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On generalized derivations in Banach algebras

by

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Abstract. We study generalized derivations G defined on a complex Banach algebra A such that the spectrum $\sigma(Gx)$ is finite for all $x \in A$. In particular, we show that if A is unital and semisimple, then G is inner and implemented by elements of the socle of A.

1. Introduction. The notion of generalized derivation is due to Brešar [6]. Let A be an algebra. A linear mapping $G : A \to A$ is called a *generalized derivation* if there exists a derivation $d : A \to A$ such that $G(xy) = (Gx)y + xdy$ for all $x, y \in A$. A linear map $T : A \to A$ is called a *left centralizer* in case $T(xy) = (Tx)y$ for all $x, y \in A$. If G is a generalized derivation determined by a derivation d, then $G-d$ is a left centralizer. Hence a generalized derivation is a sum of a derivation and a left centralizer. We say G is *inner* if there exist $a, b \in A$ such that $Gx = ax + xb$ for all $x \in A$. Obviously, if the algebra is unital then $Gx = (G1)x + dx$ for all $x \in A$; in this case G is inner if and only if d is inner [6]. For results concerning generalized derivations we refer the reader to [1, 6, 14, 16].

The purpose of this paper is to investigate generalized derivations G on a complex Banach algebra A such that the spectrum of Gx is finite for every $x \in A$. In particular, we will show that if G is a generalized derivation defined on a complex semisimple Banach algebra A such that $\sigma(Gx)$ is finite for all $x \in A$, then $GA \subseteq \operatorname{soc} A$. Our results generalize those of [7, 8] which deal with derivations d on a Banach algebra satisfying $\sharp\sigma(dx) < \infty$ for every $x \in A$. In [4, 5], one can find other conditions entailing that the range of a bounded derivation lies in the socle modulo the radical of a Banach algebra. It should be pointed out that inner generalized derivations G defined on a Banach algebra A such that $\sharp\sigma(Gx) = 1$ for every $x \in A$ were studied in [5].

Our study is closely connected with questions concerning derivations mapping into the radical. For details, we refer the readers to [17, 18], and

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the references therein. We also mention the work of Curto and Mathieu [11], where spectrally bounded generalized inner derivations were investigated.

2. The case of dense algebras. We first give some tools and notation. Let X be a vector space. As usual, $\mathcal{L}(X)$ denotes the algebra of all linear operators on X. If X is a Banach space, the Banach algebra of all bounded linear operators on X is denoted by $\mathcal{B}(X)$. The dual of X will be denoted by X^* and we will denote by $u \otimes f$ the linear operator on X defined for any $u \in X$ and $f \in X^*$ by $(u \otimes f)(x) = f(x)u$ for $x \in X$. Moreover, I denotes the identity mapping on X and $\sharp F$ denotes the cardinality of a set F. Let T be an operator on X. The point spectrum of T is $\sigma_{\mathrm{p}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not injective}\}$.

Let A be an arbitrary algebra. We denote by $\operatorname{rad} A$ the Jacobson radical of A, and by $\mathcal{Z}(A)$ the centre modulo the radical, defined by

$$\mathcal{Z}(A) = \{a \in A : ax - xa \in \operatorname{rad} A \text{ for all } x \in A\}.$$

For every $a, b \in A$ let $\delta_{a,b}$ denote the inner generalized derivation defined by $\delta_{a,b}(x) = ax + xb$ for all $x \in A$. Recall that $\delta_a = \delta_{a,-a}$ is the inner derivation determined by a.

Now let X be a Banach space and let $\mathcal{A}$ be a standard operator algebra on X. It is well-known that every derivation $d : \mathcal{A} \to \mathcal{A}$ is of the form $dS = TS - ST$ for some $T \in \mathcal{B}(X)$ [10]. Similarly, we can prove the following result.

2.1. Lemma. *Let X be a complex Banach space and $\mathcal{A}$ a dense algebra of bounded linear operators on X. Suppose that $\mathcal{A}$ is closed and contains finite rank operators. Then every generalized derivation G on $\mathcal{A}$ is of the form $GS = TS + ST'$ for some $T, T' \in \mathcal{B}(X)$.*

Proof. Let G be a generalized derivation on $\mathcal{A}$ determined by a derivation d. Since the algebra $\mathcal{A}$ is semisimple, d and G are continuous [15]. On the other hand, since $\mathcal{A}$ is a dense algebra containing finite rank operators, we check easily that $\mathcal{A}$ contains a rank one operator. Let f be a nonzero linear functional such that $u \otimes f \in \mathcal{A}$ for some $0 \neq u \in X$. Applying again the density of $\mathcal{A}$, we see that $x \otimes f \in \mathcal{A}$ for all $x \in X$. Choose $v \in X$ such that $f(v) = 1$ and define linear maps $T, T' : X \to X$ by

$$Tx = (d(x \otimes f))v, \qquad T'x = (G(x \otimes f))v$$

for all $x \in X$. We check at once that T, T' are continuous and

$$(d(Sx \otimes f))v = (dS)x + S(d(x \otimes f)v), \qquad (G(Sx \otimes f))v = (GS)x + S(d(x \otimes f)v)$$

for every $x \in \mathcal{A}$. As a result, $dS = TS - ST$ and $GS = T'S - ST$ for all $S \in \mathcal{A}$. ∎

Our proofs involve techniques that have become standard in this area: the Jacobson density theorem, its generalizations and results on locally linearly dependent operators. Let U and V be vector spaces over a field $\mathbb{F}$ and let V_0 be a finite-dimensional subspace of V. Amitsur [2] proved that if $T_1, \ldots, T_n : U \to V$ are linear operators such that $T_1 u, \ldots, T_n u$ are linearly dependent modulo V_0 for every $u \in U$, then there exist scalars $\alpha_1, \ldots, \alpha_n$, not all zero, such that $S = \alpha_1 T_1 + \cdots + \alpha_n T_n$ satisfies

$$\dim SU \leq \dim V_0 + \binom{n+1}{2} - 1.$$

Aupetit [3, p. 86] proved that if U and V are complex vector spaces and $V_0 = \{0\}$, then S can be chosen so that $\operatorname{rank} S \leq n-1$. Brešar and Šemrl [9, Theorem 2.2] extended Aupetit's result to the case of arbitrary infinite fields.

2.2. Theorem. *Let X be a complex vector space and $\mathcal{A}$ a dense algebra of linear operators on X. Suppose that there are linear operators A, B on X and an integer $n \in \mathbb{N}^*$ such that $\#\sigma_{\mathrm{p}}(AS + SB) \leq n$ for all $S \in \mathcal{A}$. Then there exist $\lambda \in \mathbb{C}$ and finite rank operators $F, F' \in \mathcal{L}(X)$ such that $A = \lambda I + F$ and $B = -\lambda I + F'$.*

Proof. If X is finite-dimensional, there is nothing to prove. So, assume that X is infinite-dimensional. Suppose first that the set

$$\{\xi_1, \ldots, \xi_{n+1}, B\xi_1, \ldots, B\xi_{n+1}\}$$

is linearly independent for some $\xi_1, \ldots, \xi_{n+1}$ in X. Then there is $S \in \mathcal{A}$ such that $S\xi_i = 0$ and $SB\xi_i = i\xi_i$. This entails that $(AS + SB)\xi_i = i\xi_i$ for each $1 \leq i \leq n+1$. Consequently, $\{1, \ldots, n+1\} \subseteq \sigma_{\mathrm{p}}(AS+SB)$, a contradiction. Thus for any $\xi_1, \ldots, \xi_{n+1}$ in X, the set $\{\xi_1, \ldots, \xi_{n+1}, B\xi_1, \ldots, B\xi_{n+1}\}$ is linearly dependent. According to [8, Lemma 3.1], there exists a finite rank operator F' such that $B = \lambda I + F'$. Let J be a basis of the subspace $F'X$ and write $A = -\lambda I + F$ for some linear operator F.

We claim that F has finite rank. Suppose this is not true and let $\xi_1, \ldots, \xi_{n+1} \in X$ be such that the set $\{F\xi_1, \ldots, F\xi_{n+1}\} \cup J$ is linearly independent. Then there exists $S \in \mathcal{A}$ such that $SJ = \{0\}$ and $SF\xi_i = i\xi_i$ for each $1 \leq i \leq n+1$. This implies that $(AS+SB)F\xi_i = (FS+SF')F\xi_i = iF\xi_i$ and hence $\#\sigma_{\mathrm{p}}(AS + SB) \geq n+1$, a contradiction. Now the result follows from [2]. ∎

For a semisimple algebra A the socle $\operatorname{soc} A$ of A is the sum of all minimal left ideals of A. If there are no minimal left ideals in A, then $\operatorname{soc} A = \{0\}$ by definition. The socle of A is a direct sum of simple ideals. Now suppose that A is a complex semisimple Banach algebra. Then every element of $\operatorname{soc} A$ has finite spectrum. Moreover, $\operatorname{soc} A$ is the largest algebraic ideal of A.

2.3. Proposition. *Let X be a complex vector space and let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(X)$ acting densely on X. Suppose that there are finite rank operators F, F' in $\mathcal{L}(X)$ satisfying $FS + SF' \in \mathcal{A}$ for all $S \in \mathcal{A}$. Then $F' \in \operatorname{soc} \mathcal{A}$ and $F\mathcal{A} \subseteq \mathcal{A}$.*

Proof. If X is finite-dimensional, we have $\mathcal{L}(X) = \mathcal{A} = \operatorname{soc} \mathcal{A}$. So suppose that X is infinite-dimensional. Write $F = \sum_{i=1}^{p} u_i \otimes \varphi_i$ and $F' = \sum_{j=1}^{r} v_j \otimes \varphi'_j$ for linearly independent sets $\{u_1, \ldots, u_{\mathrm{p}}\}, \{v_1, \ldots, v_r\}$ of vectors in X and linear functionals $\varphi_1, \ldots, \varphi_{\mathrm{p}}, \varphi'_1, \ldots, \varphi'_r$. Choose $w_1, \ldots, w_r$ in X such that the set $\{w_1, \ldots, w_r, u_1, \ldots, u_{\mathrm{p}}\}$ is linearly independent. There are $S, S' \in \mathcal{A}$ such that

$$Sv_j = w_j, \quad S'w_j = v_j, \quad \text{and} \quad S'u_i = 0 \quad (1 \le i \le p,\ 1 \le j \le r).$$

Then

$$S'(FS + SF')\xi = F'\xi$$

for all $\xi \in X$. Hence $F' = S'(FS + SF') \in \mathcal{A}$. Finally, $F\mathcal{A} \subseteq \mathcal{A}$. ∎

The above result is sharp in the following sense.

2.4. Example. Let X be a complex Banach space with a Schauder basis $\{e_n\}_{n=1}^{\infty}$ and suppose that the topological dual of X is not separable (for instance, $X = l^1$). For every integer n, denote by e_n^* the bounded linear functional on X defined by $e_n^*(e_m) = \delta_n^m$ for every $m \in \mathbb{N}^*$. Let $\mathcal{A}$ be the closed subalgebra of $\mathcal{B}(X)$ generated by $u \otimes e_n^*$ for every integer n and all $u \in X$. Observe that $\mathcal{A}$ is a dense algebra of linear operators on X. Let f be a bounded linear functional on X such that f does not lie in the closed linear span of $\{e_n^*\}$. Pick $0 \neq u \in X$ and set $F = u \otimes f$. Then it is easy to check that $F\mathcal{A} \subseteq \mathcal{A}$, but $F \notin \mathcal{A}$.

2.5. Corollary. *Let X be a complex vector space and let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(X)$ acting densely on X. Suppose that there are A, B in $\mathcal{L}(X)$ satisfying $AS + SB \in \mathcal{A}$ for all $S \in \mathcal{A}$ and there exists $n \in \mathbb{N}^*$ such that $\#\sigma_{\mathrm{p}}(AS + SB) \le n$ for all $S \in \mathcal{A}$. Then there exist finite rank operators F, F' in $\mathcal{L}(X)$ and a scalar $\lambda \in \mathbb{C}$ such that $F' \in \operatorname{soc} \mathcal{A}$, $F\mathcal{A} \subseteq \mathcal{A}$, $A = \lambda I + F$ and $B = -\lambda I + F'$.*

Proof. According to Theorem 2.2, there exist finite rank operators $F, F' \in \mathcal{L}(X)$ satisfying $A = \lambda I + F$ and $B = -\lambda I + F'$ for some scalar $\lambda \in \mathbb{C}$. Obviously, $FS + SF' \in \mathcal{A}$ for all $S \in \mathcal{A}$. Now the above proposition yields the desired result. ∎

3. The case of Banach algebras. We will denote the set of all primitive ideals in A by $\mathrm{Prim}(A)$. Recall that primitive ideals are the kernels of irreducible representations of A. For every primitive ideal P we denote by π_P an irreducible representation of A on a Banach space X_P such that

$\operatorname{Ker}\pi_P = P$. In particular, recall that the algebra A/P can be seen as a subalgebra of $\mathcal{B}(X_P)$ acting densely on X_P. If $\operatorname{Prim}(A)$ is nonempty, we will often use the following result [19, Theorem 2.2.9]:

$$\sigma(x) \cup \{0\} = \bigcup_{P \in \operatorname{Prim}(A)} \sigma(x+P) \cup \{0\}.$$

Recall that for a given linear operator T from a Banach space X into a Banach space Y, the *separating space* of T is the set

$$\mathcal{S}(T) = \{y \in Y : \text{there is a sequence } (x_n)_n \text{ in } X \text{ with } x_n \to 0 \text{ and } Tx_n \to y\}.$$

Clearly, $\mathcal{S}(T)$ is a closed subspace of Y. By the closed graph theorem, T is continuous if and only if $\mathcal{S}(T) = \{0\}$. Moreover, the map $\widehat{T} : X \to Y/\mathcal{S}(T)$ defined by $\widehat{T}(x) = Tx + \mathcal{S}(T)$ is continuous. More details can be found in [20].

3.1. Lemma. *Let A be a complex Banach algebra and let G be a continuous generalized derivation on A determined by a derivation d of A. If P is a primitive ideal of A, then $dP \subseteq P$.*

Proof. Let $\{x_k\}$ be a sequence in A such that $x_k \to 0$ and $dx_k \to y \in A$. Since G is continuous, we infer that $0 = \lim G(zx_k) = zy$ for each $z \in A$. Consequently, $Ay = \{0\}$. Let us denote by I the closed ideal

$$I = \{u \in A : Au = \{0\}\}.$$

Then $\mathcal{S}(d) \subseteq I \subseteq \operatorname{rad} A$. Consequently, the map $\bar{d} : A \to A/I$ defined by $\bar{d}a = da + I$ is continuous. Next let $u \in I$. For all $x \in A$, we have

$$0 = d(xu) = x(du).$$

It follows that $A(du) = \{0\}$, which shows that $dI \subseteq I$. Now we can define the map $\widetilde{d} : A/I \to A/I$ such that $\widetilde{d}(a+I) = da + I$. Note that $\widetilde{d}$ is continuous. According to [12, Proposition 2.7.22], $\widetilde{d}$ leaves invariant every primitive ideal of A/I. Let P be a primitive ideal of A. Then P/I is a primitive ideal of A/I. Thus, $dP \subseteq P + I = P$. ∎

An algebra is said to be *semiprime* if $\{0\}$ is the only two-sided ideal I for which $I^2 = \{0\}$. Recall that every semisimple algebra is semiprime. Note the following consequence of the above proof.

3.2. Lemma. *Let A be a complex semiprime Banach algebra and let G be a continuous generalized derivation on A determined by a derivation d of A. Then d is continuous.*

3.3. Remark. One is tempted to expect that the derivation d in Lemma 3.1 is also continuous. But this is not true in general. Indeed, it follows from [18, Example 1.1] that there exists a Banach algebra A and a discontinuous derivation d on A such that $A^2 \neq \{0\}$ and $A(dA) = (dA)A = \{0\}$.

Pick $a \in A$ such that $aA \neq \{0\}$. Let $G : A \to A$ be the left centralizer defined by $Gx = ax$. Then G is continuous. Moreover, G can be seen as a generalized derivation determined by the derivation d.

We now come to our first general result.

3.4. THEOREM. *Let A be a complex Banach algebra and let G be a continuous generalized derivation on A determined by a derivation d of A. Suppose that $\sharp\sigma(Gx) < \infty$ for all $x \in A$. Then there exists $a \in A$ such that $a + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ and $dx - \delta_a(x) \in \operatorname{rad} A$ for all $x \in A$.*

Proof. By [8, Lemma 2.1], there exists $n \in \mathbb{N}^*$ such that $\sharp\sigma(Gx) \leq n$ for all $x \in A$. If $\operatorname{Prim}(A)$ is empty, there is nothing to prove. So suppose that $\operatorname{Prim}(A)$ is nonempty and let P be a primitive ideal of A. By Lemma 3.1, $dP \subseteq P$, so denote by d_P the induced derivation on A/P.

Our next step will be to prove that d_P is of the form $d_P(S) = TS - ST$ for some linear operator T on X_P. Suppose that this is not true; then X_P is infinite-dimensional. Let $\zeta_1, \ldots, \zeta_{n+1}$ be linearly independent vectors from X_P. Applying the Jacobson density theorem and [9, Theorem 3.6] we see that there exist $x, y \in A$ such that

$$(\pi_P d(y))\zeta_i = i\zeta_i, \quad (\pi_P y)\zeta_i = 0, \quad (\pi_P x)\zeta_i = \zeta_i.$$

This implies that $(\pi_P G(xy))\zeta_i = i\zeta_i$. As a result, $\{1, \ldots, n+1\} \subseteq \sigma(G(xy))$, a contradiction.

Now let T be a linear operator on X_P such that $d_P(S) = TS - ST$ for every $S \in A/P$. Suppose that there are linearly independent vectors $\zeta_1, \ldots, \zeta_{n+1}$ in X_P such that the set $\{\zeta_1, \ldots, \zeta_{n+1}, T\zeta_1, \ldots, T\zeta_{n+1}\}$ is linearly independent. Then we can choose $x, y \in A$ such that

$$(\pi_P(y))\zeta_i = 0, \quad (\pi_P(y))(T\zeta_i) = i\zeta_i, \quad (\pi_P(x))\zeta_i = \zeta_i.$$

Thus $(\pi_P G(xy))\zeta_i = -i\zeta_i$ and $\{-1, \ldots, -(n+1)\} \subseteq \sigma(G(xy))$, a contradiction. It follows from [8, Lemma 3.1] that there exists $\lambda \in \mathbb{C}$ such that $T - \lambda I$ has finite rank. Clearly, $d_P(S) = \delta_{T-\lambda I}(S)$ for every $S \in A/P$. Thus, $\sigma(dx + P)$ is finite for all $x \in A$.

Now assume towards a contradiction that there exist distinct primitive ideals $P_1, \ldots, P_{n+1}$ of A such that $dA \nsubseteq P_i$ for $1 \leq i \leq n+1$. For each $i \in \{1, \ldots, n+1\}$, let the inner derivation d_{P_i} be implemented by the operator T_i. Then we can find $\zeta_i \in X_{P_i}$ such that the vectors $\zeta_i, T_i\zeta_i$ are linearly independent. Applying the extended Jacobson density theorem [13], we get elements $x, y \in A$ such that

$$\pi_i(y)\zeta_i = 0, \quad \pi_i(y)T_i\zeta_i = i\zeta_i, \quad \pi_i(x)\zeta_i = \zeta_i, \quad 1 \leq i \leq n+1.$$

This entails that $(\pi_i G(xy))\zeta_i = -i\zeta_i$ for each i. Hence $\{-1, \ldots, -(n+1)\} \subseteq \sigma(G(xy))$, a contradiction.

We have thereby shown that $\sigma(dx)$ is finite for all $x \in A$. Using [5, Theorem 2.4], we get the desired conclusion. ∎

3.5. Proposition. *Let A be a complex Banach algebra and let G be a continuous generalized derivation on A. Suppose that $\sharp\sigma(Gx) < \infty$ for all $x \in A$. Then there exist at most a finite number of primitive ideals P_i of A such that $G(A) \nsubseteq P_i$.*

Proof. Fix $n \in \mathbb{N}^*$ such that $\sharp\sigma(Gx) \leq n$ for all $x \in A$. Suppose that G is determined by a derivation d. It follows from Theorem 3.4 that there exists $a \in A$ such that $a + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ and $dx - \delta_a(x) \in \operatorname{rad} A$ for all $x \in A$. Further, there exist at most a finite number of primitive ideals P_i of A such that $dA \nsubseteq P_i$.

Next assume that there exist distinct primitive ideals $P_1, \dots, P_{n+1}$ of A such that $dA \subseteq \bigcap_{i=1}^{n+1} P_i$ and $GA \nsubseteq P_i$ for $1 \leq i \leq n+1$. In order to simplify the notation we write π_i, X_i instead of π_{P_i}, X_{P_i} respectively. For $1 \leq i \leq n+1$, pick $x_i \in A$ such that $Gx_i \notin P_i$ and choose $\zeta_i \in X_i$ such that $(\pi_i(Gx_i))\zeta_i \neq 0$. Applying the extended Jacobson density theorem [13], we can find $y_i \in A$ such that

$$\pi_i(y_i)((\pi_i Gx_i)\zeta_i) = i\zeta_i, \quad \pi_j(y_i)((\pi_j Gx_j)\zeta_j) = 0 \quad \text{for } j \neq i,\ 1 \leq j \leq n+1.$$

Set $x = x_1y_1 + \dots + x_{n+1}y_{n+1}$. Then $\pi_i(Gx)((\pi_i Gx_i)\zeta_i) = i(\pi_i Gx_i)\zeta_i$. We have proved that $\{1, \dots, n+1\} \subseteq \sigma(Gx)$. This contradiction completes the proof. ∎

We are now in a position to prove our main result.

3.6. Theorem. *Let A be a complex Banach algebra and let G be a continuous generalized derivation on A. Suppose that $\sharp\sigma(Gx) < \infty$ for all $x \in A$. Then $Ga + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ for all $a \in A$. Moreover, if A is unital then there are $u, v \in A$ such that $u + \operatorname{rad} A, v + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ and $(G - \delta_{u,v})A \subseteq \operatorname{rad} A$.*

Proof. Fix $n \in \mathbb{N}^*$ such that $\sharp\sigma(Gx) \leq n$ for every $x \in A$. Let G be determined by the derivation d. It follows from Theorem 3.4 that there exists $a \in A$ such that $a + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ and $dx - \delta_a(x) \in \operatorname{rad} A$ for all $x \in A$. Let P be a primitive ideal of A. Since $\pi_P(a)$ has finite rank, there exists a finite-dimensional subspace H of X_P such that $X_P = \operatorname{Ker}(\pi_P(a)) \oplus H$.

Now assume towards a contradiction that there exists $x \in A$ such that $\pi_P(Gx)$ has infinite rank. Then we check easily that there exist linearly independent vectors $\zeta_1, \dots, \zeta_{n+1}$ in $\operatorname{Ker}(\pi_P(a))$ such that the set $\{\zeta_1, \dots, \zeta_{n+1}, \pi_P(Gx)\zeta_1, \dots, \pi_P(Gx)\zeta_{n+1}\}$ is linearly independent and contained in $\operatorname{Ker} \pi_P(a)$. Now we can choose $y \in A$ such that

$$\pi_P(y)\pi_P(Gx)\zeta_i = i\zeta_i, \quad 1 \leq i \leq n+1.$$

This entails that

$$\pi_P(G(xy))(\pi_P(Gx))\zeta_i = i(\pi_P(Gx))\zeta_i$$

for each i. Consequently, $\{1,\dots,n+1\} \subseteq \sigma(G(xy))$, a contradiction.

As a result, $Gx + P \in \operatorname{soc}(A/P)$ for all $x \in A$. Now using the above proposition and [8, Proposition 2.2], we find that $Gx+\operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ for all $x \in A$.

Finally, suppose that A is unital. Then

$$Gx = (G1)x - \delta_a(x) \in \operatorname{rad} A, \qquad \forall x \in A. \blacksquare$$

3.7. Corollary. *Let A be a complex semisimple Banach algebra and let G be a generalized derivation on A. Suppose that $\sharp\sigma(Gx) < \infty$ for all $x \in A$. Then $G(A) \subseteq \operatorname{soc} A$. Moreover, if A is unital then there exist $u, v \in \operatorname{soc} A$ such that $G = \delta_{u,v}$.*

In the case of generalized inner derivations, we have the following characterization.

3.8. Theorem. *Let A be a complex Banach algebra and let $a, b \in A$. Then the following conditions are equivalent:*

(1) $\#\sigma(ax + xb) < \infty$ *for every x in A,*
(2) $ax + xb + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ *for every x in A,*
(3) *there exist $u \in \mathcal{Z}(A)$ and $a', b' \in A$ such that $a'+\operatorname{rad} A, b'+\operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ and $a = u + a'$, $b = -u - b'$.*

Proof. The implication (3)⇒(1) is clear, and (1)⇒(2) is a direct consequence of Theorem 3.6. So suppose that (2) is true. Then $\sharp\sigma(ax + xb + \operatorname{rad} A) < \infty$ for all $x \in A$. It follows from [3, Theorem 3.1.5] that $\sigma(ax+xb)$ is finite for all $x \in A$. Next we use the temporary notation $\overline{A} = A/\operatorname{rad} A$ and $x + \operatorname{rad} A = \overline{x}$ for every $x \in A$. Since the generalized derivation $\delta_{a,b}$ is determined by the inner derivation δ_{-b}, Theorem 3.4 tells us that there exists $b' \in A$ such that $\overline{b'} \in \operatorname{soc} \overline{A}$ and $\delta_{b'+b}A \subseteq \operatorname{rad} A$. Set $-u = b + b'$. Then $u \in \mathcal{Z}(A)$ and $(\delta_{a,b} - \delta_{a-u,-b'})A \subseteq \operatorname{rad} A$. Applying again [3, Theorem 3.1.5], we infer that $\sharp\sigma(\delta_{a-u,-b'}x) < \infty$ for all $x \in A$. By Theorem 3.6, $\overline{\delta_{a-u,-b'}(A)} \subseteq \operatorname{soc} \overline{A}$. Since $\overline{b'} \in \operatorname{soc} \overline{A}$, it follows that $(\overline{a-u})\overline{A} \subseteq \operatorname{soc} \overline{A}$. Now it is easy to see that the ideal of $\overline{A}$ generated by $\overline{a-u}$ is algebraic. Consequently, $\overline{a-u} \in \operatorname{soc} \overline{A}$ and (3) is proved. $\blacksquare$

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