

Using boundaries to find smooth norms

by

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Abstract. The aim of this paper is to present a tool used to show that certain Banach spaces can be endowed with C^k smooth equivalent norms. The hypothesis uses particular countable decompositions of certain subsets of B_{X^*} , namely boundaries. Of interest is that the main result unifies two quite well known results. In the final section, some new corollaries are given.

1. Introduction. We say the norm of a Banach space $(X, \|\cdot\|)$ is C^k smooth if its k th Fréchet derivative exists and is continuous at every point of $X \setminus \{0\}$. The norm is C^∞ smooth if this holds for all $k \in \mathbb{N}$. This paper is concerned with the problem of establishing sufficient conditions for when a Banach space has a C^k smooth renorming, for $k \in \mathbb{N} \cup \{\infty\}$.

DEFINITION 1.1. A subset $B \subseteq B_{X^*}$ is called a *boundary* if for each x in the unit sphere S_X , there exists $f \in B$ such that $f(x) = 1$.

EXAMPLE 1.2. The following are boundaries for any Banach space X :

- (1) The dual unit sphere S_{X^*} . This is a consequence of the Hahn–Banach Theorem.
- (2) The set of extreme points of the dual unit ball, $\text{Ext}(B_{X^*})$. This follows from the proof of the Krein–Milman Theorem [3, Fact 3.45].

Given $\varepsilon > 0$ and norms $\|\cdot\|$ and $\|\!\|\!\cdot\!\|\!$ on a Banach space X , say $\|\!\|\!\cdot\!\|\!$ ε -approximates $\|\cdot\|$ if, for all $x \in X$,

$$(1 - \varepsilon)\|x\| \leq \|\!\|x\!\|\! \leq (1 + \varepsilon)\|x\|.$$

The notion of a boundary plays an important role in this area of study. Frequently, the existence of a boundary with certain properties gives rise to the desired renormings, as seen in the following result of Hájek, which is part of a more general theorem.

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THEOREM 1.3 ([7, Theorem 1]). *If $(X, \|\cdot\|)$ admits a boundary contained in a $\|\cdot\|$ - σ -compact subset of B_{X^*} , then X admits an equivalent C^∞ smooth norm that ε -approximates $\|\cdot\|$.*

Hájek and Haydon provided another sufficient condition for when this property holds, namely when $X = C(K)$ and K is a compact Hausdorff σ -discrete space. We call a topological space K σ -discrete if $K = \bigcup_{n=0}^\infty D_n$, where each D_n is *relatively discrete*: given $x \in D_n$, there exists U_x open in K such that $U_x \cap D_n = \{x\}$.

THEOREM 1.4 ([9, Theorem 5.1]). *Let K be a σ -discrete compact space. Then, given $\varepsilon > 0$, $C(K)$ admits an equivalent C^∞ smooth norm that ε -approximates $\|\cdot\|_\infty$.*

It is worth remarking that, in certain cases, these conclusions have been strengthened. In [2], it is shown that if X has a countable boundary then X has an equivalent analytic norm which ε -approximates the original norm. Moreover, if $C(K)$ admits an analytic renorming, then K is countable [8]. For a norm $\|\cdot\|$ to be *analytic* we mean it is a real valued analytic function on $X \setminus \{0\}$. Analytic functions on Banach spaces are defined and explored in [14].

The Orlicz functions M for which the corresponding Orlicz sequence spaces l_M and Orlicz function spaces $l_M(0, 1)$, $l_M(0, \infty)$ have an equivalent C^∞ smooth norm were characterised in [12]. Furthermore, the Orlicz sequence spaces h_M with equivalent analytic norm were characterised in [10].

The main result of this paper, Theorem 2.1, generalises these results as corollaries. It also takes into account smoothness of injective tensor products, in a manner similar to that of [11]. As in the proof of [9, Theorem 5.1], the proof of Theorem 2.1 makes use of two lemmas ([9, Lemmas 5.2 and 5.3]) concerning the so-called *generalised Orlicz norm*, denoted by $\|\cdot\|_\phi$. The first lemma provides a condition implying that $\|\cdot\|_\phi$ is equivalent to $\|\cdot\|$.

DEFINITION 1.5. Let B be a set. Suppose for every element $t \in B$ there exists a convex function ϕ_t on $[0, \infty)$ with $\phi_t(0) = 0$ and $\lim_{\alpha \rightarrow \infty} \phi_t(\alpha) = \infty$ (such functions are called *Orlicz functions*). Define $\|\cdot\|_\phi$ on $l_\infty(B)$ by

$$\|f\|_\phi = \inf \left\{ \rho > 0 : \sum_{t \in B} \phi_t \left(\frac{|f(t)|}{\rho} \right) \leq 1 \right\},$$

and define $\ell_\phi(B)$ as the set of $f \in l_\infty(B)$ satisfying $\|f\|_\phi < \infty$.

LEMMA 1.6 ([9, Lemma 5.2]). *Let $\|\cdot\|_\phi$ be as in Definition 1.5. Suppose there exist $\beta > \alpha > 0$ with $\phi_t(\alpha) = 0$ and $\phi_t(\beta) \geq 1$ for all $t \in B$. Then $\ell_\phi(B) \cong \ell_\infty(B)$ and*

$$\alpha \|\cdot\|_\phi \leq \|\cdot\|_\infty \leq \beta \|\cdot\|_\phi.$$

We use $\|\cdot\|_\phi$ to define another norm on a more general space X , which we also denote by $\|\cdot\|_\phi$. The second lemma gives a sufficient condition for $\|\cdot\|_\phi$ on X to be C^k smooth. It uses the notion of local dependence on finitely many coordinates and generalises [9, Lemma 5.3].

LEMMA 1.7. *Let $\|\cdot\|_\phi$ be as in Lemma 1.6 and let $\Pi : X \rightarrow \ell_\phi(B)$ be an embedding (non-linear in general), where the map $x \mapsto \Pi(x)(t)$ is a seminorm which is C^k smooth on the set where it is non-zero, for all $t \in B$. Assume the assignment $\|x\|_\phi = \|\Pi(x)\|_\phi$ defines an equivalent norm on X . Suppose for each $x \in X$ with $\|x\|_\phi = 1$ there exists an open $U \subseteq X$ containing x , and finite $F \subseteq B$, such that $\phi_t(|y(t)|) = 0$ when $y \in U$ and $t \in B \setminus F$. Finally, assume that each ϕ_t is C^∞ smooth. Then $\|\cdot\|_\phi$ is C^k smooth on X .*

As Lemma 1.7 appears in [9, Lemma 5.3], X is taken to be a closed subspace of $\ell_\infty(B)$ and Π is the identity. The proof uses the fact that each coordinate map $x \mapsto |x(t)|$ is C^∞ smooth on the set where it is non-zero, and uses the implicit function theorem to show that $\|\cdot\|_\phi$ is also C^∞ smooth. In our case, each coordinate map is C^k smooth on the set where it is non-zero, and the same argument guarantees that $\|\cdot\|_\phi$ is C^k smooth.

The first part of the proof of Theorem 2.1 is concerned with setting up the necessary framework to apply these lemmas. The remainder uses a series of claims to prove that the hypotheses of these lemmas do in fact hold. In the final section, Theorems 1.3 and 1.4 are obtained as corollaries of Theorem 2.1, along with some other results and applications.

Before proceeding to the statement of Theorem 2.1, a key notion of w^* -locally relatively compact sets (w^* -LRC for short) needs to be introduced. This property is first studied in [6], in the context of polyhedral norms.

DEFINITION 1.8 ([6, Definition 5]). Let X be a Banach space. We call $E \subseteq X^*$ w^* -LRC if given $y \in E$, there exists a w^* -open set U such that $y \in U$ and $\overline{E \cap U}^{\|\cdot\|}$ is norm compact.

EXAMPLE 1.9 ([6, Example 6]). The following sets are w^* -LRC:

- (1) Any norm compact or w^* -relatively discrete subset of a dual space.
- (2) Given X with an unconditional basis $(e_i)_{i \in I}$ and $f \in X^*$, define

$$\text{supp}(f) = \{i \in I : f(e_i) \neq 0\}.$$

Let $E \subset X^*$ have the property that if $f, g \in E$, then $|\text{supp}(f)| = |\text{supp}(g)| < \infty$. Then E is w^* -LRC. Indeed, take $f \in E$ and define the w^* -open set $U = \{g \in X^* : 0 < |g(e_i)| < |f(e_i)| + 1, i \in \text{supp}(f)\}$. Clearly, if $g \in U \cap E$, then $\text{supp}(g) = \text{supp}(f)$. Thus $U \cap E$ is a norm bounded subset of a finite-dimensional space.

REMARK 1.10. Evidently, w^* -LRC sets are preserved under scalar multiplication. Also, the family of σ - w^* -LRC subsets of a dual Banach space forms a σ -ideal. (This is because if E is w^* -LRC and $F \subseteq E$, then F is w^* -LRC. And of course any countable union of σ - w^* -LRC sets is again σ - w^* -LRC.) But in general they do not behave well under straightforward linear and topological operations. To see this, consider the following:

- (1) Let $E = \{e_n : n \in \mathbb{N}\}$ be the usual basis of c_0 and let $F = \{0\}$. These sets are both w^* -LRC. However, 0 is a w^* -accumulation point of E and $\|e_n - e_m\|_\infty = 1$ whenever $n \neq m$, so $E \cup F$ is not w^* -LRC.
- (2) The set $E = \{\delta_\alpha + 2^{-n}\delta_{\alpha+n} : \alpha < \omega_1 \text{ is a limit ordinal}, n \in \mathbb{N}\}$ is w^* -discrete. But $\overline{E}^{\|\cdot\|} \supseteq \{\delta_\alpha : \alpha < \omega_1 \text{ is a limit ordinal}\}$. Using the fact that the ordinal ω_1 is not σ -discrete, we can see that the set $\{\delta_\alpha : \alpha < \omega_1 \text{ is a limit ordinal}\}$, and thus $\overline{E}^{\|\cdot\|}$, is not σ - w^* -LRC.
- (3) Consider the space $\ell_1 \oplus \ell_1(B_{\ell_1}) \equiv \ell_1(\mathbb{N} \cup B_{\ell_1})$. Given $x \in B_{\ell_1}$, denote by \bar{x} its canonical image in $\ell_1(\mathbb{N} \cup B_{\ell_1})$. Let $E = \{\bar{x} \pm \delta_x : x \in B_{\ell_1}\}$. This set can be shown to be w^* -discrete but we have $E + E \supseteq \{2\bar{x} : x \in B_{\ell_1}\} \equiv 2B_{\ell_1}$. A consequence of [6, Proposition 12(1)] is that for an infinite-dimensional space X , S_{X^*} cannot be covered by a countable union of w^* -LRC sets. This result extends to S_Y , where Y is any infinite-dimensional subspace of X^* . Because of this, $E + E$ is not σ - w^* -LRC.

The main result is concerned with renorming injective tensor products. Given Banach spaces X and Y , the *injective tensor product* $X \otimes_\varepsilon Y$ is the completion of the algebraic tensor product $X \otimes Y$ with respect to the norm

$$\left\| \sum_{i=1}^{\infty} x_i \otimes y_i \right\| = \sup \left\{ \sum_{i=1}^{\infty} f(x_i)g(y_i) : f \in B_{X^*}, g \in B_{Y^*} \right\}.$$

Also note the following facts. If I_Y is the identity operator on Y , then given $f \in X^*$ we define $f^Y = f \otimes I_Y$ on $X \otimes Y$ by $f^Y(\sum_{i=1}^{\infty} x_i \otimes y_i) = \sum_{i=1}^{\infty} f(x_i)y_i$. We have $\|f^Y\| = \|f\|$ and extend f^Y to the completion. Similarly define g^X for $g \in Y^*$. A useful fact is $f \otimes g = g \circ f^Y = f \circ g^X$.

Given two boundaries $N \subseteq X^*$ and $M \subseteq Y^*$, the set $\{f \otimes g : f \in N, g \in M\}$ is a boundary for $X \otimes_\varepsilon Y$. To see this, take $u \in X \otimes_\varepsilon Y$. There exist $f \in B_{X^*}$ and $g \in B_{Y^*}$ such that $\|u\| = (f \otimes g)(u) = \|g^X(u)\|$. Then there exists $\hat{f} \in N$ such that $\hat{f}(g^X(u)) = \|u\| = \|\hat{f}^Y(u)\|$. Finally, there exists $\hat{g} \in M$ such that $\hat{g}(\hat{f}^Y(u)) = (\hat{f} \otimes \hat{g})(u) = \|u\|$.

Given a Banach space Y with a C^k smooth renorming, Haydon gave a sufficient condition on X for $X \otimes_\varepsilon Y$ to have a C^k smooth renorming [11, Corollary 1]. This condition involves a type of operator that are now known as Talagrand operators. Another sufficient condition is given in the

main result below. It is worth noting that these conditions are incomparable. For example, the space $C[0, \omega_1]$ satisfies Haydon's condition but not that of Theorem 2.1. On the other hand, if we take K to be the Ciesielski–Pol space of [1], then $C(K)$ satisfies the hypothesis of Theorem 2.1 but not Haydon's condition.

2. Main result

THEOREM 2.1. *Let X and Y be Banach spaces and let (E_n) be a sequence of w^* -LRC subsets of X^* such that $E = \bigcup_{n=0}^{\infty} E_n$ is σ - w^* -compact and contains a boundary of X . Suppose further that Y has a C^k smooth norm $\|\cdot\|_Y$ for some $k \in \mathbb{N} \cup \{\infty\}$. Then $X \otimes_{\varepsilon} Y$ admits a C^k smooth renorming that ε -approximates the canonical injective tensor norm.*

Proof. The proof is based to some degree on that of [6, Theorem 7]. Given its technical nature, some of that proof is repeated here for clarity. We ask the reader to excuse any redundancy.

To begin, we can assume E is a boundary and $\overline{E_n}^{w^*} \subseteq E$ for all $n \in \mathbb{N}$. Indeed, if necessary, taking $E = \bigcup_{m=0}^{\infty} K_m$, where K_m is w^* -compact, we can consider, for all $n, m \in \mathbb{N}$,

$$E_n \cap K_m \cap B_{X^*}.$$

By [6, Proposition 12(3)] there exist w^* -open sets V_n such that if we set $A_n = \overline{E_n}^{w^*} \cap V_n$, then

$$E_n \subseteq A_n \subseteq \overline{E_n}^{\|\cdot\|} \quad \text{and} \quad A_n \text{ is } w^*\text{-LRC.}$$

Each A_n is both norm F_{σ} and norm G_{δ} . So for each $n \in \mathbb{N}$, $A_n \setminus \bigcup_{k < n} A_k$ will in particular be norm F_{σ} . Now we write

$$A_n \setminus \bigcup_{k < n} A_k = \bigcup_{m=0}^{\infty} H_{n,m},$$

where each $H_{n,m}$ is norm closed. By arrangement, we assume $H_{n,m} \subseteq H_{n,m+1}$ for all $m \in \mathbb{N}$ and, for convenience, we set $H_{n,-1} = \emptyset$. Let $\pi : \mathbb{N}^2 \rightarrow \mathbb{N}$ be a bijection and for all $i, j \in \mathbb{N}$, define

$$L_{\pi(i,j)} = H_{i,j} \setminus H_{i,j-1}.$$

Clearly E is the disjoint union of the L_n and $\overline{L_n}^{w^*} \subseteq \overline{E_p}^{w^*} \subseteq E$, where $n = \pi(p, q)$. Given $f \in E$, let

$$I(f) = \{n \in \mathbb{N} : f \in \overline{L_n}^{w^*}\} \quad \text{and} \quad n(f) = \min I(f).$$

Now fix $\varepsilon > 0$. We define $\psi : E \rightarrow (1, 1 + \varepsilon)$ by

$$\psi(f) = 1 + \frac{1}{2}\varepsilon \cdot 2^{-n(f)} \left(1 + \frac{1}{4} \sum_{i \in I(f)} 2^{-i} \right).$$

Set $\varepsilon_n = \frac{1}{96}\varepsilon \cdot 4^{-n}$. Fix n . As $\psi(L_n) \subseteq (1, 1 + \varepsilon)$, there exists a finite partition of L_n into sets J such that $\text{diam}(\psi(J)) \leq \varepsilon_n$.

Let $P = \{I \subseteq J : I \text{ is } \varepsilon_n\text{-separated}\}$. This set is non-empty because any singleton is in P . For a chain $T \subseteq P$ we have $\bigcup_{N \in T} N \in P$, so we can apply Zorn's Lemma to get $\Gamma \subseteq J$, a maximal ε_n -separated subset of J . By maximality, Γ is also an ε_n -net. And by ε_n -separation, for a totally bounded set $M \subseteq J$, the intersection $M \cap \Gamma$ is finite.

By considering the finite union of these Γ , there exists $\Gamma_n \subseteq L_n$ with the property that given $f \in L_n$ there exists $h \in \Gamma_n$ such that

$$(1) \quad |\psi(f) - \psi(h)| \leq \varepsilon_n \quad \text{and} \quad \|f - h\| \leq \varepsilon_n.$$

Moreover, if $M \subseteq L_n$ is totally bounded, $M \cap \Gamma_n$ is finite. Define $B = \bigcup_{n=0}^\infty \Gamma_n$. We are now ready to define $\|\cdot\|_\phi$ on $\ell_\infty(B)$.

For each $f \in B$ we pick a C^∞ Orlicz function ϕ_f so that

$$\begin{aligned} \phi_f(\alpha) &= 0 & \text{if } \alpha \leq \frac{1}{\psi(f)}, \\ \phi_f(\alpha) &> 1 & \text{if } \alpha \geq \frac{1}{\theta(f)}, \quad \text{where } \theta(f) = \psi(f) - \varepsilon_n. \end{aligned}$$

We define $\|\cdot\|_\phi$ with respect to these functions, as per Definition 1.5. By taking $(1 + \varepsilon)^{-1}$ and 1 as the constants in the hypothesis of Lemma 1.6 we have $l_\phi(B) \cong \ell_\infty(B)$ and $\|\cdot\|_\infty \leq \|\cdot\|_\phi \leq (1 + \varepsilon)\|\cdot\|_\infty$.

We embed $X \otimes_\varepsilon Y$ into $\ell_\infty(B)$ by setting $\Pi(u)(f) = \|f^Y(u)\|_Y$, $f \in B$. The coordinate map $u \mapsto \|f^Y(u)\|$ is a seminorm which is C^k smooth on the set where it is non-zero for each $f \in B$. Since $\|\Pi(u)\|_\infty = \|u\|$, it follows that $\|\cdot\| \leq \|\cdot\|_\phi \leq (1 + \varepsilon)\|\cdot\|$ on X .

Suppose for the sake of contradiction that the remaining hypothesis of Lemma 1.7 does not hold. Then we can find $u \in X \otimes_\varepsilon Y$ with $\|u\|_\phi = 1$, $(u_n) \subseteq X \otimes_\varepsilon Y$ with $u_n \rightarrow u$ and distinct $(f_n) \subseteq B$ such that $\phi_{f_n}(\|f_n^Y(u_n)\|) > 0$ for all n . Then $\psi(f_n)\|f_n^Y(u_n)\| > 1$ for all n .

Take a subsequence of (f_n) , again called (f_n) , such that $\psi(f_n) \rightarrow \alpha$ for some $\alpha \in \mathbb{R}$. Now take $(g_n) \subseteq S_{Y^*}$ such that $\|f_n^Y(u_n)\| = g_n(f_n^Y(u_n))$. Let $(f, g) \in B_{X^*} \times B_{Y^*}$ be an accumulation point of (f_n, g_n) in the product of the w^* -topologies. Then $f \otimes g$ is a w^* -accumulation point of $(f_n \otimes g_n)$, and $\alpha(f \otimes g)(u) \geq 1$.

The remainder of the proof is concerned with obtaining the contradiction $\alpha(f \otimes g)(u) < 1$.

CASE 1: $\alpha = 1$. With $\alpha = 1$, it is evident that $\alpha(f \otimes g)(u) = (f \otimes g)(u) \leq \|u\|$. The following claim ensures $\|u\| < 1$.

CLAIM 1. *If $v \neq 0$, then $\|v\| < \|v\|_\phi$.*

Let $\|v\| = 1$ and pick $p \in E$, $q \in S_{Y^*}$ such that $1 = (p \otimes q)(v)$. As noted above, this is possible because E and S_{Y^*} are boundaries of X and Y , respectively. By (1) above, let $r \in B$ be such that $\|p - r\| \leq \varepsilon_n$ for an appropriate n . Observe that $\theta(r)(r \otimes q)(v) \leq \|v\|_\phi$. Indeed,

$$\begin{aligned} \sum_{l \in B} \phi_l \left(\frac{\|l^Y(v)\|}{\theta(r)(r \otimes q)(v)} \right) &\geq \phi_r \left(\frac{\|r^Y(v)\|}{\theta(r)q(r^Y(v))} \right) \\ &\geq \phi_r \left(\frac{1}{\theta(r)} \right) > 1. \end{aligned}$$

Now to prove the claim, we compute

$$\begin{aligned} 1 &= (p \otimes q)(v) \\ &= (r \otimes q)(v) + ((p - r) \otimes q)(v) \\ &= \theta(r)(r \otimes q)(v) + (1 - \theta(r))(r \otimes q)(v) + ((p - r) \otimes q)(v) \\ &\leq \|v\|_\phi + (1 - \theta(r))(r \otimes q)(v) + ((p - r) \otimes q)(v). \end{aligned}$$

So we are done if $(\theta(r) - 1)(r \otimes q)(v) + ((r - p) \otimes q)(v) > 0$. Indeed,

$$\begin{aligned} \theta(r) - 1 &= \psi(r) - \varepsilon_n - 1 \\ &\geq \frac{1}{2}\varepsilon \cdot 2^{-n(r)} - \varepsilon_n \geq \frac{1}{2}\varepsilon \cdot 2^{-n} - \varepsilon_n. \end{aligned}$$

Also, $(r \otimes q)(v) = r(q^X(v)) \geq 1 - \|p - r\| \cdot \|q^X(v)\| \geq \frac{1}{2}$. Thus,

$$\begin{aligned} (\theta(r) - 1)(r \otimes q)(v) + ((r - p) \otimes q)(v) &\geq \frac{1}{4}\varepsilon \cdot 2^{-n} - \frac{1}{2}\varepsilon_n - \varepsilon_n \\ &= \frac{1}{4}\varepsilon \cdot 2^{-n} - \frac{3}{2}\varepsilon_n \\ &= \frac{1}{4}\varepsilon \cdot 2^{-n} - \frac{1}{64}\varepsilon \cdot 4^{-n} > 0, \end{aligned}$$

and the claim is proven.

CASE 2: $\alpha > 1$. We first prove $f \in E$.

Fix N large enough so that $1 + \varepsilon \cdot 2^{-N} < \frac{1}{2}(1 + \alpha)$. Because $\psi(f_n) \rightarrow \alpha$ we have $\psi(f_m) > \frac{1}{2}(1 + \alpha)$ for all m large enough. Hence, $n(f_m) < N$. Therefore, $f_m \in \bigcup_{k < N} \overline{L_k}^{w^*}$ for all such m . By w^* -closure, $f \in \bigcup_{k < N} \overline{L_k}^{w^*} \subseteq E$.

Now the aim is to prove $\psi(f) > \alpha$.

We can assume $f_n \neq f$ for all $n \in \mathbb{N}$, because the f_n are distinct.

Fix the unique m such that $f \in L_m$ and let

$$J = I(f) \cup \{k \in \mathbb{N} : k \geq m + 2\}.$$

Clearly $m \in I(f)$. Let $(p, q) \in \mathbb{N}^2$ be such that $m = \pi(p, q)$. We have $L_m \subseteq A_p$. Since A_p is w^* -LRC, there exists a w^* -open set $U \ni f$ such that $A_p \cap U$ is relatively norm compact.

By the above, $\Gamma_{\pi(p,k)} \cap U$ is finite for all $k \in \mathbb{N}$, since $\Gamma_{\pi(p,k)} \subseteq A_p$. So the set

$$V = U \setminus \left(\bigcup_{i \in \mathbb{N} \setminus J} \overline{L_i}^{w^*} \cup \left(\bigcup_{k=0}^q \Gamma_{\pi(p,k)} \setminus \{f\} \right) \right)$$

is w^* -open. Moreover, because $f \notin \bigcup_{i \in \mathbb{N} \setminus J} \overline{L_i}^{w^*}$, we have $f \in V$. We assume from now on that $f_n \in V$.

CLAIM 2a. $m \notin I(f_n)$.

If $m \in I(f_n)$, then

$$f_n \in \overline{L_m}^{w^*} \cap V \subseteq \overline{L_m \cap V}^{w^*} = \overline{L_m \cap V}^{\|\cdot\|} \subseteq \overline{L_m}^{\|\cdot\|} \subseteq H_{p,q}.$$

It follows that $f_n \in H_{p,k} \setminus H_{p,k-1} = L_{\pi(p,k)}$ for some $k \leq q$. On the other hand, $f_n \in B$, so $f_n \in L_{\pi(p,k)} \cap B = \Gamma_{\pi(p,k)}$. However, this cannot be the case, since $f_n \in V \setminus \{f\}$.

CLAIM 2b. $I(f_n) \subseteq J$.

Let $i \in I(f_n)$. If $i \notin J$, then $f_n \in \bigcup_{j \in \mathbb{N} \setminus J} \overline{L_j \cap V}^{w^*}$, but this contradicts $f_n \in V$.

CLAIM 2c. $\psi(f) - \psi(f_n) \geq \frac{1}{16}\varepsilon \cdot 4^{-m} = 6\varepsilon_m$.

First note $n(f_n) \geq n(f)$, using Claim 2b and $n(f) = \min I(f) = \min J$. There are two cases to consider. If $n(f_n) > n(f)$, then

$$\begin{aligned} \psi(f) - \psi(f_n) &\geq 1 + \frac{1}{2}\varepsilon \cdot 2^{-n(f)} - \left(1 + \frac{3}{4}\varepsilon \cdot 2^{-n(f_n)} \right) \\ &\geq \frac{1}{8}\varepsilon \cdot 2^{-n(f)} \geq \frac{1}{8}\varepsilon \cdot 2^{-m}. \end{aligned}$$

And if $n(f_n) = n(f)$, then

$$\begin{aligned} \psi(f) - \psi(f_n) &\geq \frac{1}{8}\varepsilon \cdot 2^{-n(f)} \left(\sum_{i \in I(f)} 2^{-i} - \sum_{i \in I(f_n)} 2^{-i} \right) \\ &= \frac{1}{8}\varepsilon \cdot 2^{-n(f)} \left(\sum_{i \in I(f) \setminus I(f_n)} 2^{-i} - \sum_{i \in I(f_n) \setminus I(f)} 2^{-i} \right) \\ &\geq \frac{1}{8}\varepsilon \cdot 2^{-n(f)} \left(2^{-m} - \sum_{i \in J \setminus I(f)} 2^{-i} \right) \\ &\geq \frac{1}{8}\varepsilon \cdot 2^{-n(f)} \cdot 2^{-m-1} \geq \frac{1}{16}\varepsilon \cdot 4^{-m}. \end{aligned}$$

CLAIM 2d. For $h \in B$, $\|h \otimes g\|_\phi \leq 1/\theta(h)$.

If $|(h \otimes g)(v)| > 1/\theta(h)$, then

$$\sum_{l \in B} \phi_l(\|l^Y(v)\|) \geq \phi_h(\|h^Y(v)\|) \geq \phi_h((h \otimes g)(v)) > 1, \quad \text{so } \|v\|_\phi > 1.$$

Hence $\|h \otimes g\|_\phi = \sup\{|(h \otimes g)(v)| : \|v\|_\phi \leq 1\} \leq 1/\theta(h)$.

We can now prove $\alpha(f \otimes g)(x) < 1$. By (1), take $h \in B$ such that $\|f - h\| \leq \varepsilon_n$ and $|\psi(f) - \psi(h)| \leq \varepsilon_n$. We then have

$$\begin{aligned} \alpha\|f \otimes g\|_\phi &\leq \alpha(\|h \otimes g\|_\phi + \|(f - h) \otimes g\|_\phi) \\ &\leq \alpha(\|h \otimes g\|_\phi + \|(f - h) \otimes g\|) \\ &\leq \alpha\left(\frac{1}{\theta(h)} + \varepsilon_n\right). \end{aligned}$$

So we are done if $\alpha(1/\theta(h) + \varepsilon_n) < 1$. Now,

$$\begin{aligned} 1 - \frac{\alpha}{\theta(h)} - \alpha\varepsilon_n > 0 &\Leftrightarrow \theta(h) - \alpha - \varepsilon_n\theta(h)\alpha > 0 \\ &\Leftrightarrow \psi(h) - \varepsilon_n - \alpha - \varepsilon_n\theta(h)\alpha > 0. \end{aligned}$$

By Claim 2c, we have $\psi(h) - \varepsilon_n - \alpha \geq 4\varepsilon_n$ and since $\theta(h), \alpha < 2$, it follows that $\varepsilon_n\theta(h)\alpha < 4\varepsilon_n$. Hence $\alpha\|f \otimes g\|_\phi < 1$, and so $\alpha(f \otimes g)(u) < 1$. ■

3. Applications

COROLLARY 3.1. *Suppose X has a σ - w^* -LRC and σ - w^* -compact boundary. Then X has a C^∞ renorming.*

Proof. Apply Theorem 2.1 to $X \otimes_\varepsilon \mathbb{R} = X$. ■

We can now prove Theorems 1.3 and 1.4 as corollaries of Corollary 3.1.

Proof of Theorem 1.3. Every norm compact subset of X^* is trivially w^* -LRC. The result follows from Corollary 3.1. ■

Proof of Theorem 1.4. Let $K = \bigcup_{n=0}^\infty D_n$, where each D_n is relatively discrete. Let δ_t be the usual evaluation functional, $\delta_t(f) = f(t)$. Then $E_n = \{\pm\delta_t : t \in D_n\}$ is w^* -relatively discrete and so w^* -LRC. Moreover, $E = \bigcup_{n=0}^\infty E_n$ is a w^* -compact boundary of $C(K)$, since for any $f \in C(K)$, there exists $t \in K$ such that $\|f\|_\infty = |f(t)|$, by compactness. ■

The corollaries below are new results. Before presenting them, a definition and a theorem appearing in [6] are needed.

DEFINITION 3.2 ([6, Definition 2]). Let X be a Banach space. We say a set $F \subseteq X^*$ is a *relative boundary* if, whenever $x \in X$ satisfies $\sup\{f(x) : f \in F\} = 1$, there exists $f \in F$ such that $f(x) = 1$.

EXAMPLE 3.3. Any boundary and any w^* -compact set are relative boundaries.

THEOREM 3.4 ([6, Theorem 4]). *Let X be a Banach space and suppose we have sets $S_n \subseteq S_X$ and an increasing sequence $H_n \subseteq B_{X^*}$ of relative boundaries such that $S_X = \bigcup_{n=0}^\infty S_n$ and the numbers*

$$b_n = \inf \{ \sup \{ h(x) : h \in H_n \} : x \in S_n \}$$

are strictly positive and converge to 1. Then for a suitable sequence $(a_n)_{n=0}^\infty$ of numbers the set $F = \bigcup_{n=0}^\infty a_n(H_n \setminus H_{n-1})$ is a boundary of an equivalent norm.

Given a Banach space with an unconditional basis $(e_i)_{i \in I}$ and $x = \sum_{i \in I} x_i e_i$, let $e_i^*(x) = x_i$. For $\sigma \subseteq I$, let P_σ denote the projection given by $P_\sigma(x) = \sum_{i \in \sigma} e_i^*(x) e_i$.

COROLLARY 3.5. *Let X have a monotone unconditional basis $(e_i)_{i \in I}$, with associated projections P_σ , $\sigma \subseteq I$, and suppose we can write $S_X = \bigcup_{n=1}^\infty S_n$ in such a way that the numbers*

$$c_n = \inf \{ \sup \{ \|P_\sigma(x)\| : \sigma \subseteq I, |\sigma| = n \} : x \in S_n \}$$

are strictly positive and converge to 1. Then X admits an equivalent C^∞ smooth norm.

Proof. Let $H_n = \{h \in B_{X^*} : |\text{supp}(h)| \leq n\}$. Each H_n is a relative boundary because it is w^* -compact. Note that given $x \in S_n$ and $\sigma \subseteq I$ with $|\sigma| = n$,

$$\|P_\sigma(x)\| = \sup \{ f(P_\sigma(x)) : f \in B_{X^*} \} = \sup \{ P_\sigma^* f(x) : f \in B_{X^*} \}.$$

Of course, $|\text{supp}(P_\sigma^* f)| \leq n$ for all $f \in B_{X^*}$. And by monotonicity, $\|P_\sigma^*\| = 1$. So $P_\sigma^*(f) \in H_n$. Therefore,

$$\begin{aligned} 0 < c_n &= \inf \{ \sup \{ \|P_\sigma(x)\| : \sigma \subseteq I, |\sigma| = n \} : x \in S_n \} \\ &= \inf \{ \sup \{ P_\sigma^* f(x) : f \in B_{X^*}, \sigma \subseteq I, |\sigma| = n \} : x \in S_n \} \\ &= \inf \{ \sup \{ h(x) : h \in H_n \} : x \in S_n \} = b_n. \end{aligned}$$

Thus, (b_n) is a strictly positive sequence converging to 1.

The set $H_n \setminus H_{n-1}$ is w^* -LRC by Example 1.9(2).

By Theorem 3.4, there exists a sequence $(a_n)_{n=0}^\infty$ for which the set $F = \bigcup_{n=0}^\infty a_n(H_n \setminus H_{n-1})$ is a σ - w^* -LRC and σ - w^* -compact boundary for an equivalent norm $\|\| \cdot \|\|$. By Corollary 3.1, X will admit an equivalent C^∞ smooth norm that ε -approximates $\|\| \cdot \|\|$. ■

COROLLARY 3.6. *Let X be a Banach space with a monotone unconditional basis $(e_i)_{i \in I}$ and suppose for each $x \in S_X$ there exists $\sigma \subset I$, $|\sigma| < \infty$, such that $\|P_\sigma(x)\| = 1$. Then X admits an equivalent C^∞ smooth norm that ε -approximates the original norm.*

Proof. Let $H_n = \{h \in B_{X^*} : |\text{supp}(h)| \leq n\}$. As mentioned in the proof of Corollary 3.5, each H_n is w^* -compact and is a finite union of w^* -LRC sets. Now take $x \in S_X$ and σ such that $\|P_\sigma(x)\| = 1$. Then there is $f \in B_{X^*}$ with

$$1 = \|P_\sigma(x)\| = f(P_\sigma(x)) = P_\sigma^* f(x).$$

Because $(e_i)_{i \in I}$ is monotone, $\|P_\sigma^*\| = 1$ and so $P_\sigma^* f \in H_{|\sigma|}$. Therefore, the set $H = \bigcup_{n=0}^\infty H_n$ is a boundary satisfying the hypothesis of Corollary 3.1. ■

Using Corollary 3.5 we can obtain new examples of spaces with equivalent C^∞ smooth renormings.

EXAMPLE 3.7. Let $\mathbb{N} = \bigcup_{n=0}^\infty A_n$, where each A_n is finite, and let $p = (p_n)$ be an unbounded increasing sequence of real numbers with $p_n \geq 1$. For each sequence $x = (x_n)$ of real numbers define

$$\Phi(x) = \sup \left\{ \sum_{n=0}^\infty \sum_{k \in B_n} |x(k)|^{p_n} : B_n \subset A_n, \text{ and the } B_n \text{ are pairwise disjoint} \right\}.$$

Indeed, we define $\ell_{A,p}$ as the space of sequences x where $\Phi(x/\lambda) < \infty$ for some $\lambda > 0$, with norm $\|x\| = \inf\{\lambda > 0 : \Phi(x/\lambda) \leq 1\}$. Define the subspace $h_{A,p}$ as the norm closure of the linear space generated by the basis $e_n(k) = \delta_{n,k}$. Now [6, Example 16] provides an appropriate sequence (S_n) of subsets of S_X so that the assumption of Corollary 3.5 holds.

EXAMPLE 3.8. Let M be an Orlicz function with

$$M(t) > 0 \quad \text{for all } t > 0, \quad \text{and} \quad \lim_{t \rightarrow 0} \frac{M(K(t))}{M(t)} = \infty$$

for some constant $K > 0$. Let $h_M(\Gamma)$ be the space of all real functions x defined on Γ with $\sum_{\gamma \in \Gamma} M(x_\gamma/\rho) < \infty$ for all $\rho > 0$, with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{\gamma \in \Gamma} M\left(\frac{x_\gamma}{\rho}\right) \leq 1 \right\}.$$

The canonical unit vector basis $(e_\gamma)_{\gamma \in \Gamma}$ of functions $e_\gamma(\beta) = \delta_{\gamma,\beta}$ is unconditionally monotone. [6, Example 18] provides suitable subsets of S_X to ensure the hypothesis of Corollary 3.5 holds.

The final example concerns the predual of a Lorentz sequence space $d(w, 1, A)$, for an arbitrary set A .

Let $w = (w_n) \in c_0 \setminus \ell_1$ with each w_n strictly positive and $w_0 = 1$. We define $d(w, 1, A)$ as the space of $x : A \rightarrow \mathbb{R}$ for which

$$\|x\| = \sup \left\{ \sum_{j=0}^\infty w_j |x(a_j)| : (a_j) \subseteq A \text{ is a sequence of distinct points} \right\} < \infty.$$

The canonical predual $d_*(w, 1, A)$ of $d(w, 1, A)$ is given by the space of functions $y : A \rightarrow \mathbb{R}$ for which $\bar{y} = (\bar{y}_k) \in c_0$, where

$$\bar{y}_k = \sup \left\{ \frac{\sum_{i=0}^{k-1} |y(a_i)|}{\sum_{i=0}^{k-1} w_i} : a_0, a_1, \dots, a_{k-1} \text{ are distinct points of } A \right\},$$

with norm $\|y\| = \|\bar{y}\|_\infty$. We can see that $(e_a)_{a \in A}$ is a monotone unconditional basis for both $d(w, 1, A)$ and $d_*(w, 1, A)$. The separable version of $d_*(w, 1, A)$ was first introduced in [13].

EXAMPLE 3.9. The space $X = d_*(w, 1, A)$ has a C^∞ smooth equivalent renorming that ε -approximates the original norm.

Indeed, let $y \in S_X$. Since $\bar{y} \in c_0$, there exists $k \in \mathbb{N}$ such that $\bar{y}^k = 1$. It can also be shown that $y \in c_0(A)$ and thus the supremum in the definition of \bar{y}^k is attained. Consequently, there exist $a_0, a_1, \dots, a_{k-1} \in A$ such that

$$1 = \bar{y}^k = \frac{\sum_{i=0}^{k-1} |y(a_i)|}{\sum_{i=0}^{k-1} w_i}.$$

Setting $\sigma = \{a_0, a_1, \dots, a_{k-1}\}$, we have $\|P_\sigma(y)\| = 1$. By Corollary 3.6, X has a C^∞ smooth equivalent renorming that ε -approximates the original norm.

REMARK 3.10. The space $X = d_*(w, 1, A)$ for A uncountable is a new example of a space with a C^∞ smooth renorming. It is not yet known if X has an analytic renorming.

REMARK 3.11. In Theorem 2.1 and Corollary 3.1 we cannot drop the σ - w^* -compactness condition in general, and expect an equivalent norm of any order of smoothness that depends locally on finitely many coordinates. In [4], it is shown that $C_0(\omega_1)$ has no such norm. On the other hand, $C_0(\omega_1)$ admits an equivalent norm supporting a boundary that is w^* -discrete (this follows from [5, Theorem 10]).

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