 2K=J(E)$ for every isometry $T:E\to E$.*

If Fact 3 is not true, there is an isometry $T:E\to E$ with $d(T,g_r)=q<2K$. By Lemma 1.13, A is contained in a ball of radius $q/2<K$, which gives a contradiction.

Since $\dim E<\infty$ and since $g_r:E\to E$ is a continuous nearisometry, g_r is surjective by [Bo, 4.1]; the surjectivity can also be seen directly. Let $\varepsilon>0$. By the definition 1.4 of $H_{\mathrm{c}}(E)$, there is an isometry $T:E\to E$ with $d(T,g_r)\le(H_{\mathrm{c}}(E)+\varepsilon)\lambda$. Applying Fact 3 and letting $\varepsilon\to 0$ and $\lambda\to 1$ yields $J(E)\le H_{\mathrm{c}}(E)$. The proposition is now proved in the case $\dim E<\infty$.

Finally assume that $\dim E=\infty$. We must show that $H_{\mathrm{c}}(E)\ge J(E)=\sqrt{2}$. Assume that $H_{\mathrm{c}}(E)=c<\sqrt{2}$. Choose an integer n such that $J(\mathbb{R}^n)=\sqrt{2n/(n+1)}=M>c$, and let F be a linear subspace of E with $\dim F=n$. Let $1<\lambda<M/c$. The proof of the finite-dimensional case gives a continuous surjective λ-nearisometry $g:F\to F$ such that $d(g,\mathrm{id})<\infty$ and such that $d(T,g)\ge M$ for every isometry $T:F\to F$ (Fact 3).

Let $P:E\to F$ and $P':E\to F^\perp$ be the orthogonal projections. Define $f:E\to E$ by $fx=gPx+P'x$. Then f is a continuous surjective λ-nearisometry. Since $c\lambda<M$, there is a surjective isometry $S:E\to E$ with $d(S,f)=c'<M$. Since $d(f,\mathrm{id})=d(g,\mathrm{id})<\infty$, we have $d(S,\mathrm{id})<\infty$. By Lemma 1.12, S is of the form $Sx=x+w$ for some $w\in E$. Setting $Tx=x+Pw$ we obtain an isometry $T:F\to F$. For each $x\in F$ we have

$$|gx-Tx|=|P(fx-Sx)|\le|fx-Sx|\le c',$$

and hence $d(g,T)\le c'$. Since $d(g,T)\ge M>c'$, this is a contradiction and proves the proposition. ■

3. Banach spaces

3.1. CONJECTURE. *The equality $H(E) = J(E)$ of 1.11 holds for all Banach spaces E.*

Šemrl [Še, 1.1] proved that $H(E) = J(E) = 2$ for the space $E = C(X)$ where X is a countable compact space with a single cluster point. Thus E is isometrically isomorphic to the space c of all convergent sequences with the sup-norm. In fact, he constructed a surjective 1-nearisometry $f : E \to E$ such that $d(T, f) \geq 2$ for all surjective isometries $T : E \to E$.

We give a short proof for the equality in the space c_0 of all sequences converging to 0.

3.2. THEOREM. *$H(c_0) = J(c_0) = 2$.*

Proof. Let $0 < q \leq 1/2$. We define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = \begin{cases} -t & \text{for } t \in [-q, 1-q], \\ t - 1 + 2q & \text{for } t \notin [-q, 1-q]. \end{cases}$$

Then g is clearly a bijective 1-nearisometry. Define a map $f : c_0 \to c_0$ by $(fx)_j = g(x_j)$. As $j \to \infty$, we have $x_j \to 0$, and thus $g(x_j) \to 0$. Hence the sequence fx is indeed in c_0. Moreover, f is a bijective 1-nearisometry. Its inverse is given by $(f^{-1}x)_j = g^{-1}(x_j)$.

Let $\lambda > H(c_0)$. There is a surjective isometry $T : c_0 \to c_0$ with $d(T, f) \leq \lambda$. Since $|g(t) - t| \leq 2$ for all $t \in \mathbb{R}$, we have $d(f, \mathrm{id}) \leq 2$. Hence $d(T, \mathrm{id}) \leq 2 + \lambda < \infty$. By Lemma 1.12, T is of the form $Tx = x + w$ for some $w \in c_0$. Thus

$$\|x + w - fx\| \leq \lambda \tag{3.3}$$

for all $x \in c_0$, where $\|\cdot\|$ is the norm of c_0.

Let k be a positive integer and let $x \in c_0$ be the sequence with $x_k = 1 - q$ and $x_j = 0$ for $j \neq k$. Since $fx = -x$, (3.3) gives $\|2x + w\| \leq \lambda$, and hence $|2 - 2q + w_k| \leq \lambda$. As $k \to \infty$, this yields $|2 - 2q| \leq \lambda$. Letting $q \to 0$ and $\lambda \to H(c_0)$ gives $H(c_0) \geq 2$, and the theorem follows from (1.10). ■

3.4. *Summary.* The conjecture $H(E) = J(E)$ is true if E is a Hilbert space (1.11) or if $E = c$ ([Še, 1.1]) or $E = c_0$ (3.2). Moreover, by (1.10) it holds whenever $J(E) = 1$, for example, for $E = l_\infty$. In addition, we have proved that $H(E) = J(E) = 4/3$ for $E = \mathbb{R}^2$ with a regular hexagon as the unit disk.

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