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The method of rotation and Marcinkiewicz integrals on product domains

by

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Abstract. We give some rather weak sufficient condition for L^p boundedness of the Marcinkiewicz integral operator μ_Ω on the product spaces $\mathbb{R}^n \times \mathbb{R}^m$ $(1 < p < \infty)$, which improves and extends some known results.

1. Introduction. Let $\mathbb{R}^N$ $(N \geq 2, N = n$ or $m)$ be the N-dimensional Euclidean space and S^{N-1} be the unit sphere in $\mathbb{R}^N$. For nonzero $x \in \mathbb{R}^N$, we set $x' = x/|x|$. E. M. Stein [16] defined a higher dimensional analogue of the Marcinkiewicz integral on $\mathbb{R}^N$ by

$$\nu_\Omega(f)(x) := \Big(\int_0^\infty |F_s(x)|^2 \frac{ds}{s^3}\Big)^{1/2}$$

where

$$F_s(x) = \int_{|v| \leq s} \frac{\Omega(v')}{|v|^{N-1}} f(x - v)\, dv,$$

and Ω is a homogeneous function of degree zero whose restriction to S^{N-1} is in $L^1(S^{N-1})$ and satisfies the cancellation condition

$$\int_{S^{N-1}} \Omega(x')\, dx' = 0.$$

It is well known that the Marcinkiewicz integral is an important special case of the Littlewood–Paley–Stein functions and that it plays a key role in harmonic analysis. Readers can consult [4, 5, 8, 13–17], among numerous references, for its development and applications. In particular, it is closely related to the singular integral operator T_Ω introduced by Calderón and

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Zygmund [2], where

$$T_\Omega(f)(x) = \text{p.v.} \int\limits_{\mathbb{R}^N} \frac{\Omega(v')}{|v|^N} f(x-v)\,dv$$

with Ω satisfying the same conditions as in ν_Ω.

In their famous papers [2] and [3], Calderón and Zygmund proved that the operator T_Ω is bounded on L^p, $1 < p < \infty$, provided $\Omega \in L\log^+ L$. Moreover, the size condition $\Omega \in L\log^+ L$ is the best possible in the sense of Orlicz spaces (see [20]).

Recently, some authors began to study the Marcinkiewicz integral on the product spaces $\mathbb{R}^n \times \mathbb{R}^m$, which is defined by

$$(1)\quad \begin{aligned} \mu_\Omega(f)(x,y) &:= \Big(\iint\limits_{\mathbb{R}^2_+} |F^\Omega_{t,s}(x,y)|^2 \frac{dt\,ds}{(ts)^3}\Big)^{1/2}, \\ F^\Omega_{t,s}(x,y) &:= \iint\limits_{|u|\le t,\,|v|\le s} \frac{\Omega(u',v')}{|u|^{n-1}|v|^{m-1}} f(x-u,y-v)\,du\,dv, \end{aligned}$$

where $\Omega \in L^1(S^{n-1}\times S^{m-1})$, and

$$(2)\quad \int\limits_{S^{n-1}} \Omega(x',y')\,dx' = \int\limits_{S^{m-1}} \Omega(x',y')\,dy' = 0 \ (\forall (x',y') \in S^{n-1}\times S^{m-1}).$$

See [7, 10, 11, 21].

Below, we list some known results on the function μ_Ω.

Theorem A [21]. *If Ω satisfies the cancellation condition* (2) *and*

$$\sup_{\xi\in S^{n-1},\,\eta\in S^{m-1}} \iint\limits_{S^{n-1}\times S^{m-1}} |\Omega(u',v')| \Big\{ \log\frac{1}{|u'\xi|} + \log\frac{1}{|v'\eta|} + \log\frac{1}{|u'\xi|}\log\frac{1}{|v'\eta|}\Big\}^{\alpha+1} du'\,dv' < \infty$$

where $\alpha > 0$, then μ_Ω is bounded on $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ for $p \in \big(\frac{2\alpha+2}{2\alpha+1}, 2\alpha+2\big)$.

Theorem B [9, 10]. *If $\Omega \in L(\log^+ L)^2(S^{n-1}\times S^{m-1})$ and satisfies the cancellation condition* (2), *then μ_Ω is bounded on $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ for $1<p<\infty$.*

As mentioned above, to obtain the L^p boundedness of the singular integral operator T_Ω, the best size condition in the sense of the Orlicz space norm is $\Omega \in L\log^+ L$. However, the Marcinkiewicz integral has weaker singularity than that of T_Ω so that the size condition on Ω can be weakened. In the one-parameter case $\mathbb{R}^n$, T. Walsh [18] obtained the following theorem.

Theorem C [18]. *For $\Omega \in L(\log^+ L)^{1/r}(\log^+\log^+ L)^{2-4/r'}(S^{n-1})$ and satisfying* (2), *μ_Ω is L^p bounded, where $p \in (1,\infty)$, $r = \min(p,p')$, $1/q' + 1/q = 1$.*

REMARK. Walsh considered a slightly more general operator. But there are no essential differences between it and the Marcinkiewicz integral.

The main purpose of this paper is to extend Walsh's result to the product Marcinkiewicz integral μ_Ω. We will prove the following theorem.

THEOREM 1. *If* $\Omega \in L^1(S^{n-1} \times S^{m-1})$ *and satisfies the cancellation condition* (2), *then*

$$\|\mu_\Omega(f)\|_p \le C_p(1 + \|\Omega\|_{L(\log^+ L)^{2/r}(\log^+ \log^+ L)^{8(1-2/r')}(S^{n-1}\times S^{m-1})})\|f\|_p,$$

where $r = \min(p, p')$, $1 < p < \infty$, $1/p' + 1/p = 1$.

Clearly this theorem is an improvement of Theorem C. Our proof will combine the basic ideas used in [18] and the rotation method by Calderón and Zygmund. But in the product case, the latter is more difficult to apply, in particular in the case of $p \neq 2$. Also, we need some new estimates for terms involving product kernels. We will prove the easy case $p = 2$ in the second section. After giving some technical lemmas in Section 3, we will prove the case $p \neq 2$ in the fourth section. Throughout this paper, we always use C to denote a positive constant independent of the essential variables and functions. It may be different at different occurrences.

2. Proof of Theorem 1 for $p = 2$**.** Define

$$\varphi(u, v) = \varphi^\Omega(u, v) = \frac{\Omega(u', v')}{|u|^{n-1}|v|^{m-1}}\, \chi_{|u|\le 1}(|u|)\chi_{|v|\le 1}(|v|). \tag{3}$$

Then

$$\mu_\Omega(f)(x, y) \equiv |\varphi_{t,s} * f(x, y)|_{\widetilde{\mathcal{H}}}$$

where $\varphi_{t,s}(u, v) = t^{-n}s^{-m}\varphi(u/t, v/s)$ and $\widetilde{\mathcal{H}} = L^2\big((\mathbb{R}^1_+)^2, \frac{dt}{t}\frac{ds}{s}\big)$. By Plancherel's formula, it is sufficient to estimate

$$\widehat{\varphi}(tx, sy) = \iint\limits_{S^{n-1}\times S^{m-1}} \Omega(u', v')\psi(tx \cdot u')\psi(sy \cdot v')\, du'\, dv' \tag{4}$$

for $|x| = |y| = 1$, where

$$\psi(t) = \int\limits_{[0,1]} \exp(-itr)\, dr = \frac{\sin t + i(\cos t - 1)}{t}.$$

For the convenience of notation, we denote the Hilbert spaces $L^2((0 \le t \le 1), dt/t)$ and $L^2((0 \le t \le 1, 0 \le s \le 1), dtds/(ts))$ by $\mathcal{H}(t \le 1)$ and $\widetilde{\mathcal{H}}\,(t \le 1,\ s \le 1)$ respectively; similarly, we define $\widetilde{\mathcal{H}}(t > 1,\ s \le 1)$, $\mathcal{H}(t > 1)$, $\widetilde{\mathcal{H}}(t \le 1,\ s > 1)$, $\widetilde{\mathcal{H}}(t > 1,\ s > 1)$, etc.

Now, for $t \le 1$, $s \le 1$, by the cancellation property of Ω,

$$|\widehat{\varphi}(tx, sy)|_{\widetilde{\mathcal{H}}(t\le 1, s\le 1)} \le C\|\Omega\|_1.$$

For $t > 1$, $s \le 1$, by the cancellation property of Ω,

$$\begin{aligned}
|\widehat{\varphi}(tx, sy)|_{\widetilde{\mathcal{H}}(t>1,s\le 1)} &\le \Big| \iint_{S^{n-1}\times S^{m-1}} |\Omega(u', v')| \cdot |\psi(tx\cdot u')|\, s\, du'\, dv' \Big|_{\widetilde{\mathcal{H}}(t>1,s\le 1)} \\
&\le \int_{S^{m-1}} \Big| \int_{S^{n-1}} |\Omega(u', v')| \cdot |\psi(tx\cdot u')|\, du' \Big|_{\mathcal{H}(t>1)} dv' \\
&\le \int_{S^{m-1}} \Big(\Big| \int_{|x\cdot u'|\le 1/t} |\Omega(u', v')| \cdot |\psi(tx\cdot u')|\, du' \Big|_{\mathcal{H}(t>1)} \\
&\qquad + \Big| \int_{|x\cdot u'|>1/t} |\Omega(u', v')| \cdot |\psi(tx\cdot u')|\, du' \Big|_{\mathcal{H}(t>1)} \Big) dv' \\
&=: \int_{S^{m-1}} (I(v') + II(v'))\, dv'.
\end{aligned}$$

Note that for the Young function $\Phi(t) = t(\log(t+2))^{1/2}$, its Young conjugate is $\Psi(t) = t\varphi^{-1}(t) - \Phi(\varphi^{-1}(t)) \le Ct\exp(t^2)$ where $\varphi(t) = \Phi'(t)$ and φ^{-1} is the inverse function of φ. Thus, by Hölder's inequality (see [22, Chapter 4]), we have

$$\begin{aligned}
I(v') &\le \int_{S^{n-1}} |\Omega(u', v')| \big|\chi_{|x\cdot u'|\le 1/t}(t) |\psi(tx\cdot u')|\big|_{\mathcal{H}(t>1)}\, du' \\
&\le C \int_{S^{n-1}} |\Omega(u', v')| (\log|x\cdot u'|^{-1})^{1/2}\, du' \\
&\le C \|\Omega(u', v')\|_{L_\Phi(S^{n-1},du')} \|(\log|x\cdot u'|^{-1})^{1/2}\|_{L_\Psi(S^{n-1},du')} \\
&\le C(\|\Omega(\cdot, v')\|_{L(\log^+ L)^{1/2}(S^{n-1})} + \|\Omega(\cdot, v')\|_{L^1(S^{n-1})}).
\end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}
II(v') &\le \int_{S^{m-1}} |\Omega(u', v')| \Big| \chi_{|x\cdot u'|>1/t}(t) |\psi(tx\cdot u')|\Big|_{\mathcal{H}(t>1)}\, du' \\
&\le C\|\Omega(\cdot, v')\|_{L^1(S^{n-1})}.
\end{aligned} \tag{6}$$

Hence, we have

$$\begin{aligned}
&|\widehat{\varphi}(tx, sy)|_{\widetilde{\mathcal{H}}(t>1,s\le 1)} \\
&\qquad \le C(\|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1}\times S^{m-1})} + \|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}).
\end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned}
&|\widehat{\varphi}(tx, sy)|_{\widetilde{\mathcal{H}}(t\le 1,s>1)} \\
&\qquad \le C(\|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1}\times S^{m-1})} + \|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}).
\end{aligned} \tag{8}$$

For $t > 1$, $s > 1$, we have

$$(9)\qquad \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v')\psi(tx\cdot u')\psi(sy\cdot v')\,du'\,dv'$$
$$=\Big(\int_{|x\cdot u'|\le t^{-1}} + \int_{|x\cdot u'|>t^{-1}}\Big)\Big(\int_{|y\cdot v'|\le s^{-1}} + \int_{|y\cdot v'|>s^{-1}}\Big)\Omega(u',v')\psi(tx\cdot u')\psi(sy\cdot v')\,du'\,dv'.$$

Of the above four terms, we shall only estimate the first and the last one; the other terms can be treated similarly. For the first term, by Hölder's inequality, we have

$$(10)\qquad \Big|\iint_{|x\cdot u'|\le t^{-1},\,|y\cdot v'|\le s^{-1}} \Omega(u',v')\psi(tx\cdot u')\psi(sy\cdot v')\,du'\,dv'\Big|_{\widetilde{\mathcal H}(t>1,s>1)}$$
$$\le C\Big|\int_{|y\cdot v'|\le s^{-1}}\Big|\int_{|x\cdot u'|\le t^{-1}}|\Omega(u',v')|\,du'\Big|_{\mathcal H(t>1)}dv'\Big|_{\mathcal H(s>1)}$$
$$\le C\Big|\int_{|y\cdot v'|\le s^{-1}}(\|\Omega(\cdot,v')\|_{L(\log^+ L)^{1/2}} + \|\Omega(\cdot,v')\|_{L^1})\,dv'\Big|_{\mathcal H(s>1)}$$
$$\le C(\|\Omega\|_{L\log^+ L(S^{n-1}\times S^{m-1})} + \|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}).$$

For the last term,

$$(11)\qquad \Big|\iint_{|x\cdot u'|>t^{-1},\,|y\cdot v'|>s^{-1}} \Omega(u',v')\psi(tx\cdot u')\psi(sy\cdot v')\,du'\,dv'\Big|_{\widetilde{\mathcal H}(t>1,s>1)}$$
$$\le \iint_{S^{n-1}\times S^{m-1}} |\Omega(u',v')|\cdot|\psi(tx\cdot u')|_{\mathcal H(t>|x\cdot u'|^{-1})}|\psi(sy\cdot v')|_{\mathcal H(s>|y\cdot v'|^{-1})}\,du'\,dv'$$
$$\le C\|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}.$$

Form (4)–(11), we get the assertion of Theorem 1 for $p=2$, i.e.

$$(12)\qquad \|\mu_\Omega(f)\|_2 \le C(1+\|\Omega\|_{L(\log^+ L)})\|f\|_2.$$

3. Some basic lemmas. In this section, we shall mainly discuss some properties of Littlewood–Paley–Stein g-functions.

3.1. *Preliminary lemmas.* We first recall some basic results about the Littlewood–Paley–Stein g-function, which is defined by

$$(13)\qquad \begin{aligned} &g_\sigma(f)(x) := |T_\sigma(f)(x)|_{\mathcal H},\\ &T_\sigma : f\mapsto T_\sigma(f)(x) = \sigma_t * f(x)|_{t\in\mathbb R^1_+} = K_\sigma * f(x),\\ &K_\sigma(x) = \sigma_t(x)|_{t\in\mathbb R^1_+} \in \mathcal H = L^2(\mathbb R^1_+, dt/t),\end{aligned}$$

where σ is an $L^1(\mathbb R)$ function. We have the following known result.

LEMMA 2 [17]. *If*

$$\|\sigma\|_1 < \infty, \qquad \int_{\mathbb{R}} \sigma(t)\,dt = 0, \qquad |K_\sigma(x)|_{\mathcal{H}} \le B|x|^{-1},$$
$$\int_{|x|>2|y|} |K_\sigma(x-y) - K_\sigma(x)|_{\mathcal{H}}\,dx \le B \qquad (\forall y \ne 0),$$

then

$$\|g_\sigma(f)\|_{WL^1} \le C(\|\sigma\|_1 + B)\|f\|_1, \qquad \|g_\sigma(f)\|_p \le C(pp')(\|\sigma\|_1 + B)\|f\|_p,$$

where $1 < p < \infty$ *and* $1/p' + 1/p = 1$.

In the product space case, the Littlewood–Paley–Stein g-function of one dimension is defined by

$$\begin{aligned} &\widetilde{g}_\varsigma(f)(x,y) := |\widetilde{T}_\varsigma(f)(x,y)|_{\widetilde{\mathcal{H}}}, \\ &\widetilde{T}_\varsigma : f \mapsto \widetilde{T}_\varsigma(f)(x,y) = \varsigma_{t,s} * f(x,y)|_{(t,s)\in(\mathbb{R}^1_+)^2} = \widetilde{K}_\varsigma * f(x,y), \\ &\widetilde{K}_\varsigma(x) = \varsigma_{t,s}(x,y)|_{(t,s)\in(\mathbb{R}^1_+)^2} \in \widetilde{\mathcal{H}} = L^2((\mathbb{R}^1_+)^2, dtds/(ts)), \end{aligned} \tag{14}$$

where ς is an $L^1(\mathbb{R}\times\mathbb{R})$ function. We have

LEMMA 3. *If* $\|\varsigma\|_1 < \infty$, ς *is odd both in* t *and in* s, *and there exists* $\alpha \in (0,1]$ *such that*

$$|\widetilde{K}_\varsigma(x,y)|_{\widetilde{\mathcal{H}}} \le B|x|^{-1}|y|^{-1},$$
$$|\widetilde{K}_\varsigma(x+h,y) - \widetilde{K}_\varsigma(x,y)|_{\widetilde{\mathcal{H}}} + |\widetilde{K}_\varsigma(y,x+h) - \widetilde{K}_\varsigma(y,x)|_{\widetilde{\mathcal{H}}} \le B|h|^\alpha|x|^{-1-\alpha}|y|^{-1},$$
$$\begin{aligned} |\widetilde{K}_\varsigma(x+h,y+k) - \widetilde{K}_\varsigma(x+h,y) - \widetilde{K}_\varsigma(x,y+k) + \widetilde{K}_\varsigma(x,y)|_{\widetilde{\mathcal{H}}} \\ \le B|h|^\alpha|k|^\alpha|x|^{-1-\alpha}|y|^{-1-\alpha} \end{aligned}$$

for all $|x| \ge 2|h|$, $|y| \ge 2|k|$, *then*

$$\|\widetilde{g}_\varsigma(f)\|_p \le C(pp')^6(\|\varsigma\|_1 + B)\|f\|_p, \qquad 1 < p < \infty.$$

This lemma can be viewed as a vector-valued generalization of boundedness of singular integrals on the product space $\mathbb{R}\times\mathbb{R}$ (see [11, 12]).

3.2. *Some special g-functions.* Set $\chi(t) = \chi_{(0,1)}(t)$, take $\lambda \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(\lambda) \subset [1,2]$ and $\int_{\mathbb{R}} \lambda(t)\,dt = 1$, let $\varrho = \chi - \lambda$, and define

$$\begin{aligned} &\sigma^{(1,u',v')}(t) = |t|^{n-1}\mathcal{R}'\bigg(\lambda(|\cdot|)\,\frac{\Omega(\cdot,v')}{|\cdot|^{n-1}}\bigg)(tu'), \\ &\sigma^{(2,u',v')}(s) = |s|^{m-1}\mathcal{R}''\bigg(\lambda(|\circ|)\,\frac{\Omega(u',\circ)}{|\circ|^{m-1}}\bigg)(sv'), \\ &\varsigma^{(u',v')}(t,s) = |t|^{n-1}|s|^{m-1}\mathcal{R}'\mathcal{R}''\bigg(\lambda(|\cdot|)\lambda(|\circ|)\,\frac{\Omega(\cdot,\circ)}{|\cdot|^{n-1}|\circ|^{m-1}}\bigg)(tu', sv'), \end{aligned} \tag{15}$$

where $\mathcal{R}'$ and $\mathcal{R}''$ are the Riesz transforms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then it is easy to see that $\sigma^{(i,u',v')}$ is odd, and $\varsigma^{(u',v')}(t,s)$ is odd both in t and in s.

LEMMA 4. *T_ϱ, $T_{\sigma^{(i,u',v')}}$ are bounded on $L^p(\mathbb{R})$, and $\widetilde{T}_{\varsigma^{(u',v')}}$ is bounded on $L^p(\mathbb{R}\times\mathbb{R})$; furthermore,*

$$\|T_\varrho\|_{p,p} \le C(pp'),$$
$$\|T_{\sigma^{(i,u',v')}}\|_{p,p} \le C(pp')(1+\Omega_i^*(x',y')),$$
$$\|\widetilde{T}_{\varsigma^{(u',v')}}\|_{p,p} \le C(pp')^6(1+\Omega^*(x',y')),$$

where $1<p<\infty$, $i=1,2$, and

$$\begin{aligned}\Omega_k^*(u',v') &= \sup_{r\in\mathbb{R},\, j=0,1} \frac{|\partial_r^j(\sigma^{(k,u',v')}(r))|}{(1+r)^{-2-j}}, \qquad k=1,2,\\ \Omega^*(u',v') &= \sup_{r,s\in\mathbb{R},\, i,j=0,1} \frac{|\partial_r^j\partial_s^i(\varsigma^{(u',v')}(r,s))|}{(1+r)^{-2-j}(1+s)^{-2-i}}.\end{aligned} \tag{16}$$

Proof. By Lemmas 2–3, it is enough to check that K_ϱ, $K_{\sigma^{(i,u',v')}}$ $(i=1,2)$ satisfy the conditions of Lemma 2, and $\widetilde{K}_{\varsigma^{(u',v')}}$ is as in Lemma 3. We shall check that in Subsections 3.2.1–3.2.3 below.

3.2.1. *Estimates of K_ϱ.* First, we note that $\varrho\in L^1(\mathbb{R})$, $\int\varrho(t)\,dt=0$ and

$$|K_\varrho(x)|_{\mathcal{H}}^2 = \int_0^\infty \left|t^{-1}\varrho\left(\frac{x}{t}\right)\right|^2 \frac{dt}{t} \le C\int_{2/|x|}^\infty t^{-2}\,\frac{dt}{t} \le C|x|^{-2}.$$

In addition, we have

$$\begin{aligned}&\int_{|x|>2|y|} |K_\varrho(x-y)-K_\varrho(x)|_{\mathcal{H}}\,dx = |y|\int_{|x|>2} |K_\varrho(x|y|-y)-K_\varrho(x|y|)|_{\mathcal{H}}\,dx\\ &= \int_{|x|>2}\left(\int_0^\infty \left|(\chi-\lambda)\left(\frac{x-y'}{t}\right)-(\chi-\lambda)\left(\frac{x}{t}\right)\right|^2\frac{dt}{t^3}\right)^{1/2} dx\\ &\le C\int_{|x|>2}\left(\int_{\min(1/|x-y'|,1/|x|)}^{\max(1/|x-y'|,1/|x|)} t\,dt\right)^{1/2} dx + C\int_{|x|>2}\left(\int_{|x|/2}^{2|x|}\|\lambda'\|_\infty^2\,\frac{dt}{t^5}\right)^{1/2} dx \le C.\end{aligned}$$

So, K_ϱ satisfies the conditions of Lemma 2.

3.2.2. *Estimates of $K_{\sigma^{(i,u',v')}}$.* We shall only consider the case $i=1$. Note that $\sigma^{(1,u',v')}\in L^1(\mathbb{R})$, $\int\sigma^{(1,u',v')}(t)\,dt=0$. Now, by definition of Ω_1^*

(see (16)) we have

$$|K_{\sigma^{(1,u',v')}}(x)|^2_{\mathcal{H}} = \int_0^\infty \left| t^{-1}\mathcal{R}'(\lambda(|\cdot|)\Omega(\cdot,v'))\left(\frac{x}{t}\,u'\right)\right|^2 \frac{dt}{t}$$

$$= |x|^{-2}\int_0^\infty \left|t^{-1}\sigma^{(1,u',v')}\left(\frac{x'}{t}\right)\right|^2 \frac{dt}{t}$$

$$\le |x|^{-2}\int_0^\infty \left|t^{-1}\Omega_1^*(u',v')\left(1+\frac{1}{t}\right)^{-2}\right|^2 \frac{dt}{t} \le C(\Omega_1^*(u',v')|x|^{-1})^2.$$

In addition, for $|x| \ge 2|h|$,

$$|K_{\sigma^{(1,u',v')}}(x-h) - K_{\sigma^{(1,u',v')}}(x)|_{\mathcal{H}}$$

$$= \left(\int_0^\infty \left|\mathcal{R}'(\lambda(|\cdot|)\Omega(\cdot,v'))\left(\frac{x-h}{t}\,u'\right) - \mathcal{R}'(\lambda(|\cdot|)\Omega(\cdot,v'))\left(\frac{x}{t}\,u'\right)\right|^2 \frac{dt}{t^3}\right)^{1/2}$$

$$= \left(\int_0^\infty \left|\frac{d\sigma^{(1,u',v')}}{dr}\left(\frac{x-\theta h}{t}\right)\right|^2 \frac{dt}{t^5}\right)^{1/2}$$

$$\le |x|^{-2}|h|\left(\int_0^\infty \left|\Omega_1^*(u',v')\left(1+\frac{1}{t}\right)^{-3}\right|^2 \frac{dt}{t^5}\right)^{1/2} \le C\Omega_1^*(u',v')|x|^{-2}|h|.$$

Thus, $K_{\sigma^{(1,u',v')}}$ satisfies the conditions of Lemma 2.

3.2.3. *Estimates of* $\widetilde{K}_{\varsigma^{(u',v')}}$. By the oddness of $\widetilde{K}_{\varsigma^{(u',v')}}$, we can assume $x > 0$ and $y > 0$. Now, by the definition of Ω^* (see (16)),

(17) $$|\widetilde{K}_{\varsigma^{(u',v')}}(x,y)|^2_{\widetilde{\mathcal{H}}}$$

$$= \iint_{(\mathbb{R}^1_+)^2} \left| t^{-1}s^{-1}\left(\frac{x}{t}\right)^{n-1}\left(\frac{y}{s}\right)^{m-1} \right.$$

$$\left. \times\, \mathcal{R}'\mathcal{R}''\left(\frac{\lambda(|\cdot|)\lambda(|\circ|)\Omega(\cdot,\circ)}{|\cdot|^{n-1}|\circ|^{m-1}}\right)\left(\frac{x}{t}\,u',\frac{y}{s}\,v'\right)\right|^2 \frac{dt\,ds}{ts}$$

$$\le \iint_{(\mathbb{R}^1_+)^2} \left|t^{-1}s^{-1}\left(1+\left|\frac{x}{t}\right|\right)^{-2}\left(1+\left|\frac{y}{s}\right|\right)^{-2}\right|^2 \frac{dt\,ds}{ts}\,(\Omega^*(u',v'))^2$$

$$\le C|x|^{-2}|y|^{-2}\iint_{(\mathbb{R}^1_+)^2} \left|\left(1+\left|\frac{1}{t}\right|\right)^{-2}\left(1+\left|\frac{1}{s}\right|\right)^{-2}\right|^2 \frac{dt\,ds}{t^3s^3}\,(\Omega^*(u',v'))^2$$

$$\le C|x|^{-2}|y|^{-2}(\Omega^*(u',v'))^2.$$

On the other hand, for $|x| \geq 2|h|$,

$$
\begin{aligned}
(18)\quad & |\widetilde{K}_{\varsigma^{(u',v')}}(x+h,y) - \widetilde{K}_{\varsigma^{(u',v')}}(x,y)|_{\widetilde{\mathcal{H}}}^2 \\
&\leq \iint_{(\mathbb{R}^1_+)^2} \left| \varsigma^{(u',v')}\left(\frac{x+h}{t}, \frac{y}{s}\right) - \varsigma^{(u',v')}\left(\frac{x}{t}, \frac{y}{s}\right) \right|^2 \frac{dt\,ds}{t^3 s^3} \\
&\leq |h|^2 \iint_{(\mathbb{R}^1_+)^2} \left| D_x \varsigma^{(u',v')}\left(\frac{x+\theta h}{t}, \frac{y}{s}\right) \right|^2 \frac{dt\,ds}{t^5 s^3} \\
&\leq |h|^2 \iint_{(\mathbb{R}^1_+)^2} \left| \left(1 + \left|\frac{x+\theta h}{t}\right|\right)^{-3} \left(1 + \left|\frac{y}{s}\right|\right)^{-2} \right|^2 \frac{dt\,ds}{t^5 s^3} \, (\Omega^*(u',v'))^2 \\
&\leq C|h|^2 |x|^{-4} |y|^{-2} (\Omega^*(u',v'))^2.
\end{aligned}
$$

Similarly, for $|y| \geq 2|h|$,

$$
(19)\quad |\widetilde{K}_{\varsigma^{(u',v')}}(x+h,y) - \widetilde{K}_{\varsigma^{(u',v')}}(x,y)|_{\widetilde{\mathcal{H}}} \leq C|h| \cdot |x|^{-2} |y|^{-1} \Omega^*(u',v').
$$

For $|x| \geq 2|h|$ and $|y| \geq 2|k|$,

$$
\begin{aligned}
(20)\quad & |\widetilde{K}_{\varsigma^{(u',v')}}(x+h,y+k) - \widetilde{K}_{\varsigma^{(u',v')}}(x+h,y) \\
&\qquad - \widetilde{K}_{\varsigma^{(u',v')}}(x,y+k) + \widetilde{K}_{\varsigma^{(u',v')}}(x,y)|_{\widetilde{\mathcal{H}}}^2 \\
&\leq |h|^2 |k|^2 \iint_{(\mathbb{R}^1_+)^2} \left| D_x D_y \varsigma^{(u',v')}\left(\frac{x+\theta_1 h}{t}, \frac{y+\theta_2 k}{s}\right) \right|^2 \frac{dt\,ds}{t^5 s^5} \\
&\leq |h|^2 |k|^2 \iint_{(\mathbb{R}^1_+)^2} \left(1 + \left|\frac{x}{t}\right|\right)^{-3} \left(1 + \left|\frac{y}{s}\right|\right)^{-3} \frac{dt\,ds}{t^5 s^5} \, (\Omega^*(u',v'))^2 \\
&\leq C|h|^2 |k|^2 |x|^{-4} |y|^{-4} (\Omega^*(u',v'))^2.
\end{aligned}
$$

All the above estimates (17)–(20) ensure that the kernel function $\widetilde{K}_{\varsigma^{(u',v')}}$ satisfies the conditions stated in Lemma 3. ∎

3.3. *Integrability of Ω^*'s*

LEMMA 5. *We have*

$$
\int_{S^{n-1}} \Omega_1^*(x',y')\,dx' \leq C(1 + \|\Omega(\cdot,y')\|_{L\log^+ L(S^{n-1})}),
$$

which also means that for fixed $y' \in S^{m-1}$, $\sigma^{(1,x',y')} \in L^1(\mathbb{R})$ for almost every $x' \in S^{n-1}$. A similar result holds for Ω_2^*.

LEMMA 6. *We have*

$$\iint\limits_{S^{n-1}\times S^{m-1}} \Omega^*(x',y')\,dx'\,dy' \le C(1+\|\Omega\|_{L(\log^+ L)^2(S^{n-1}\times S^{m-1})}),$$

which also means that $\varsigma^{(x',y')} \in L^1(\mathbb{R}\times\mathbb{R}^1)$ *for almost every* $(x',y') \in S^{n-1}\times S^{m-1}$.

3.3.1. *Proof of Lemma 5.* For $|x| \le 1/2$,

$$\begin{aligned} D_x^\alpha \mathcal{R}'(\lambda(|\cdot|)|\cdot|^{1-n}\Omega(\cdot,y'))(x) &= \int\limits_{\mathbb{R}^n} D_x^\alpha \frac{c_n(x-u)}{|x-u|^{n+1}}\cdot\lambda(|u|)|u|^{1-n}\Omega(u,y')\,du \\ &= O'(1)\|\Omega(\cdot,y')\|_{L^1(S^{n-1})}, \end{aligned}$$

where $O'(1)$ is a function depending only on n,α,x,y' and satisfies $|O'(1)| \le C_{n,\alpha}$, thus, for $|s| \le 1/2$ and $j=0,1,2$,

$$\left|\frac{d^j\sigma^{(1,x',y')}(s)}{ds^j}\right| \le C\|\Omega(\cdot,y')\|_{L^1(S^{n-1})}. \tag{21}$$

For $|x| > 4$,

$$\begin{aligned} &D_x^\alpha \mathcal{R}'(\lambda(|\cdot|)|\cdot|^{1-n}\Omega(\cdot,y'))(x) \\ &\quad= \int\limits_{\mathbb{R}^n}\left(D_x^\alpha\frac{c_n(x-u)}{|x-u|^{n+1}} - D_x^\alpha\frac{c_n x}{|x|^{n+1}}\right)\lambda(u)|u|^{1-n}\Omega(u,y')\,du \\ &\quad= |x|^{-n-1-|\alpha|}O'(1)\|\Omega(\cdot,y')\|_{L^1(S^{n-1})}, \end{aligned}$$

thus, for $|s| \ge 4$ and $j=0,1,2$,

$$\left|\frac{d^j\sigma^{(1,x',y')}(s)}{ds^j}\right| \le Cs^{-2-j}\|\Omega(\cdot,y')\|_{L^1(S^{n-1})}. \tag{22}$$

For $1/2 \le |x| \le 4$, let $r=|x|$; then

$$\begin{aligned} &\frac{d^j}{ds^j}\mathcal{R}'(\lambda(|\cdot|)|\cdot|^{1-n}\Omega(\cdot,y'))(x) \\ &\qquad= s^{-j}\mathcal{R}'\left(\Omega(\cdot,y')\sum_{k=0}^{j} a_k^{(j)}|\cdot|^{k+1-n}\,\frac{d^k\lambda(|\cdot|)}{ds^k}\right)(x) \end{aligned}$$

for $j=0,1,2$, where $a_k^{(j)}$'s are some constants. Let

$$\lambda_j(s) = \sum_{k=0}^{j} a_k^{(j)} s^{k+1-n}\,\frac{d^k\lambda(s)}{ds^k}.$$

Then $\lambda_j \in C_c^\infty(\mathbb{R}^1_+)$ because $\lambda \in C_c^\infty(\mathbb{R})$ and $\operatorname{supp}(\lambda) \subset [1,2]$. By integration by parts, we have

$$\left|\frac{d^j}{ds^j}\mathcal{R}'(\lambda(|\cdot|)|\cdot|^{1-n}\Omega(\cdot,y'))(x)\right|$$
$$\le \left|\frac{d^j}{ds^j}\mathcal{R}'(\lambda(|\cdot|)|\cdot|^{1-n}\Omega(\cdot,y'))(x'/2)\right| + \int_{1/2}^{4}\left|\frac{d^{j+1}}{ds^{j+1}}\mathcal{R}'(\lambda(|\cdot|)|\cdot|^{1-n}\Omega(\cdot,y'))(sx')\right| ds$$
$$\le C\|\Omega(\cdot,y')\|_{L^1(S^{n-1})} + C\int_{1/2}^{4}|\mathcal{R}'(\lambda_{j+1}(|\cdot|)\Omega(\cdot,y'))(sx')|\,ds,$$

thus, for $1/2 \le |s| \le 4$ and $j = 0,1$,

$$\left|\frac{d^j\sigma^{(1,x',y')}(s)}{ds^j}\right| \le C\|\Omega(\cdot,y')\|_{L^1(S^{n-1})} + C\sum_{j=0}^{1}\int_{1/2}^{4}|\mathcal{R}'(\lambda_{j+1}(|\cdot|)\Omega(\cdot,y'))(sx')|\,ds. \tag{23}$$

Combining (21)–(23), we get

$$\Omega_1^*(x',y') \le C\|\Omega(\cdot,y')\|_{L^1(S^{n-1})} + C\sum_{j=1}^{2}\int_{1/2}^{4}|\mathcal{R}'(\lambda_j(|\cdot|)\Omega(\cdot,y'))(sx')|\,ds. \tag{24}$$

By $L\log^+ L(\mathbb{R}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}^n)$ boundedness of $\mathcal{R}'$, we have

$$\int_{1/2\le|x|\le4}|\mathcal{R}'(\lambda_j(|\cdot|)\Omega(\cdot,y'))(x)|\,dx \le C(1+\|\Omega(\cdot,y')\|_{L\log^+ L(S^{n-1})}). \tag{25}$$

From (24)–(25), we get the assertion of the lemma ∎

3.3.2. *Proof of Lemma 6.* Let us first introduce some notations. Define $\Lambda = \{-1,0,1\}$, $E_{-1} = \{t \in \mathbb{R}^1 : |t| < 1/2\}$, $E_0 = \{t \in \mathbb{R}^1 : 1/2 \le |t| \le 4\}$, $E_1 = \{t \in \mathbb{R}^1 : |t| > 4\}$. Then

$$\Omega^*(x',y') \le \sum_{(\gamma,\delta)\in\Lambda\times\Lambda,\, i,j=0,1}\ \sup_{(t,s)\in E_\gamma\times E_\delta}(1+|t|)^{2+j}(1+|s|)^{2+i} \times|\partial_t^j\partial_s^i(\varsigma^{(x',y')}(t,s))| =: \sum_{(\gamma,\delta)\in\Lambda\times\Lambda,\, i,j=0,1}\Omega^*_{\gamma,\delta}(x',y'). \tag{26}$$

For $\Omega^*_{-1,-1}$, noting that $\mathrm{supp}(\lambda) \subset [1,2]$, so for $(t,s) \in E_{-1}\times E_{-1}$, we have

$$|D_x^\alpha D_y^\beta \mathcal{R}'\mathcal{R}''(\lambda(|\cdot|)\lambda(|\circ|)\Omega(\cdot,\circ)|\cdot|^{1-n}|\circ|^{1-m})(x,y)|$$
$$\le \iint\limits_{\mathbb{R}^n\times\mathbb{R}^m} \left| D_x^\alpha \frac{c_n(x-u)}{|x-u|^{n+1}} \cdot \frac{\lambda(|u|)}{|u|^{n-1}} \cdot D_y^\beta \frac{c_m(y-v)}{|y-v|^{m+1}} \cdot \frac{\lambda(|v|)}{|v|^{m-1}}\, \Omega(u',v') \right| du\, dv$$
$$\le C_{n,m}\|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}.$$

Hence

$$\iint\limits_{S^{n-1}\times S^{m-1}} |\Omega^*_{-1,-1}(x',y')|\, dx'dy' \le C_{n,m}\|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}. \tag{27}$$

For $\Omega^*_{-1,0}$, following the same idea of the proof of Lemma 5, we find that for $i=0,1,2$,

$$|D_x^\alpha \partial_s^i \mathcal{R}'\mathcal{R}''(\lambda(|\cdot|)\lambda(|\circ|)\Omega(\cdot,\circ)|\cdot|^{1-n}|\circ|^{1-m})(x,sy')|$$
$$\le |D_x^\alpha \mathcal{R}'\mathcal{R}''(\lambda(|\cdot|)\lambda^{(i)}(|\circ|)\Omega(\cdot,\circ)|\cdot|^{1-n})(x,sy')|,$$

where $\lambda^{(i)}(s) = \sum_{k=0}^i b_k^{(i)} s^{k+1-m} d^k\lambda(s)/ds^k \in C_c^\infty(\mathbb{R}^1_+)$ and $b_k^{(i)}$'s are some constants depending only on m,k,i. By integration by parts, we deduce that for $|\alpha|\le 1$, $i=0,1$,

$$|D_x^\alpha \mathcal{R}'\mathcal{R}''(\lambda(|\cdot|)\lambda^{(i)}(|\circ|)\Omega(\cdot,\circ)|\cdot|^{1-n})(x,sy')|$$
$$\le |D_x^\alpha \mathcal{R}'\mathcal{R}''(\lambda(|\cdot|)\lambda^{(i)}(|\circ|)\Omega(\cdot,\circ)|\cdot|^{1-n})(x,y'/2)|$$
$$+ \int\limits_{1/2}^{4} |D_x^\alpha \mathcal{R}'\mathcal{R}''(\lambda(|\cdot|)\lambda^{(i+1)}(|\circ|)\Omega(\cdot,\circ)|\cdot|^{1-n})(x,sy')|\, ds$$
$$\le C_{n,m}\|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}$$
$$+ \int\limits_{1/2}^{4} \left| D_x^\alpha \mathcal{R}'\mathcal{R}''\left(\frac{\lambda(|\cdot|)\lambda^{(i+1)}(|\circ|)\Omega(\cdot,\circ)}{|\cdot|^{n-1}}\right)(x,sy')\right| ds,$$

while for $|\alpha|\le 1$ and $i=0,1$,

$$\int\limits_{1/2}^{4} |D_x^\alpha \mathcal{R}'\mathcal{R}''(\lambda(|\cdot|)\lambda^{(i+1)}(|\circ|)\Omega(\cdot,\circ)|\cdot|^{1-n})(x,sy')|\, ds$$
$$\le \int\limits_{\mathbb{R}^n} \left| D_x^\alpha \frac{c_n(x-u)}{|x-u|^{n+1}} \cdot \frac{\lambda(|u|)}{|u|^{n-1}} \right| \cdot \int\limits_{1/2}^{4} |\mathcal{R}''(\lambda^{(i+1)}(|\circ|)\Omega(\cdot,\circ))(x,sy')|\, ds\, du$$
$$\le C \int\limits_{|u|\le 2} \int\limits_{1/2}^{4} |\mathcal{R}''(\lambda^{(i+1)}(|\circ|)\Omega(\cdot,\circ))(x,sy')|\, ds\, du.$$

By local boundedness of $\mathcal{R}''$, we have

$$\iint_{S^{n-1}\times S^{m-1}} |\Omega^*_{-1,0}(x',y')|\,dx'\,dy' \le C_{n,m}(1+\|\Omega\|_{L\log^+ L(S^{n-1}\times S^{m-1})}). \tag{28}$$

Similarly, we have

$$\iint_{S^{n-1}\times S^{m-1}} |\Omega^*_{0,-1}(x',y')|\,dx'\,dy' \le C_{n,m}(1+\|\Omega\|_{L\log^+ L(S^{n-1}\times S^{m-1})}). \tag{29}$$

For $\Omega^*_{0,0}$, by integration by parts, we infer that for $i,j=0,1$,

$$\begin{aligned}
&|\partial_r^j\partial_s^i\mathcal{R}'\mathcal{R}''(\lambda(|\cdot|)\lambda(|\circ|)\Omega(\cdot,\circ)|\cdot|^{1-n}|\circ|^{1-m})(rx',sy')| \\
&\quad\le \left|\partial_r^j\partial_s^i\mathcal{R}'\mathcal{R}''\left(\lambda(|\cdot|)\lambda(|\circ|)\,\frac{\Omega(\cdot,\circ)}{|\cdot|^{n-1}|\circ|^{m-1}}\right)(x'/2,y'/2)\right| \\
&\qquad+\int_{1/2}^{4}\left|\partial_r^j\partial_s^{i+1}\mathcal{R}'\mathcal{R}''\left(\lambda(|\cdot|)\lambda(|\circ|)\,\frac{\Omega(\cdot,\circ)}{|\cdot|^{n-1}|\circ|^{m-1}}\right)(x'/2,sy')\right|ds \\
&\qquad+\int_{1/2}^{4}\left|\partial_r^{j+1}\partial_s^{i}\mathcal{R}'\mathcal{R}''\left(\lambda(|\cdot|)\lambda(|\circ|)\,\frac{\Omega(\cdot,\circ)}{|\cdot|^{n-1}|\circ|^{m-1}}\right)(rx',y'/2)\right|dr \\
&\qquad+\iint_{[1/2,4]^2}\left|\partial_r^{j+1}\partial_s^{i+1}\mathcal{R}'\mathcal{R}''\left(\lambda(|\cdot|)\lambda(|\circ|)\frac{\Omega(\cdot,\circ)}{|\cdot|^{n-1}|\circ|^{m-1}}\right)(rx',sy')\right|ds\,dr \\
&\quad=: I(x',y')+II(x',y')+III(x',y')+IV(x',y').
\end{aligned}$$

Similarly to (27), we have

$$\iint_{S^{n-1}\times S^{m-1}} \sup_{1/2\le r\le 4,\,1/2\le s\le 4} I(x',y')\,dx'dy' \le C_{n,m}\|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}. \tag{30}$$

And, similarly to (28)–(29), we obtain

$$\begin{aligned}\iint_{S^{n-1}\times S^{m-1}} \sup_{1/2\le r\le 4,\,1/2\le s\le 4} A(x',y')\,dx'\,dy'& \\ \le C_{n,m}(1+\|\Omega\|_{L\log^+ L(S^{n-1}\times S^{m-1})}),&\end{aligned} \tag{31}$$

where $A=II$ or III. To estimate IV, by the local boundedness of $\mathcal{R}'\mathcal{R}''$ from $L(\log^+ L)^2(\mathbb{R}^n\times\mathbb{R}^m)$ to $L^1_{\mathrm{loc}}(\mathbb{R}^n\times\mathbb{R}^m)$ (see [6]) and the above estimation method, it is not difficult to show that

$$(32)\quad \iint\limits_{S^{n-1}\times S^{m-1}} \sup_{1/2\le r\le 4,\,1/2\le s\le 4} III(x',y')\,dx'\,dy'$$
$$\le \iint\limits_{1/2\le|x|\le 4,\,1/2\le|y|\le 4} |\mathcal{R}'\mathcal{R}''(\lambda_{j+1}(|\cdot|)\lambda^{(i+1)}(|\circ|)\Omega(\cdot,\circ))(x,y)|\,dx\,dy$$
$$\le C_{n,m}(1+\|\Omega\|_{L(\log^+ L)^2(S^{n-1}\times S^{m-1})}).$$

Combining (30)–(32), we get

$$(33)\quad \iint\limits_{S^{n-1}\times S^{m-1}} |\Omega^*_{0,0}(x',y')|\,dx'dy' \le C_{n,m}(1+\|\Omega\|_{L(\log^+ L)^2(S^{n-1}\times S^{m-1})}).$$

For $\Omega^*_{1,1}$, by cancellation properties of Ω, we have

$$|D^\alpha_x D^\beta_y \mathcal{R}'\mathcal{R}''(\lambda(|\cdot|)\lambda(|\circ|)\Omega(\cdot,\circ)|\cdot|^{1-n}|\circ|^{1-m})(x,y)|$$
$$\le \iint\limits_{\mathbb{R}^n\times\mathbb{R}^m} \bigg| D^\alpha_x\bigg(\frac{c_n(x-u)}{|x-u|^{n+1}} - \frac{c_n x}{|x|^{n+1}}\bigg) D^\beta_y\bigg(\frac{c_m(y-v)}{|y-v|^{m+1}} - \frac{c_m y}{|y|^{m+1}}\bigg)$$
$$\times \frac{\lambda(|u|)\lambda(|v|)\Omega(u',v')}{|u|^{n-1}|v|^{m-1}}\bigg|\,du\,dv$$
$$\le C_{n,m}|x|^{-n-1-|\alpha|}|y|^{-m-1-|\beta|}\|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}.$$

Hence,

$$(34)\quad \iint\limits_{S^{n-1}\times S^{m-1}} |\Omega^*_{1,1}(x',y')|\,dx'\,dy' \le C_{n,m}\|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}.$$

Combining all the above estimation methods, we can easily prove

$$(35)\quad \iint\limits_{S^{n-1}\times S^{m-1}} |A(x',y')|\,dx'\,dy' \le C_{n,m}(1+\|\Omega\|_{L\log^+ L(S^{n-1}\times S^{m-1})}),$$

where $A=\Omega^*_{1,0}$ or $\Omega^*_{0,1}$.

From (26), (27)–(29), (33)–(35), we get Lemma 6 easily. ∎

4. Proof of Theorem 1 for $1<p<\infty$. We have $\mu_\Omega(f)(x,y)\equiv \widetilde{g}_{\varphi^\Omega}(f)(x,y)$ where φ^Ω is defined in (3). Without loss of generality, we suppose Ω is even both in the first and in the second variable (for the odd case, things are much easier to deal with). Now,

$$\varphi^\Omega_{t,s}*f(x,y) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^m} \frac{t^{-1}s^{-1}\Omega(u',v')}{|u|^{n-1}|v|^{m-1}}\,\chi_{|u|\le t}(u)\chi_{|v|\le s}(v)f(x-u,y-v)\,du\,dv$$
$$= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^m} \frac{\Omega(u',v')}{|u|^{n-1}|v|^{m-1}}\,(\varrho_t(|u|)\varrho_s(|v|)+\lambda_t(|u|)\varrho_s(|v|)$$
$$+\varrho_t(|u|)\lambda_s(|v|)+\lambda_t(|u|)\lambda_s(|v|))f(x-u,y-v)\,du\,dv$$

$$= \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v') \int_{\mathbb{R}^2} \varrho_t(t')\varrho_s(s') f(x-t'u', y-s'v')\, dt'\, ds'\, du'\, dv'$$
$$+ \frac{1}{2} \iint_{S^{n-1}\times S^{m-1}} \iint_{\mathbb{R}^2} \sigma_t^{(1,u',v')}(t')\varrho_s(s')\mathcal{R}' f(x-t'u', y-s'v')\, dt'\, ds'\, du'\, dv'$$
$$+ \frac{1}{2} \iint_{S^{n-1}\times S^{m-1}} \iint_{\mathbb{R}^2} \varrho_t(t')\sigma_s^{(2,u',v')}(s')\mathcal{R}'' f(x-t'u', y-s'v')\, dt'\, ds'\, du'\, dv'$$
$$+ \frac{1}{4} \iint_{S^{n-1}\times S^{m-1}} \iint_{\mathbb{R}^2} \varsigma_{t,s}^{(u',v')}(t',s')\mathcal{R}'\mathcal{R}'' f(x-t'u', y-s'v')\, dt'\, ds'\, du'\, dv'.$$

Decompose $\mathbb{R}^n = L(u') + L(u')^\perp$, $\mathbb{R}^m = L(v') + L(v')^\perp$ and $f_{x',y'}^{u',v'}(a,b) = f(x'+au', y'+bv')$, and let K'_ϱ, K''_ϱ denote the K_ϱ acting on the first and the second variables respectively. Then

$$\begin{aligned}(36)\qquad \widetilde{K}_{\varphi^\Omega} * f(x,y) = &\iint_{S^{n-1}\times S^{m-1}} \Omega(u',v') K'_\varrho K''_\varrho * f_{x',y'}^{u',v'}(a,b)\, du'\, dv' \\ &+ \int_{S^{n-1}\times S^{m-1}} \frac{1}{2} K'_{\sigma^{(1,u',v')}} K''_\varrho * (\mathcal{R}'f)_{x',y'}^{u',v'}(a,b)\, du'\, dv' \\ &+ \iint_{S^{n-1}\times S^{m-1}} \frac{1}{2} K'_\varrho K''_{\sigma^{(2,u',v')}} * (\mathcal{R}''f)_{x',y'}^{u',v'}(a,b)\, du'\, dv' \\ &+ \iint_{S^{n-1}\times S^{m-1}} \frac{1}{4} \widetilde{K}_{\varsigma^{(u',v')}} * (\mathcal{R}'\mathcal{R}''f)_{x',y'}^{u',v'}(a,b)\, du'\, dv'.\end{aligned}$$

Thus, by the rotation method, Lemmas 4–6, and the boundedness of the Riesz transform, we get

$$\|\mathcal{R}'(f)\|_p + \|\mathcal{R}''(f)\|_p \le C_{n,m}(pp')\|f\|_p,$$
$$\|\mathcal{R}'\mathcal{R}''(f)\|_p \le C_{n,m}(pp')^2\|f\|_p;$$

furthermore,

$$(37)\qquad \|\mu_\Omega(f)\|_p = \big\||\widetilde{K}_{\varphi^\Omega} * f|_{\widetilde{\mathcal{H}}}\big\|_p \le C_{n,m}(pp')^8(1 + \|\Omega\|_{L\log^+ L^2})\|f\|_p.$$

Now, for $\Omega \in L(\log^+ L)^{2/q}$ where $q \in (1,2]$, define

$$\Omega'_z(x',y') = \Omega(x',y')(1 + \log^+|\Omega(x',y')|)^{z-2/q'},$$
$$\Omega_z(x',y') = \Omega'_z(x',y') + \frac{1}{\sigma(S^{n-1})\sigma(S^{m-1})} \iint_{S^{n-1}\times S^{m-1}} \Omega'_z(x',y')\, dx'\, dy'$$
$$- \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} \Omega'_z(x',y')\, dx' - \frac{1}{\sigma(S^{m-1})} \int_{S^{m-1}} \Omega'_z(x',y')\, dy',$$

where z is a complex number and $\sigma(S^k)$ is the surface area of S^k. Notice that Ω_z satisfies (2) and

$$\sup_{\mathrm{Re}(z)\le 2}\|\Omega_z\|_1 \le C \iint\limits_{S^{n-1}\times S^{m-1}} |\Omega(x',y')|(1+|\Omega(x',y')|^{2/q})\,dx'\,dy'.$$

By (37), for $\mathrm{Re}(z)=0$, $1<s<2$,

$$\|\mu_{\Omega_z}(f)\|_{L^s(\mathbb{R}^n\times\mathbb{R}^m)} \le C(s-1)^{-8}(1+\|\Omega_z\|_{L(\log^+ L)^2})\|f\|_{L^s(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{38}$$

By (12), for $\mathrm{Re}(z)=1$,

$$\|\mu_{\Omega_z}(f)\|_2 \le C(1+\|\Omega_z\|_{L\log^+ L})\|f\|_2. \tag{39}$$

But it is easy to see that

$$\sup_{\mathrm{Re}(z)=0}(1+\|\Omega_z\|_{L(\log^+ L)^2})+\sup_{\mathrm{Re}(z)=1}(1+\|\Omega_z\|_{L\log^+ L}) \le C(1+\|\Omega\|_{L(\log^+ L)^{2/q}}).$$

So, for

$$\frac{1}{s}=\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q'}\right)\Big/\left(\frac{1}{2}-\frac{1}{q'}\right),\quad \text{i.e.}\quad \frac{1}{p}=\left(1-\frac{2}{q'}\right)\frac{1}{s}+\frac{2}{q'}\frac{1}{2},$$

by (38)–(39), the vector-valued interpolation of Stein for analytic families of operators and the fact that $\Omega=\Omega_{2/q'}$, we get

$$\|\mu_\Omega(f)\|_p \le C\left(\frac{1}{q}-\frac{1}{p}\right)^{-8(1-2/q')}(1+\|\Omega\|_{L(\log^+ L)^{2/q}})\|f\|_p. \tag{40}$$

Replacing Ω by $A\Omega$ in (40) for $A>0$, we get

$$\|\mu_\Omega(f)\|_p \le C\left(\frac{1}{q}-\frac{1}{p}\right)^{-8(1-2/q')}(A^{-1}+\|\Omega\|_{L(\log^+(AL))^{2/q}})\|f\|_p. \tag{41}$$

Let $j_0=[p']+1$ and define

$$E_{j_0}=\{(x',y')\in S^{n-1}\times S^{m-1} : \log^+|\Omega(x',y')|\le 2^{j_0}\},$$
$$E_j=\{(x',y')\in S^{n-1}\times S^{m-1} : 2^j<\log^+|\Omega(x',y')|\le 2^{j+1}\},\ j>j_0,$$

$$\Omega_j'(x',y')=\Omega(x',y')\chi_{E_j}(x',y'),$$

$$\Omega_j(x',y')=\Omega_j'(x',y')+\frac{1}{\sigma(S^{n-1})\sigma(S^{m-1})}\iint\limits_{S^{n-1}\times S^{m-1}}\Omega_j'(x',y')\,dx'\,dy'$$
$$-\frac{1}{\sigma(S^{n-1})}\int\limits_{S^{n-1}}\Omega_j'(x',y')\,dx'-\frac{1}{\sigma(S^{m-1})}\int\limits_{S^{m-1}}\Omega_j'(x',y')\,dy'.$$

Then Ω_j satisfies (2) and $\Omega=\sum_{j=j_0}^{\infty}\Omega_j$. Applying (41) to Ω_j with $1/q=1/p+1/j$ and $A=j^{10}$, we get

$$(42)\qquad \|\mu_\Omega(f)\|_p \le \sum_{j=j_0}^{\infty} \|\mu_{\Omega_j}(f)\|_p \le C\sum_{j=j_0}^{\infty} j^{8(1-2/q')}(j^{-10}+\|\Omega_j\|_{L(\log^+(j^{10}L))^{2/q}})\|f\|_p \le C(1+\|\Omega\|_{L(\log^+ L)^{2/p}(\log^+\log^+ L)^{8(1-2/p')}})\|f\|_p,$$

which finishes the case $1 < p \le 2$ of Theorem 1.

For $2 < p < \infty$, replacing q by q' in the proof of (41), we can show

$$\|\mu_\Omega(f)\|_p \le C\left(\frac{1}{q}-\frac{1}{p}\right)^{-8(1-2/q)}(A^{-1}+\|\Omega\|_{L(\log^+(AL))^{2/q'}})\|f\|_p.$$

Then, taking $j_0 = [p]+1$, $1/q = 1/p + 1/j$ and $A = j^{10}$, we can also get (42) for $2 < p < \infty$. ■

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