STUDIA MATHEMATICA 210 (2) (2012)

Numerical radius inequalities for 2×2 operator matrices

by

Omar Hirzallah (Zarqa), Fuad Kittaneh (Amman) and
Khalid Shebrawi (Salt)

Abstract. We derive several numerical radius inequalities for 2×2 operator matrices. Numerical radius inequalities for sums and products of operators are given. Applications of our inequalities are also provided.

1. Introduction. Let $\mathfrak{B}(\mathcal{H})$ be the space of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. The *numerical range* of $A\in\mathfrak{B}(\mathcal{H})$, denoted by $W(A)$, is the subset of the complex numbers given by

$$W(A)=\{\langle Ax,x\rangle : x\in\mathcal{H},\ \|x\|=1\}.$$

The *numerical radius* of A, $w(A)$, is defined by

$$w(A)=\sup\{|\lambda| : \lambda\in W(A)\}.$$

It is well-known that $w(\cdot)$ defines a norm on $\mathfrak{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for $A\in\mathfrak{B}(\mathcal{H})$, we have

$$\frac{1}{2}\|A\|\le w(A)\le\|A\|. \tag{1.1}$$

Also, it is known that $w(\cdot)$ is weakly unitarily invariant, that is,

$$w(U^*AU)=w(A) \tag{1.2}$$

for every unitary $U\in\mathfrak{B}(\mathcal{H})$. For other properties of the numerical radius, the reader is referred to [7] and [8]. Recent numerical radius inequalities for commutators of operators and operator matrices have been given in [9] and [10].

The following numerical radius inequalities for a product of operators have been given in [6]: If $A,B\in\mathfrak{B}(\mathcal{H})$, then

$$\max(\|A+B\|^2,\|A-B\|^2)-\big\||A^*|^2+|B^*|^2\big\|\le 2w(AB^*) \tag{1.3}$$

2010 *Mathematics Subject Classification*: Primary 47A12, 47A30, 47A63; Secondary 15A60, 47B15.
Key words and phrases: numerical range, numerical radius, operator norm.

DOI: 10.4064/sm210-2-1

and

$$\big\| |A^*|^2 + |B^*|^2 \big\| - \min(\|A+B\|^2, \|A-B\|^2) \le 2w(AB^*), \tag{1.4}$$

where $|X| = (X^*X)^{1/2}$.

The aim of this paper is to present new numerical radius inequalities for 2×2 operator matrices. In Section 2, we give an inequality stronger than (1.3), and a corresponding reverse type inequality. Moreover, we derive sharp estimates for the numerical radii of 2×2 operator matrices. One of the applications of our results is a generalization of (1.1). In Section 3, we give numerical radius inequalities for sums and products of operators from which (1.4) follows as a special case.

2. Inequalities for the numerical radii of 2×2 operator matrices. In this section, we introduce new inequalities for the numerical radii of 2×2 operator matrices in $\mathfrak{B}(\mathcal{H}\oplus\mathcal{H})$. A general 2×2 operator matrix in $\mathfrak{B}(\mathcal{H}\oplus\mathcal{H})$ is an operator of the form $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where $A, B, C, D \in \mathfrak{B}(\mathcal{H})$. The operator $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]$ is called the *diagonal part* of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]$ is the *off-diagonal part*. It is well-known (see, e.g., [3, p. 107]) that

$$w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \le w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right), \tag{2.1}$$

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \le w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right), \tag{2.2}$$

which are pinching type inequalities. Moreover, it is known (see, e.g., [4, p. 81]) that

$$w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) = \max(w(A), w(D)), \tag{2.3}$$

while several estimates for $w\left(\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]\right)$ have been given in [10].

For $A \in \mathfrak{B}(\mathcal{H})$ with $A^2 = 0$, it is known (see, e.g., [12]) that

$$w(A) = \tfrac{1}{2}\|A\|. \tag{2.4}$$

Since $\left[\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right]^2 = \left[\begin{smallmatrix} 0 & 0 \\ C & 0 \end{smallmatrix}\right]^2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, the subadditivity of the numerical radius $w(\cdot)$ and the inequalities (2.1), (2.2), together with the identities (2.3) and (2.4), imply that

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le \max(w(A), w(D)) + \frac{\|B\| + \|C\|}{2}$$

and

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \ge \max\left(\max(w(A), w(D)), w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right).$$

In particular, if $C = D = 0$, then

$$\max\left(w(A), \frac{\|B\|}{2}\right) \le w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \le w(A) + \frac{\|B\|}{2}. \tag{2.5}$$

Now, we start by presenting numerical radius inequalities for 2×2 operator matrices. In order to do this, we need the following two lemmas containing a series of new inequalities for the usual operator norm.

LEMMA 2.1. *Let $A, B \in \mathfrak{B}(\mathcal{H})$. Then*

$$\begin{aligned} &\max(\|A+B\|^2, \|A-B\|^2) \\ &\ge \max\big(\|A^2+B^2\|, \big\||A|^2+|B|^2\big\|, \big\||A^*|^2+|B^*|^2\big\|\big) + \frac{\big|\|A+B\|^2 - \|A-B\|^2\big|}{2} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\min(\|A+B\|^2, \|A-B\|^2) \\ &\ge \max\big(\|A^2+B^2\|, \big\||A|^2+|B|^2\big\|, \big\||A^*|^2+|B^*|^2\big\|\big) - \frac{\big|\|A+B\|^2 - \|A-B\|^2\big|}{2}. \end{aligned} \tag{2.7}$$

Proof. To prove (2.6), observe that

$$\begin{aligned} &\max(\|A+B\|^2, \|A-B\|^2) \\ &\qquad = \frac{1}{2}\big(\|A+B\|^2 + \|A-B\|^2 + \big|\|A+B\|^2 - \|A-B\|^2\big|\big) \\ &\qquad \ge \frac{1}{2}\big(\|(A+B)^2\| + \|(A-B)^2\| + \big|\|A+B\|^2 - \|A-B\|^2\big|\big) \\ &\qquad \ge \frac{1}{2}\big(\|(A+B)^2 + (A-B)^2\| + \big|\|A+B\|^2 - \|A-B\|^2\big|\big) \\ &\qquad = \|A^2+B^2\| + \frac{\big|\|A+B\|^2 - \|A-B\|^2\big|}{2}. \end{aligned} \tag{2.8}$$

On the other hand,

$$\begin{aligned} &\max(\|A+B\|^2, \|A-B\|^2) \\ &\qquad = \frac{1}{2}\big(\|A+B\|^2 + \|A-B\|^2 + \big|\|A+B\|^2 - \|A-B\|^2\big|\big) \\ &\qquad = \frac{1}{2}\big(\big\||A+B|^2\big\| + \big\||A-B|^2\big\| + \big|\|A+B\|^2 - \|A-B\|^2\big|\big) \\ &\qquad \ge \frac{1}{2}\big(\big\||A+B|^2 + |A-B|^2\big\| + \big|\|A+B\|^2 - \|A-B\|^2\big|\big) \\ &\qquad = \big\||A|^2 + |B|^2\big\| + \frac{\big|\|A+B\|^2 - \|A-B\|^2\big|}{2}. \end{aligned} \tag{2.9}$$

Replacing A and B by A^* and B^*, respectively, in (2.9) we obtain

$$\max(\|A+B\|^2, \|A-B\|^2) \geq \big\||A^*|^2+|B^*|^2\big\| + \frac{\big|\|A+B\|^2-\|A-B\|^2\big|}{2}. \tag{2.10}$$

Now, (2.6) follows from (2.8)–(2.10).

The inequality (2.7) can be proved by a similar argument. ■

LEMMA 2.2. *Let* $A, B \in \mathfrak{B}(\mathcal{H})$. *Then*

$$\max(\|A+B\|^2, \|A-B\|^2) \leq \min\big(\big\||A|^2+|B|^2\big\| + \|A^*B+B^*A\|, \big\||A^*|^2+|B^*|^2\big\| + \|AB^*+BA^*\|\big) \tag{2.11}$$

and

$$\min(\|A+B\|^2, \|A-B\|^2) \geq \max\big(\big\||A|^2+|B|^2\big\| - \|A^*B+B^*A\|, \big\||A^*|^2+|B^*|^2\big\| - \|AB^*+BA^*\|\big). \tag{2.12}$$

Proof. Observe that $(A\pm B)^*(A\pm B) = |A|^2+|B|^2 \pm (A^*B+B^*A)$, and so

$$\begin{aligned}\|A\pm B\|^2 &= \|(A\pm B)^*(A\pm B)\| \\ &= \big\||A|^2+|B|^2 \pm (A^*B+B^*A)\big\| \\ &\leq \big\||A|^2+|B|^2\big\| + \|A^*B+B^*A\|.\end{aligned}$$

Consequently,

$$\max(\|A+B\|^2, \|A-B\|^2) \leq \big\||A|^2+|B|^2\big\| + \|A^*B+B^*A\|. \tag{2.13}$$

Replacing A and B by A^* and B^*, respectively, in (2.13), we obtain

$$\max(\|A+B\|^2, \|A-B\|^2) \leq \big\||A^*|^2+|B^*|^2\big\| + \|AB^*+BA^*\|. \tag{2.14}$$

Now, (2.11) follows from (2.13) and (2.14).

The inequality (2.12) follows by a similar argument. ■

REMARK. Since A^*B+B^*A is self-adjoint, we have

$$\begin{aligned}\|A^*B+B^*A\| &= w(A^*B+B^*A) \\ &\leq w(A^*B)+w(B^*A) \\ &= 2w(A^*B) \quad (\text{since } w(A^*B)=w(B^*A)).\end{aligned}$$

Similarly,

$$\|AB^*+BA^*\| \leq 2w(AB^*). \tag{2.15}$$

In view of (2.15), it is evident that the inequality (2.11) is stronger than (1.3).

As employed in the proof of Lemma 2.1, it is known that for real numbers a and b, we have

$$\max(a,b) = \frac{a+b}{2} + \frac{|a-b|}{2}. \tag{2.16}$$

We now derive operator norm inequalities related to (2.16).

Lemma 2.3. *Let $X, Y \in \mathfrak{B}(\mathcal{H})$. Then*

$$\begin{aligned} &\max(\|X\|^2, \|Y\|^2) \\ &\geq \frac{\max\big(\|X^2+Y^2\|, \||X|^2+|Y|^2\|, \||X^*|^2+|Y^*|^2\|\big)}{2} + \frac{\big|\|X\|^2-\|Y\|^2\big|}{2}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\min(\|X\|^2, \|Y\|^2) \\ &\geq \frac{\max\big(\|X^2+Y^2\|, \||X|^2+|Y|^2\|, \||X^*|^2+|Y^*|^2\|\big)}{2} - \frac{\big|\|X\|^2-\|Y\|^2\big|}{2}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\max(\|X\|^2, \|Y\|^2) \\ &\leq \frac{1}{2}\min\big(\||X|^2+|Y|^2\|+\||X|^2-|Y|^2\|, \||X^*|^2+|Y^*|^2\|+\||X^*|^2-|Y^*|^2\|\big), \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\min(\|X\|^2, \|Y\|^2) \\ &\geq \frac{1}{2}\max\big(\||X|^2+|Y|^2\|-\||X|^2-|Y|^2\|, \||X^*|^2+|Y^*|^2\|-\||X^*|^2-|Y^*|^2\|\big). \end{aligned} \tag{2.20}$$

Proof. The inequalities (2.17)–(2.20) follow, respectively, from Lemmas 2.1 and 2.2 by letting $A = X + Y$ and $B = X - Y$. ∎

Now, we need the following lemma [2].

Lemma 2.4. *Let $X, Y \in \mathfrak{B}(\mathcal{H})$ be nonzero operators. Then the relation $\|X+Y\| = \|X\| + \|Y\|$ holds if and only if $\|X\|\,\|Y\| \in \overline{W(X^*Y)}$, where the bar denotes closure.*

A particular case of the inequality (2.17) says that if $X, Y \in \mathfrak{B}(\mathcal{H})$, then

$$\frac{\|X^2+Y^2\|}{2} + \frac{\big|\|X\|^2-\|Y\|^2\big|}{2} \leq \max(\|X\|^2, \|Y\|^2),$$

and so

$$\frac{\|X^2+Y^2\|}{2} \leq \max(\|X\|^2, \|Y\|^2). \tag{2.21}$$

Based on Lemma 2.4, a condition for equality in (2.21) can be stated:

THEOREM 2.5. *Let $X, Y \in \mathfrak{B}(\mathcal{H})$ be nonzero operators. Then*

$$\frac{\|X^2+Y^2\|}{2} = \max(\|X\|^2, \|Y\|^2) \quad \textit{if and only if} \quad w(X^{*2}Y^2) = \max(\|X\|^4, \|Y\|^4).$$

Proof. Suppose that $\|X^2+Y^2\|/2 = \max(\|X\|^2, \|Y\|^2)$. Since

$$\frac{\|X^2+Y^2\|}{2} \le \frac{\|X^2\|+\|Y^2\|}{2} \le \frac{\|X\|^2+\|Y\|^2}{2} \le \max(\|X\|^2, \|Y\|^2),$$

we have

$$\frac{\|X\|^2+\|Y\|^2}{2} = \max(\|X\|^2, \|Y\|^2), \tag{2.22}$$

$$\|X^2\|+\|Y^2\| = \|X\|^2+\|Y\|^2, \tag{2.23}$$

$$\|X^2+Y^2\| = \|X^2\|+\|Y^2\|. \tag{2.24}$$

It follows from (2.22) and (2.23) that

$$\|X^2\| = \|X\|^2 = \|Y\|^2 = \|Y^2\|. \tag{2.25}$$

The equality (2.24), together with Lemma 2.4 and (2.25), implies that

$$\max(\|X\|^4, \|Y\|^4) = \|X^2\|\,\|Y^2\| \in \overline{W(X^{*2}Y^2)}, \tag{2.26}$$

and so

$$\max(\|X\|^4, \|Y\|^4) \le w(X^{*2}Y^2). \tag{2.27}$$

This inequality, together with

$$w(X^{*2}Y^2) \le \|X^{*2}Y^2\| \le \|X\|^2\|Y\|^2 \le \max(\|X\|^4, \|Y\|^4),$$

implies that $w(X^{*2}Y^2) = \max(\|X\|^4, \|Y\|^4)$. This proves the "only if" part.

For the "if" part, suppose that

$$w(X^{*2}Y^2) = \max(\|X\|^4, \|Y\|^4). \tag{2.28}$$

Then, by the second inequality of (1.1), we have

$$\begin{aligned} w(X^{*2}Y^2) &\le \|X^{*2}Y^2\| \le \|X^{*2}\|\,\|Y^2\| = \|X^2\|\,\|Y^2\| \\ &\le \|X\|^2\|Y\|^2 \le \frac{\|X\|^4+\|Y\|^4}{2} \le \max(\|X\|^4, \|Y\|^4). \end{aligned} \tag{2.29}$$

The inequality (2.29), together with (2.28), implies that

$$\|X\|^2\|Y\|^2 = \frac{\|X\|^4+\|Y\|^4}{2}, \tag{2.30}$$

$$\|X^2\|\,\|Y^2\| = \|X\|^2\|Y\|^2, \tag{2.31}$$

$$w(X^{*2}Y^2) = \|X^2\|\,\|Y^2\|. \tag{2.32}$$

The equalities (2.30) and (2.31) imply that

$$\|X^2\| = \|X\|^2 = \|Y\|^2 = \|Y^2\|. \tag{2.33}$$

Also, (2.32) implies that $\|X^2\|\,\|Y^2\| \in \overline{W(X^{*2}Y^2)}$, and so

$$\begin{aligned}\frac{\|X^2+Y^2\|}{2} &= \frac{\|X^2\|+\|Y^2\|}{2} && \text{(by Lemma 2.4)}\\ &= \max(\|X\|^2,\|Y\|^2) && \text{(by (2.33))},\end{aligned}$$

as required. ■

Based on (1.1) and (2.6), an upper bound for the numerical radius of the general 2×2 operator matrix $\left[\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right]$ can be derived:

THEOREM 2.6. *Let $A,B,C,D \in \mathfrak{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} A & B\\ C & D\end{bmatrix}\right) \le \min(\alpha,\beta),$$

where

$$\begin{aligned}\alpha &= \sqrt{\frac{\|A+B\|^2+\|A-B\|^2}{2}} + \sqrt{\frac{\|C+D\|^2+\|C-D\|^2}{2}},\\ \beta &= \sqrt{\frac{\|A+C\|^2+\|A-C\|^2}{2}} + \sqrt{\frac{\|B+D\|^2+\|B-D\|^2}{2}}.\end{aligned}$$

Proof. We have

$$\begin{aligned}(2.34)\qquad w\left(\begin{bmatrix} A & B\\ 0 & 0\end{bmatrix}\right) &\le \left\|\begin{bmatrix} A & B\\ 0 & 0\end{bmatrix}\right\| \qquad \text{(by the second inequality of (1.1))}\\ &= \left\|\begin{bmatrix} A & B\\ 0 & 0\end{bmatrix}\begin{bmatrix} A^* & 0\\ B^* & 0\end{bmatrix}\right\|^{1/2} = \left\|\begin{bmatrix} |A^*|^2+|B^*|^2 & 0\\ 0 & 0\end{bmatrix}\right\|^{1/2}\\ &= \||A^*|^2+|B^*|^2\|^{1/2}\end{aligned}$$

and so

$$\begin{aligned}(2.35)\qquad w\left(\begin{bmatrix} A & B\\ 0 & 0\end{bmatrix}\right) &\le \sqrt{\max(\|A+B\|^2,\|A-B\|^2) - \frac{|\|A+B\|^2-\|A-B\|^2|}{2}} \qquad \text{(by (2.6))}\\ &= \sqrt{\frac{\|A+B\|^2+\|A-B\|^2}{2}}.\end{aligned}$$

For the general case, let $U = \left[\begin{smallmatrix} 0 & I\\ I & 0\end{smallmatrix}\right]$. Then U is unitary, and it follows from the subadditivity of the numerical radius $w(\cdot)$ that

$$(2.36)\quad w\left(\begin{bmatrix} A & B\\ C & D\end{bmatrix}\right) \le w\left(\begin{bmatrix} A & B\\ 0 & 0\end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & 0\\ C & D\end{bmatrix}\right)$$

$$= w\left(\begin{bmatrix} A & B\\ 0 & 0\end{bmatrix}\right) + w\left(U^*\begin{bmatrix} D & C\\ 0 & 0\end{bmatrix}U\right) \quad \text{(by (1.2))}$$

$$\le \sqrt{\frac{\|A+B\|^2+\|A-B\|^2}{2}} + \sqrt{\frac{\|C+D\|^2+\|C-D\|^2}{2}} \text{ (by (2.35))}$$

$$= \alpha.$$

Also, by applying (2.36) to $\left[\begin{smallmatrix} A^* & C^*\\ B^* & D^*\end{smallmatrix}\right]$, and observing that $w\left(\left[\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right]\right) = w\left(\left[\begin{smallmatrix} A^* & C^*\\ B^* & D^*\end{smallmatrix}\right]\right)$, we have

$$(2.37)\quad w\left(\begin{bmatrix} A & B\\ C & D\end{bmatrix}\right) = w\left(\begin{bmatrix} A^* & C^*\\ B^* & D^*\end{bmatrix}\right)$$

$$\le \sqrt{\frac{\|A^*+C^*\|^2+\|A^*-C^*\|^2}{2}} + \sqrt{\frac{\|B^*+D^*\|^2+\|B^*-D^*\|^2}{2}}$$

$$= \sqrt{\frac{\|A+C\|^2+\|A-C\|^2}{2}} + \sqrt{\frac{\|B+D\|^2+\|B-D\|^2}{2}} = \beta.$$

Now, the result follows from (2.36) and (2.37). ∎

Another upper bound for the numerical radius of the operator $\left[\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right]$ is as follows.

THEOREM 2.7. *Let $A, B, C, D \in \mathfrak{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} A & B\\ C & D\end{bmatrix}\right) \le \min(\alpha,\beta),$$

where

$$\alpha = \sqrt{\min(\|A+B\|^2, \|A-B\|^2) + 2w(AB^*)} + \sqrt{\min(\|D+C\|^2, \|D-C\|^2) + 2w(DC^*)},$$

$$\beta = \sqrt{\min(\|A+C\|^2, \|A-C\|^2) + 2w(A^*C)} + \sqrt{\min(\|B+D\|^2, \|B-D\|^2) + 2w(D^*B)}.$$

Proof. Since the operators AA^*+BB^* and $(A\pm B)(A\pm B)^*$ are positive, and since $AA^*+BB^* = (A\pm B)(A\pm B)^* \mp (AB^*+BA^*)$, we have

$$\begin{aligned}\|AA^*+BB^*\| &= w(AA^*+BB^*)\\ &= w((A\pm B)(A\pm B)^* \mp (AB^*+BA^*))\\ &\le w((A\pm B)(A\pm B)^*) + w(AB^*+BA^*)\\ &= \|(A\pm B)(A\pm B)^*\| + w(AB^*+BA^*)\end{aligned}$$

$$
\begin{aligned}
&= \|A \pm B\|^2 + w(AB^* + BA^*) \\
&\le \|A \pm B\|^2 + w(AB^*) + w(BA^*) \\
&= \|A \pm B\|^2 + 2w(AB^*) \quad (\text{since } w(AB^*) = w(BA^*)),
\end{aligned}
$$

and so

$$\|AA^* + BB^*\| \le \min(\|A+B\|^2, \|A-B\|^2) + 2w(AB^*). \tag{2.38}$$

The inequality (2.34), together with (2.38), implies that

$$w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \le \sqrt{\min(\|A+B\|^2, \|A-B\|^2) + 2w(AB^*)}. \tag{2.39}$$

Now, the proof of the general case can be obtained by an argument similar to that used in the proof of Theorem 2.6. ∎

An application of Theorem 2.7 yields

Corollary 2.8. *Let* $A, B, C, D \in \mathfrak{B}(\mathcal{H})$.

(a) *If* $AB^* = 0$ *and* $DC^* = 0$, *then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le \min(\|A+B\|, \|A-B\|) + \min(\|D+C\|, \|D-C\|). \tag{2.40}$$

(b) *If* $A^*C = 0$ *and* $D^*B = 0$, *then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le \min(\|A+C\|, \|A-C\|) + \min(\|B+D\|, \|B-D\|). \tag{2.41}$$

We need the following lemma concerning the numerical radii of certain 2×2 operator matrices. This lemma has been used in [9] and [10].

Lemma 2.9 (see, e.g., [1]). *Let* $A, B \in \mathfrak{B}(\mathcal{H})$. *Then*

$$w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \max(w(A+B), w(A-B)).$$

Using Lemma 2.9, a particular case of Corollary 2.8 can be stated:

Corollary 2.10. *Let* $A, B \in \mathfrak{B}(\mathcal{H})$. *If* $AB^* = 0$ *or* $A^*B = 0$, *then*

$$\max(w(A+B), w(A-B)) \le 2\min(\|A+B\|, \|A-B\|). \tag{2.42}$$

Proof. The result follows from (2.40) and (2.41), by letting $C = B$, $D = A$, and using Lemma 2.9. ∎

Remark. It should be mentioned here that (2.42) is not true for arbitrary operators $A, B \in \mathfrak{B}(\mathcal{H})$. To see this, consider a nonzero $A \in \mathfrak{B}(\mathcal{H})$, and let $B = A$.

An upper bound for the numerical radius of the operator $\left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$ has been given in (2.39). Now we present a lower bound for this numerical radius.

Theorem 2.11. *Let $A, B \in \mathfrak{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \geq \frac{1}{2}\sqrt{\max(\|A+B\|^2, \|A-B\|^2) - 2w(AB^*)}. \tag{2.43}$$

Proof. We have

$$\begin{aligned} w^2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) &\geq \frac{1}{4}\left\|\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right\|^2 \quad \text{(by the first inequality of (1.1))} \\ &= \frac{1}{4}\|AA^* + BB^*\| \\ &\geq \frac{1}{4}(\max(\|A+B\|^2, \|A-B\|^2) - 2w(AB^*)) \quad \text{(by (1.3))}. \end{aligned} \tag{2.44}$$

It remains to show that the right-hand side of (2.44) is nonnegative. In fact, it is shown in [5] that

$$2\|AB^*\| \leq \|A^*A + B^*B\|. \tag{2.45}$$

This inequality, together with the second inequality of (1.1), implies that

$$2w(AB^*) \leq \|A^*A + B^*B\|. \tag{2.46}$$

It follows from (2.6) applied to A^* and B^* that

$$\begin{aligned} \max(\|A+B\|^2, \|A-B\|^2) &= \max(\|A^*+B^*\|^2, \|A^*-B^*\|^2) \\ &\geq \|A^*A + B^*B\| + \frac{1}{2}\big|\|A^*+B^*\|^2 - \|A^*-B^*\|^2\big| \\ &= \|A^*A + B^*B\| + \frac{1}{2}\big|\|A+B\|^2 - \|A-B\|^2\big| \\ &\geq 2w(AB^*) + \frac{1}{2}\big|\|A+B\|^2 - \|A-B\|^2\big| \quad \text{(by (2.46))}, \end{aligned} \tag{2.47}$$

and so

$$\max(\|A+B\|^2, \|A-B\|^2) - 2w(AB^*) \geq \frac{1}{2}\big|\|A+B\|^2 - \|A-B\|^2\big|. \tag{2.48}$$

Now, it follows from (2.48) that the right-hand side of (2.44) is nonnegative, and so

$$w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \geq \frac{1}{2}\sqrt{\max(\|A+B\|^2, \|A-B\|^2) - 2w(AB^*)},$$

as required. ∎

Remark. If $AB^* = 0$, then (2.39) and (2.43) imply that

$$\frac{1}{2}\max(\|A+B\|, \|A-B\|) \leq w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \leq \min(\|A+B\|, \|A-B\|).$$

In particular, letting $B = 0$, we obtain (1.1).

Now, we need the following lemma, which has also been used in [9] and [10].

LEMMA 2.12 (see, e.g., [1]). *Let* $A, B \in \mathfrak{B}(\mathcal{H})$. *Then*

$$w\left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\right) = \max(w(A+iB), w(A-iB)).$$

In the following result, we derive another lower bound for the numerical radius of the operator $\left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$. In fact, a special case of this result has been given in [9], using a different analysis.

THEOREM 2.13. *Let* $A, B \in \mathfrak{B}(\mathcal{H})$. *Then*

$$w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \geq \frac{1}{2}\max(w(A \pm B), w(A \pm iB)). \tag{2.49}$$

Proof. Let $U = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$. Then U is unitary, and

$$\begin{aligned} \max(w(A+B), w(A-B)) &= w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \quad \text{(by Lemma 2.9)} \\ &= w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + U^*\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}U\right) \\ &\leq w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) + w\left(U^*\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}U\right) \\ &= 2w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \quad \text{(by (1.2))}. \end{aligned} \tag{2.50}$$

Moreover, let $V = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$. Then V is unitary, and

$$\begin{aligned} &\max(w(A+iB), w(A-iB)) \\ &\quad = w\left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\right) \quad \text{(by Lemma 2.12)} \\ &\quad = w\left(V^*\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}V + U^*\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}U\right) \\ &\quad \leq w\left(V^*\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}V\right) + w\left(U^*\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}U\right) \\ &\quad = 2w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \quad \text{(by (1.2))}. \end{aligned} \tag{2.51}$$

Now, the result follows from (2.50) and (2.51). ∎

To give further upper bounds for the numerical radius of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, we need the following compression inequality.

Lemma 2.14 (see, e.g., [11]). *Let $A, B, C, D \in \mathfrak{B}(\mathcal{H})$. Then*

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \le \left\| \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right\|.$$

Theorem 2.15. *Let $A, B, C, D \in \mathfrak{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \le \min(\alpha, \beta),$$

where

$$\alpha = \frac{1}{\sqrt{2}} \sqrt{\|A\|^2 + \|B\|^2 + \sqrt{(\|A\|^2 - \|B\|^2)^2 + 4\|AB^*\|^2}} + \frac{1}{\sqrt{2}} \sqrt{\|C\|^2 + \|D\|^2 + \sqrt{(\|C\|^2 - \|D\|^2)^2 + 4\|DC^*\|^2}}$$

and

$$\beta = \frac{1}{\sqrt{2}} \sqrt{\|A\|^2 + \|C\|^2 + \sqrt{(\|A\|^2 - \|C\|^2)^2 + 4\|A^*C\|^2}} + \frac{1}{\sqrt{2}} \sqrt{\|B\|^2 + \|D\|^2 + \sqrt{(\|B\|^2 - \|D\|^2)^2 + 4\|D^*B\|^2}}$$

Proof. First, we prove that

(2.52)
$$w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) \le \frac{1}{\sqrt{2}} \sqrt{\|A\|^2 + \|B\|^2 + \sqrt{(\|A\|^2 - \|B\|^2)^2 + 4\|AB^*\|^2}}.$$

We have

(2.53)
$$\begin{aligned} w^2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) &\le \left\| \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right\|^2 \quad \text{(by the second inequality of (1.1))} \\ &= \left\| \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} \right\| \\ &\le \left\| \begin{bmatrix} \|A^*A\| & \|A^*B\| \\ \|B^*A\| & \|B^*B\| \end{bmatrix} \right\| \quad \text{(by Lemma 2.14)} \end{aligned}$$

$$= \left\| \begin{bmatrix} \|A\|^2 & \|A^*B\| \\ \|A^*B\| & \|B\|^2 \end{bmatrix} \right\|$$
$$= \frac{1}{2}\Big(\|A\|^2 + \|B\|^2 + \sqrt{(\|A\|^2 - \|B\|^2)^2 + 4\|A^*B\|^2}\Big),$$

where the last equality follows from the fact that the usual operator norm of a positive (semidefinite) matrix is its largest eigenvalue. Now, (2.52) follows from (2.15).

The proof of the result for the general operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ can be obtained by an argument similar to that given in the proof of Theorem 2.6. ∎

An application of Theorem 2.15 yields

COROLLARY 2.16. *Let $A, B, C, D \in \mathfrak{B}(\mathcal{H})$.*

(a) *If $AB^* = 0$ and $DC^* = 0$, then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le \max(\|A\|, \|B\|) + \max(\|C\|, \|D\|).$$

(b) *If $A^*C = 0$ and $D^*B = 0$, then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le \max(\|A\|, \|C\|) + \max(\|B\|, \|D\|).$$

REMARKS. (1) Comparing the upper bounds for the numerical radius of $\left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$ given in (2.5), (2.39), and (2.52), it can be seen that none of them is uniformly better than the others. This can be demonstrated as follows: Let

$$\alpha = \sqrt{\min(\|A+B\|^2, \|A-B\|^2) + 2w(AB^*)},$$
$$\beta = \frac{1}{\sqrt{2}}\sqrt{\|A\|^2 + \|B\|^2 + \sqrt{(\|A\|^2 - \|B\|^2)^2 + 4\|A^*B\|^2}},$$
$$\gamma = w(A) + \frac{\|B\|}{2}.$$

If we take, for example, $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$, then $\alpha > \beta > \gamma$. On the other hand, if $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & -1 \\ -1 & 1 \end{smallmatrix}\right]$, then $\gamma > \beta > \alpha$.

(2) Comparing the lower bounds for the numerical radius of $\left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$ given in Theorems 2.11, 2.13, and (2.5), it can be seen that none of them is uniformly better than the others. This can be demonstrated by constructing suitable 2×2 matrices. We leave the details to the reader.

(3) Our numerical radius inequalities for 2×2 operator matrices are sharp. For example, in Theorems 2.6, 2.7, and 2.15, equality holds if we choose $A = I$, $B = C = D = 0$. In Theorem 2.11, equality holds if we choose $A = 0$, $B = I$.

3. Inequalities for the numerical radii of sums and products of operators. In this section, we introduce numerical radius inequalities for sums and products of operators. First, we need the following simple lemma on complex numbers.

LEMMA 3.1. *Let* $z_0, \dots, z_{n-1} \in \mathbb{C}$, *and let* $u_j = e^{2ij\pi/n}$, $j = 0, \dots, n-1$, *be the* n*th roots of unity. Then*

$$\sum_{k=0}^{n-1} |z_k|^2 = \frac{1}{n} \sum_{k=0}^{n-1} \Big| \sum_{j=0}^{n-1} u_j^k z_j \Big|^2.$$

Our result for sums of operators can be stated as follows.

THEOREM 3.2. *Let* $A_0, \dots, A_{n-1} \in \mathfrak{B}(\mathcal{H})$. *Then*

$$\sum_{k=0}^{n-1} w^2(A_k) \ge \frac{1}{n^2} \max_{\Gamma \subseteq \{0,\dots,n-1\}} w^2\Big(\sum_{k\in\Gamma} \sum_{j=0}^{n-1} u_j^k A_j \Big). \tag{3.1}$$

Proof. Let $x \in \mathcal{H}$ with $\|x| = 1$, and let $\Gamma \subseteq \{0, \dots, n-1\}$. Then

$$\sum_{k=0}^{n-1} |\langle A_k x, x\rangle|^2 = \frac{1}{n} \sum_{k=0}^{n-1} \Big| \sum_{j=0}^{n-1} u_j^k \langle A_j x, x\rangle \Big|^2 \quad \text{(by Lemma 3.1)} \tag{3.2}$$

$$\ge \frac{1}{n} \sum_{k\in\Gamma} \Big| \sum_{j=0}^{n-1} \langle u_j^k A_j x, x\rangle \Big|^2 \ge \frac{1}{n^2} \Big| \sum_{k\in\Gamma} \sum_{j=0}^{n-1} \langle u_j^k A_j x, x\rangle \Big|^2.$$

Now,

$$\sum_{k=0}^{n-1} w^2(A_k) = \sum_{k=0}^{n-1} \sup\{|\langle A_j x, x\rangle|^2 : x \in \mathcal{H},\ \|x\| = 1\} \tag{3.3}$$

$$\ge \sup\Big\{ \sum_{k=0}^{n-1} |\langle A_k x, x\rangle|^2 : x \in \mathcal{H}, \|x\| = 1 \Big\}$$

$$\ge \frac{1}{n^2} \sup\Big\{ \Big| \sum_{k\in\Gamma} \sum_{j=0}^{n-1} \langle u_j^k A_j x, x\rangle \Big|^2 : x \in \mathcal{H}, \|x\| = 1 \Big\} \quad \text{(by (3.2))}$$

$$= \frac{1}{n^2} \sup\Big\{ \Big| \Big\langle \sum_{k\in\Gamma} \sum_{j=0}^{n-1} u_j^k A_j x, x \Big\rangle \Big|^2 : x \in \mathcal{H}, \|x\| = 1 \Big\}$$

$$= \frac{1}{n^2} w^2\Big(\sum_{k\in\Gamma} \sum_{j=0}^{n-1} u_j^k A_j \Big)$$

for all $\Gamma \subseteq \{0, \dots, n-1\}$. Now, the result follows from (3.3). ∎

The following result, related to (2.11) and (2.12), gives lower bounds for the numerical radius of a product of operators. It can be easily seen that (1.4) follows from this result as a special case.

THEOREM 3.3. *Let $A, B \in \mathfrak{B}(\mathcal{H})$. Then*

$$w(AB^*) \geq \max(\alpha, \beta),$$

where

$$\alpha = \frac{1}{2}\big|\|AA^* + BB^*\| - \min(\|A+B\|^2, \|A-B\|^2)\big|,$$
$$\beta = \frac{1}{4}\big|\|A+B\|^2 - \|A-B\|^2\big|.$$

Proof. By (2.38), we have

$$2w(AB^*) \geq \|AA^* + BB^*\| - \min(\|A+B\|^2, \|A-B\|^2). \tag{3.4}$$

Also, Lemma 2.2, together with (2.15), implies that

$$2w(AB^*) \geq \min(\|A+B\|^2, \|A-B\|^2) - \|AA^* + BB^*\|. \tag{3.5}$$

Consequently, (3.4) and (3.5) imply that

$$w(AB^*) \geq \alpha. \tag{3.6}$$

On the other hand, (2.6) and (2.11) imply that

$$\begin{aligned}\|AA^* + BB^*\| &+ \|AB^* + BA^*\| \\ &\geq \max(\|A+B\|^2, \|A-B\|^2) \\ &\geq \|AA^* + BB^*\| + \frac{\big|\|A+B\|^2 - \|A-B\|^2\big|}{2},\end{aligned}$$

and so

$$\|AB^* + BA^*\| \geq \frac{1}{2}\big|\|A+B\|^2 - \|A-B\|^2\big|.$$

This inequality, together with (2.15), implies that

$$w(AB^*) \geq \beta. \tag{3.7}$$

Now, the result follows from (3.6) and (3.7). ∎

An application of Theorem 3.3 yields

COROLLARY 3.4. *Let $T \in \mathfrak{B}(\mathcal{H})$ with the Cartesian decomposition $T = A + iB$. Then*

$$w(AB) \geq \frac{1}{2}\|T\|^2 - \frac{1}{4}\|T^*T + TT^*\|. \tag{3.8}$$

Proof. This follows by applying Theorem 3.3 to the operators A and iB, and observing that $\|A \pm iB\| = \|T\|$ and $2(A^2 + B^2) = T^*T + TT^*$. ∎

A fundamental inequality for the numerical radius is the power inequality, which says that for $T \in \mathfrak{B}(\mathcal{H})$,

$$w(T^n) \leq (w(T))^n \tag{3.9}$$

for $n = 1, 2, \ldots$ (see, e.g., [8, p. 118]).

Now, we need the following lemma, which includes refinements of (1.1).

LEMMA 3.5 ([13]). *Let $T \in \mathfrak{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\|TT^* + T^*T\| \leq w^2(T) \leq \frac{1}{2}\|TT^* + T^*T\|.$$

We close this paper with the following result, a reverse type inequality of the power inequality (3.9), when $n = 2$.

THEOREM 3.6. *Let $T \in \mathfrak{B}(\mathcal{H})$ with the Cartesian decomposition $T = A + iB$. Then*

$$w(T^2) + 2\min(\|A\|^2, \|B\|^2) \geq w^2(T).$$

Proof. Replacing A and B by T and T^*, respectively, in Theorem 3.3, we obtain

$$\begin{aligned} w(T^2) &\geq \frac{1}{2}\big|\|TT^* + T^*T\| - \min\big(\|T + T^*\|^2, \|T - T^*\|^2\big)\big| \\ &\geq \frac{1}{2}\big(\|TT^* + T^*T\| - \min\big(\|T + T^*\|^2, \|T - T^*\|^2\big)\big) \\ &= \frac{1}{2}\big(\|TT^* + T^*T\| - 4\min(\|A\|^2, \|B\|^2)\big). \\ &\geq w^2(T) - 2\min(\|A\|^2, \|B\|^2) \quad \text{(by Lemma 3.5).} \ \blacksquare \end{aligned}$$

Acknowledgements. The authors are grateful to the referee for his suggestions.

References

[1] W. Bani-Domi and F. Kittaneh, *Norm equalities and inequalities for operator matrices*, Linear Algebra Appl. 429 (2008), 57–67.

[2] M. Barraa and M. Boumazgour, *Inner derivations and norm equality*, Proc. Amer. Math. Soc. 130 (2001), 471–467.

[3] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.

[4] R. Bhatia, *Positive Definite Matrices*, Princeton Univ. Press, 2007.

[5] R. Bhatia and F. Kittaneh, *On the singular values of a product of operators*, SIAM J. Matrix Anal. Appl. 11 (1990), 272–277.

[6] S. S. Dragomir, *Inequalities for the norm and numerical radius of composite operators in Hilbert spaces*, RGMIA Res. Rep. Coll. 8 (suppl.) (2005), art. 11.

[7] K. E. Gustafson and D. K. M. Rao, *Numerical Range*, Springer, New York, 1997.

[8] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer, New York, 1982.

[9] O. Hirzallah, F. Kittaneh, and K. Shebrawi, *Numerical radius inequalities for commutators of Hilbert space operators*, Numer. Funct. Anal. Optim. 32 (2011), 739–749.

[10] O. Hirzallah, F. Kittaneh, and K. Shebrawi, *Numerical radius inequalities for certain* 2×2 *operator matrices*, Integral Equations Operator Theory 71 (2011), 129–147.

[11] J. C. Hou and H. K. Du, *Norm inequalities of positive operator matrices*, Integral Equations Operator Theory 22 (1995), 281–294.

[12] F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. 158 (2003), 11–17.

[13] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math. 168 (2005), 73–80.

Omar Hirzallah
Department of Mathematics
Hashemite University
Zarqa, Jordan
E-mail: o.hirzal@hu.edu.jo

Fuad Kittaneh
Department of Mathematics
University of Jordan
Amman, Jordan
E-mail: fkitt@ju.edu.jo

Khalid Shebrawi
Department of Applied Sciences
Al-Balqa' Applied University
Salt, Jordan
E-mail: khalid@bau.edu.jo

Received November 2, 2011
Revised version August 5, 2012 (7355)