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Homotonic algebras

by

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Abstract. An algebra $\mathcal{A}$ of real- or complex-valued functions defined on a set $\mathbf{T}$ shall be called *homotonic* if $\mathcal{A}$ is closed under taking absolute values, and for all f and g in $\mathcal{A}$, the product $f \times g$ satisfies $|f \times g| \le |f| \times |g|$. Our main purpose in this paper is two-fold: to show that the above definition is equivalent to an earlier definition of homotonicity, and to provide a simple inequality which characterizes submultiplicativity and strong stability for weighted sup norms on homotonic algebras.

1. Definition and examples. Throughout this paper, let $\mathcal{A}$ denote a (finite- or infinite-dimensional) algebra over a field $\mathbb{F}$, either $\mathbb{R}$ or $\mathbb{C}$, of $\mathbb{F}$-valued functions defined on a given set $\mathbf{T}$. As usual, addition and scalar multiplication in $\mathcal{A}$ will be defined pointwise, i.e., for all f and g in $\mathcal{A}$, and all α in $\mathbb{F}$,

$$(f+g)(t) = f(t) + g(t), \qquad (\alpha f)(t) = \alpha f(t).$$

Multiplication, often not pointwise, will be denoted by $\times$.

Definition 1.1. Let $\mathcal{A}$ be as above. We say that $\mathcal{A}$ is *homotonic* if:

(i) $\mathcal{A}$ is closed under taking absolute values, i.e., $f \in \mathcal{A}$ implies $|f| \in \mathcal{A}$.
(ii) For any two elements f and g in $\mathcal{A}$, we have $|f \times g| \le |f| \times |g|$.

Here, for every $f \in \mathcal{A}$, the function $|f|$ is defined for each $t \in \mathbf{T}$ by $|f|(t) = |f(t)|$; and for real-valued functions f and g, the notation $f \le g$ will have the usual meaning, namely, $f(t) \le g(t)$ for all $t \in \mathbf{T}$.

We point out that Definition 1.1 does not require $\mathcal{A}$ to be associative.

We also note that property (ii) implies that the product of non-negative functions in $\mathcal{A}$ is non-negative.

Examples of homotonic algebras are not hard to come by.

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Example 1.1. Clearly, the algebra of all (bounded or not) $\mathbb{F}$-valued functions defined on a given set $\mathbf{T}$, with pointwise multiplication, is homotonic.

Example 1.2 (cf. [AG2]). A more interesting example of a homotonic algebra is given by $\mathbb{F}^{n\times n}$, the algebra of all $n \times n$ matrices over $\mathbb{F}$ with the usual matrix operations. This algebra consists, of course, of all $\mathbb{F}$-valued functions on the set

$$\mathbf{T} = \{(j,k) : j,k = 1,\ldots,n\}.$$

Example 1.3 ([G]). To further illustrate homotonicity, fix positive constants p and κ, and let $\mathcal{C}_{p,\kappa}(\mathbb{F})$ be the associative (and, in fact, commutative) algebra over $\mathbb{F}$ of all continuous, p-periodic, $\mathbb{F}$-valued functions on $\mathbb{R}$, where the product of f and g in $\mathcal{C}_{p,\kappa}(\mathbb{F})$ is defined by the convolution

$$(f * g)(t) = \kappa\int_0^p f(t-x)g(x)\,dx, \qquad t \in \mathbb{R}.$$

Surely, if f belongs to $\mathcal{C}_{p,\kappa}(\mathbb{F})$, so does $|f|$. Moreover, if f and g are members of $\mathcal{C}_{p,\kappa}(\mathbb{F})$, then

$$\begin{aligned}|f * g|(t) = |(f*g)(t)| = \kappa\Big|\int_0^p f(t-x)g(x)\,dx\Big| &\le \kappa\int_0^p |f(t-x)g(x)|\,dx\\ = \kappa\int_0^p |f|(t-x)|g|(x)\,dx &= (|f| * |g|)(t),\end{aligned}$$

hence $\mathcal{C}_{p,\kappa}(\mathbb{F})$ is homotonic.

This example is a convenient prototype of many instances of algebras of functions defined on a locally compact abelian group where multiplication is a scalar multiple of convolution defined with respect to Haar measure on the group.

Example 1.4. Let $\mathcal{A}$ be a homotonic algebra, and let $\mathcal{A}^+$ be the algebra obtained by replacing the original product $f \times g$ in $\mathcal{A}$ by the Jordan product

$$f \circ g \equiv \tfrac{1}{2}(f \times g + g \times f).$$

Then it is not hard to see that $\mathcal{A}^+$ is also homotonic. Indeed, if $\mathcal{A}$ is closed under taking absolute values, then so is $\mathcal{A}^+$. Further, if f and g are elements of $\mathcal{A}$ then, by the homotonicity of $\mathcal{A}$,

$$|f\circ g| = \tfrac{1}{2}|f\times g+g\times f| \le \tfrac{1}{2}(|f\times g|+|g\times f|) \le \tfrac{1}{2}(|f|\times|g|+|g|\times|f|) = |f|\circ|g|.$$

This example gives rise to straightforward constructions of *non-associative* homotonic algebras. For instance, take $\mathcal{A}$ to be $\mathbb{F}^{n\times n}$ ($n \ge 2$), and consider $\mathbb{F}^{n\times n+}$, obtained by adopting the Jordan product $A \circ B \equiv \frac{1}{2}(AB + BA)$. For

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus O_{n-2}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus O_{n-2},$$

O_{n-2} denoting the $(n-2)\times(n-2)$ zero matrix, we have

$$\begin{aligned}(A \circ B) \circ B &= \tfrac{1}{2}(AB + BA) \circ B = \tfrac{1}{4}[(AB + BA)B + B(AB + BA)] \\ &= \tfrac{1}{4}(AB^2 + 2BAB + B^2A) = \tfrac{1}{2}B\end{aligned}$$

and

$$A \circ (B \circ B) = A \circ B^2 = \tfrac{1}{2}(AB^2 + B^2A) = 0.$$

Hence, $\mathbb{F}^{n\times n+}$ fails to be associative, although $\mathbb{F}^{n\times n}$ is.

EXAMPLE 1.5. We note that if $\mathcal{B}$ is a subalgebra of a homotonic algebra $\mathcal{A}$, then evidently, $\mathcal{B}$ is homotonic if and only if $\mathcal{B}$ is closed under taking absolute values. For instance, consider the matrix algebra

$$\mathcal{A}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} \tag{1.1}$$

with the usual matrix operations. Since this subalgebra of $\mathbb{R}^{2\times 2}$ is not closed under taking absolute values, it is not homotonic.

In the case where $\mathbb{F} = \mathbb{R}$, we can replace condition (ii) in Definition 1.1 by a simpler condition:

THEOREM 1.1. *Let $\mathcal{A}$ be an algebra over $\mathbb{R}$ of real-valued functions defined on a given set $\mathbf{T}$. Then $\mathcal{A}$ is homotonic if and only if*:

(i) *$\mathcal{A}$ is closed under taking absolute values, i.e., $f \in \mathcal{A}$ implies $|f| \in \mathcal{A}$.*

$(\mathrm{ii})_{\mathbb{R}}$ *For each pair of non-negative functions f and g in $\mathcal{A}$, the product $f \times g$ is also non-negative.*

Proof. If condition (ii) holds then, for any non-negative functions f and g in $\mathcal{A}$, we have

$$f \times g = |f| \times |g| \geq |f \times g| \geq 0,$$

so $(\mathrm{ii})_{\mathbb{R}}$ is established. Hence, in order to complete the proof, it suffices to show that (i) and $(\mathrm{ii})_{\mathbb{R}}$ imply (ii).

Indeed, in view of (i), for each u in $\mathcal{A}$, the non-negative functions $u_+ \equiv \frac{1}{2}(|u|+u)$ and $u_- \equiv \frac{1}{2}(|u|-u)$ are both in $\mathcal{A}$. Moreover, we have $u = u_+ - u_-$ and $|u| = u_+ + u_-$. Thus, for every u and v in $\mathcal{A}$,

$$|u|\times|v| = (u_+ + u_-)\times(v_+ + v_-) = u_+\times v_+ + u_-\times v_- + u_+\times v_- + u_-\times v_+. \tag{1.2}$$

and

$$u\times v = (u_+ - u_-)\times(v_+ - v_-) = u_+\times v_+ + u_-\times v_- - u_+\times v_- - u_-\times v_+. \tag{1.3}$$

By $(\mathrm{ii})_{\mathbb{R}}$, the products $u_+ \times v_+$, $u_- \times v_-$, $u_+ \times v_-$ and $u_- \times v_+$ are all non-negative. So, comparing (1.2) and (1.3), we get

$$\begin{aligned}|u \times v| &= |u_+ \times v_+ + u_- \times v_- - u_+ \times v_- - u_- \times v_+| \\ &\le u_+ \times v_+ + u_- \times v_- + u_+ \times v_- + u_- \times v_+ = |u| \times |v|,\end{aligned}$$

and the proof follows. ■

Example 1.6. To illustrate this theorem, consider the familiar real vector space

$$\mathbb{R}^2 = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{R}\}.$$

For all (α, β) and (γ, δ) in $\mathbb{R}^2$, define multiplication by

$$(\alpha, \beta) \times (\gamma, \delta) = (\alpha\gamma - \beta\delta, \alpha\delta + \beta\gamma), \tag{1.4}$$

which makes $\mathbb{R}^2$ into a 2-dimensional algebra over the reals. Surely, $\mathbb{R}^2$ is closed under taking absolute values, i.e., condition (i) holds. We observe, however, that if α, β, γ and δ are positive numbers with $\alpha\gamma < \beta\delta$, then the first component of the product $(\alpha, \beta) \times (\gamma, \delta)$ is negative; so condition $(\mathrm{ii})_{\mathbb{R}}$ fails, and by Theorem 1.1, our algebra is not homotonic.

The mapping

$$(\alpha, \beta) \mapsto \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{R},$$

shows that the above algebra is an algebraically isomorphic image of the algebra $\mathcal{A}_2(\mathbb{R})$ defined in (1.1). In fact, the reader must have noticed by now that both these algebras are algebraically isomorphic to the complex numbers

$$\mathbb{C} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}$$

viewed as a 2-dimensional algebra over $\mathbb{R}$.

2. An earlier equivalent definition of homotonic algebras. The notion of homotonicity was first introduced in [AG2] in connection with functionals acting on a linear space $\mathbf{V}$ over $\mathbb{C}$ of bounded complex-valued functions defined on a given set $\mathbf{T}$. In the same paper, the idea of homotonicity was extended to mappings from $\mathbf{V}$ into $\mathbf{V}$, and then to multiplication with which $\mathbf{V}$ was given the structure of an associative algebra.

Adapting the definitions in [AG2], the term *homotonic algebra* was coined in [G]. There, an associative algebra of bounded $\mathbb{F}$-valued functions defined on $\mathbf{T}$ is called *homotonic* if:

(i) $\mathcal{A}$ is closed under taking absolute values.
(ii)′ For any four elements f_1, f_2, g_1 and g_2 in $\mathcal{A}$ such that $|f_1| \le g_1$ and $|f_2| \le g_2$, we have $|f_1 \times f_2| \le g_1 \times g_2$.

The name "homotonic" was chosen in this earlier definition because *homo* indicates that multiplication preserves the relation $|f| \leq g$ and *tonic* reflects the fact that this relation is about order.

We shall now show that even in the general case where $\mathcal{A}$ is not necessarily associative and the functions in $\mathcal{A}$ are not necessarily bounded, the old and new definitions of homotonicity coincide. More precisely, we post:

THEOREM 2.1. *Let $\mathcal{A}$ be an algebra over $\mathbb{F}$ of $\mathbb{F}$-valued functions defined on a set $\mathbf{T}$. Then $\mathcal{A}$ is homotonic if and only if conditions* (i) *and* (ii)$'$ *hold.*

Proof. Putting $f_1 = f$, $f_2 = g$, $g_1 = |f|$ and $g_2 = |g|$, we immediately observe that (ii)$'$ implies (ii). So assume that (i) and (ii) hold, and let us prove (ii)$'$, thus forcing the desired result.

If f and g are non-negative functions in $\mathcal{A}$, then by (ii),

$$f \times g = |f| \times |g| \geq |f \times g| \geq 0;$$

hence, as in the proof of Theorem 1.1, (ii) implies (ii)$_{\mathbb{R}}$. Let u, v and w be real-valued functions in $\mathcal{A}$ with $u \leq v$ and $w \geq 0$. Then, by (ii)$_{\mathbb{R}}$,

$$v \times w - u \times w = (v - u) \times w \geq 0;$$

so

$$u \leq v \text{ and } w \geq 0 \ \Rightarrow\ u \times w \leq v \times w. \tag{2.1}$$

Analogously, we get

$$u \leq v \text{ and } w \geq 0 \ \Rightarrow\ w \times u \leq w \times v. \tag{2.2}$$

Suppose now that f_1, f_2, g_1 and g_2 are arbitrary functions in $\mathcal{A}$ which satisfy $|f_1| \leq g_1$ and $|f_2| \leq g_2$. Then appealing to (ii), (2.1) and (2.2) (in that order), we obtain

$$|f_1 \times f_2| \leq |f_1| \times |f_2| \leq g_1 \times |f_2| \leq g_1 \times g_2,$$

and we are done. ∎

3. Submultiplicative weighted sup norms on homotonic algebras. Our study of homotonic algebras is motivated mainly by the following theorem which provides a simple characterization of submultiplicativity for weighted sup norms.

Here, as usual, we call a norm on an algebra $\mathcal{A}$ *submultiplicative* if

$$\|f \times g\| \leq \|f\| \, \|g\| \qquad \text{for all } f, g \in \mathcal{A}.$$

THEOREM 3.1 (cf. [AG2, Theorem 4.2]). *Let $\mathcal{A}$ be a homotonic algebra over $\mathbb{F}$ of $\mathbb{F}$-valued functions defined on a set $\mathbf{T}$. Let w be a fixed positive function on $\mathbf{T}$ (not necessarily in $\mathcal{A}$) such that w_{-1}, defined by $w_{-1}(t) = 1/w(t)$ for all $t \in \mathbf{T}$, is an element of $\mathcal{A}$. Assume that*

$$\sup_{t \in \mathbf{T}} w(t)|f(t)| < \infty \qquad \textit{for all } f \in \mathcal{A}.$$

Then the weighted sup norm

$$\|f\|_{w,\infty} \equiv \sup_{t\in\mathbf{T}} w(t)|f(t)|, \qquad f \in \mathcal{A}, \tag{3.1}$$

is submultiplicative on $\mathcal{A}$ *if and only if*

$$w_{-1} \times w_{-1} \le w_{-1}. \tag{3.2}$$

Proof. Suppose that $\|\cdot\|_{w,\infty}$ is submultiplicative. Since w_{-1} is a member of $\mathcal{A}$, it follows that

$$\|w_{-1} \times w_{-1}\|_{w,\infty} \le \|w_{-1}\|_{w,\infty}^2 = 1; \tag{3.3}$$

hence,

$$|w_{-1} \times w_{-1}| \le w_{-1}.$$

Since w_{-1} is a positive function, the homotonicity of $\mathcal{A}$ implies $w_{-1} \times w_{-1} \ge 0$; thus

$$w_{-1} \times w_{-1} = |w_{-1} \times w_{-1}| \le w_{-1} \tag{3.4}$$

and (3.2) is in the bag.

Conversely, let (3.2) hold. Set

$$\lambda \equiv \sup\{\|f \times g\|_{w,\infty} : f, g \in \mathcal{A},\ \|f\|_{w,\infty} = \|g\|_{w,\infty} = 1\},$$

and observe that $\|\cdot\|_{w,\infty}$ is submultiplicative if and only if $\lambda \le 1$. Select $f, g \in \mathcal{A}$ with $\|f\|_{w,\infty} = \|g\|_{w,\infty} = 1$; hence

$$|f| \le w_{-1} \quad \text{and} \quad |g| \le w_{-1}. \tag{3.5}$$

Since $\mathcal{A}$ is homotonic, Theorem 2.1 guarantees that condition (ii)$'$ holds. By (3.5), therefore,

$$|f \times g| \le w_{-1} \times w_{-1},$$

so aided by (3.4), we get

$$|f \times g| \le w_{-1}.$$

Consequently,

$$\|f \times g\|_{w,\infty} \le \|w_{-1}\|_{w,\infty} = 1;$$

whence $\lambda \le 1$, and the proof is complete. ∎

Example 3.1 (cf. [AG1, Theorem 1]). To illustrate Theorem 3.1, let us revisit $\mathbb{F}^{n\times n}$, the algebra of $n \times n$ matrices over $\mathbb{F}$ with the usual matrix operations. Let $W = (\omega_{jk})$ be a fixed $n \times n$ matrix of positive entries ω_{jk}, and consider the weighted sup norm

$$\|A\|_{W,\infty} = \max_{j,k} \omega_{jk}|\alpha_{jk}|, \qquad A = (\alpha_{jk}) \in \mathbb{F}^{n\times n}. \tag{3.6}$$

Let W_{-1} be the Hadamard inverse of W, that is, the matrix whose (j,k) entry is $1/\omega_{jk}$. Then, by the theorem, $\|\cdot\|_{W,\infty}$ is submultiplicative if and only if

$$(W_{-1})^2 \le W_{-1}, \tag{3.7}$$

where $(W_{-1})^2$ is the usual square of W_{-1}, and where the inequality in (3.7) is construed entrywise. For instance (cf. [GS, Corollary 1.1]), selecting $W = \mu E$, where μ is a positive constant and E is the matrix all of whose entries are 1, we easily find that the norm in (3.6) is multiplicative if and only if

$$\mu \ge n.$$

In other words, the norm

$$\|A\|_{\mu,\infty} \equiv \mu \max_{j,k} |\alpha_{jk}|, \qquad A = (\alpha_{jk}) \in \mathbb{F}^{n\times n},$$

is submultiplicative if and only if $\mu \ge n$.

Surely, the results in this example remain valid when the sup norm in (3.6) is applied to the *non-associative* algebra $\mathbb{F}^{n\times n+}$ defined in Example 1.4.

EXAMPLE 3.2 ([G]). Falling back on the algebra $\mathcal{C}_{p,\kappa}(\mathbb{F})$ of Example 1.3, we let w be a continuous, p-periodic, positive function on $\mathbb{R}$. Then, evidently, w_{-1} belongs to $\mathcal{C}_{p,\kappa}(\mathbb{F})$; so by Theorem 3.1, the w-weighted sup norm

$$\|f\|_{w,\infty} = \max_{0\le t\le p} w(t)|f(t)|, \qquad f \in \mathcal{C}_{p,\kappa}(\mathbb{F}),$$

is submultiplicative if and only if $w_{-1} * w_{-1} \le w_{-1}$; that is, precisely when

$$\kappa \int_0^p \frac{dx}{w(t-x)w(x)} \le \frac{1}{w(t)}, \qquad 0 \le t \le p.$$

In particular, we see that the usual sup norm

$$\|f\|_\infty = \max_{0\le t\le p} |f(t)|, \qquad f \in \mathcal{C}_{p,\kappa}(\mathbb{F}),$$

is submultiplicative if and only if $\kappa p \le 1$.

Our next example involves an algebra of *unbounded* functions where the weight function w is *not* a member of $\mathcal{A}$.

EXAMPLE 3.3. Set $\mathbf{T} = (0,\infty)$, and let $\mathcal{A}$ be the real vector space of all functions on $\mathbf{T}$ of the form $f(t) = \alpha t$ where α is a real constant. For each f and g in $\mathcal{A}$, define the product $f \times g$ by

$$(f \times g)(t) = \frac{f(t)g(t)}{t}, \qquad t \in \mathbf{T},$$

thus making $\mathcal{A}$ into a homotonic algebra which is a faithful image of $\mathbb{R}$. Let $w : \mathbf{T} \to \mathbb{R}$ be the positive unbounded function $w(t) = \nu t^{-1}$ where ν is a positive constant. Note that w is not an element of $\mathcal{A}$ but w_{-1} is. With this choice of w, and for each $f(t) = \alpha t$ in $\mathcal{A}$, we have

$$\sup_{t\in\mathbf{T}} w(t)|f(t)| = \nu|\alpha| < \infty.$$

Hence, by Theorem 3.1, the weighted sup norm

$$\|f\|_{w,\infty} = \sup_{t\in\mathbf{T}} w(t)|f(t)|$$

is submultiplicative on $\mathcal{A}$ if and only if $w_{-1} \times w_{-1} \le w_{-1}$; that is, if and only if $\nu \ge 1$.

4. Strongly stable weighted sup norms on homotonic algebras. As usual, whether the algebra $\mathcal{A}$ is associative or not, we define powers of each element $f \in \mathcal{A}$ inductively by

$$f^1 = f; \quad f^k = f^{k-1} \times f, \quad k = 2, 3, 4, \ldots$$

Having powers at our disposal, we say that a norm $\|\cdot\|$ on $\mathcal{A}$ is *strongly stable* if

$$\|f^k\| \le \|f\|^k \quad \text{for all } f \in \mathcal{A} \text{ and } k = 1, 2, 3, \ldots$$

With these definitions, we can easily characterize strong stability for weighted sup norms on homotonic algebras.

Theorem 4.1 (cf. [AG2, Theorem 4.2]). *Let $\mathcal{A}$ be a homotonic algebra over $\mathbb{F}$ of $\mathbb{F}$-valued functions defined on a set $\mathbf{T}$. Let w be a fixed positive function on $\mathbf{T}$ (not necessarily in $\mathcal{A}$) such that w_{-1} belongs to $\mathcal{A}$. Assume that*

$$\sup_{t\in\mathbf{T}} w(t)|f(t)| < \infty \quad \textit{for all } f \in \mathcal{A}.$$

Then the weighted sup norm $\|\cdot\|_{w,\infty}$ in (3.1) is strongly stable if and only if

$$w_{-1} \times w_{-1} \le w_{-1}.$$

Proof. If $w_{-1} \times w_{-1} \le w_{-1}$, then by Theorem 3.1, $\|\cdot\|_{w,\infty}$ is submultiplicative, hence strongly stable since for all f in $\mathcal{A}$,

$$\|f^k\| = \|f^{k-1} \times f\| \le \|f^{k-1}\|\,\|f\|, \quad k = 2, 3, 4, \ldots$$

Conversely, if $\|\cdot\|_{w,\infty}$ is strongly stable, then

$$\|f \times f\|_{w,\infty} \le \|f\|_{w,\infty}^2 \quad \text{for all } f \in \mathcal{A}.$$

So setting $f = w_{-1}$, we get (3.3) and $w_{-1} \times w_{-1} \le w_{-1}$ follows. ∎

Theorems 3.1 and 4.1 show, of course, that in the homotonic case, submultiplicativity and strong stability are equivalent for weighted sup norms. It thus follows that the examples presented in Section 3 are also relevant here, in the sense that in each of those examples, the condition given for submultiplicativity is also necessary and sufficient for strong stability.

We conclude by remarking that in general, a strongly stable norm on a (homotonic) algebra may fail to be submultiplicative. A familiar example is the numerical radius,

$$r(A) = \max\{|(Ax, x)| : x \in \mathbb{C}^n,\ (x, x) = 1\},$$

defined on $\mathbb{C}^{n\times n}$ $(n \geq 2)$, with respect to a given inner product $(\cdot,\cdot)$ on $\mathbb{C}^n$. It is well known (e.g., [H, Chapter 17]) that r is a norm on $\mathbb{C}^{n\times n}$ which is not submultiplicative; on the other hand, the celebrated Berger Inequality [B, P] tells us that r is strongly stable.

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