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A "hidden" characterization of approximatively polyhedral convex sets in Banach spaces

by

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Abstract. A closed convex subset C of a Banach space X is called *approximatively polyhedral* if for each $\varepsilon > 0$ there is a polyhedral (= intersection of finitely many closed half-spaces) convex set $P \subset X$ at Hausdorff distance $< \varepsilon$ from C. We characterize approximatively polyhedral convex sets in Banach spaces and apply the characterization to show that a connected component $\mathcal{H}$ of the space $\mathrm{Conv}_{\mathsf{H}}(X)$ of closed convex subsets of X endowed with the Hausdorff metric is separable if and only if $\mathcal{H}$ contains a polyhedral convex set.

1. Introduction. In [1] the authors proved that a closed convex subset C of a complete linear metric space X is polyhedral in its linear hull if and only if no infinite subset $A \subset X \setminus C$ is hidden behind C in the sense that $[a, b] \cap C \neq \emptyset$ for any distinct points $a, b \in A$. In this paper we shall prove a similar "hidden" characterization of approximatively polyhedral subsets in Banach spaces, simultaneously giving a characterization of separable components of the space $\mathrm{Conv}_{\mathsf{H}}(X)$ of non-empty closed convex subsets of a Banach space X, endowed with the *Hausdorff metric*

$$\mathsf{d}_{\mathsf{H}}(A, B) = \max\Big\{\sup_{a\in A} \mathsf{dist}(a, B), \sup_{b\in B} \mathsf{dist}(b, A)\Big\} \in [0, \infty].$$

Here $\mathsf{dist}(a, B) = \inf_{b\in B} \|a - b\|$ stands for the distance from a point $a \in X$ to a subset $B \subset X$ of the Banach space X.

It is well-known that for each $C \in \mathrm{Conv}_{\mathsf{H}}(X)$ the Hausdorff distance d_{H} restricted to the set

$$\mathcal{H}_C = \{A \in \mathrm{Conv}_{\mathsf{H}}(X) : \mathsf{d}_{\mathsf{H}}(A, C) < \infty\}$$

is a metric (see [10, Ch. 2]). The resulting metric space $(\mathcal{H}_C, \mathsf{d}_{\mathsf{H}})$ will be called the *Hausdorff metric component* (or just *component*) of C in $\mathrm{Conv}_{\mathsf{H}}(X)$.

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In fact, the present investigation was motivated by the problem of calculating the density of components of the space $\mathrm{Conv}_{\mathsf{H}}(X)$ and detecting closed convex subsets $C \subset X$ with $\mathcal{H}_C$ separable. In this paper we shall characterize such sets C in terms of approximative polyhedrality as well as in "hidden" terms resembling those from [1].

A convex subset C of a Banach space X is called

- *a closed half-space* if $C = f^{-1}([a,\infty))$ for some non-zero linear continuous functional $f : X \to \mathbb{R}$ and some $a \in \mathbb{R}$;
- *polyhedral* if C can be written as the intersection of a finite family of closed half-spaces in X;
- *approximatively polyhedral* if for every $\varepsilon > 0$ there is a closed polyhedral subset $P \subset X$ with $\mathsf{d}_{\mathsf{H}}(C, P) < \varepsilon$.

Observe that the whole space X is polyhedral, being the intersection of the empty family of closed half-spaces ([1]).

It is well-known that each compact convex subset of a Banach space is approximatively polyhedral (see [7] for more information on that topic). This is not necessarily true for non-compact closed convex sets. For example, the convex parabola

$$P = \{(x,y) \in \mathbb{R}^2 : y \geq x^2\}$$

is not approximatively polyhedral in $\mathbb{R}^2$, while the convex hyperbola

$$H = \{(x,y) \in \mathbb{R}^2 : y \geq \sqrt{x^2+1}\}$$

is approximatively polyhedral.

Next, we introduce some "hidden" properties of convex sets. Following [1], we say that a subset C of a linear space X *hides* a set $A \subset X$ if for any two distinct points $a, b \in A$ the segment

$$[a,b] = \{ta + (1-t)b : t \in [0,1]\}$$

meets C.

A convex subset C of a Banach space X is called

- *hiding* if C hides some infinite set $A \subset X \setminus C$;
- *positively hiding* if C hides some infinite set $A \subset X \setminus C$ such that $\inf_{a\in A} \mathsf{dist}(a, C) > 0$;
- *infinitely hiding* if C hides some infinite set $A \subset X \setminus C$ such that $\sup_{a\in A} \mathsf{dist}(a, C) = \infty$.

It is clear that each infinitely hiding set is positively hiding and each positively hiding set is hiding.

([1]) The polyhedrality of Banach spaces (as closed convex subsets of themselves) should not be mixed with the classical notion of polyhedrality of Banach spaces studied in the geometric theory of Banach spaces [4, §6].

By [1], a closed convex subset C of a complete linear metric space X is hiding if and only if C is not polyhedral in its closed linear hull. So, both the parabola and the hyperbola are hiding (being non-polyhedral). Yet, the parabola is infinitely hiding (but not approximatively polyhedral) while the hyperbola is not positively hiding (but is approximatively polyhedral).

It turns out that approximative polyhedrality and positive or infinite hiding properties are mutually exclusive, and can be characterized via properties of the characteristic cone of a given convex set.

Let us recall that the *characteristic cone* of a convex subset C in a linear topological space X is the set V_C of all vectors $v \in X$ such that for every point $c \in C$ the ray $c + \bar{\mathbb{R}}_+ v = \{c + tv : t \geq 0\}$ lies in C. Here $\bar{\mathbb{R}}_+ = [0, \infty)$. The cone V_C is closed in X if C is closed or open in X (see Lemma 2.2).

The main result of this paper is the following characterization theorem that will be used in the paper [2] devoted to recognizing the topological structure of the space $\mathrm{Conv}_{\mathsf{H}}(X)$. In the finite-dimensional case, the equivalence of conditions (1)–(3) was proved by Victor Klee [9].

Theorem 1.1. *For a closed convex subset C of a Banach space X the following conditions are equivalent:*

(1) *C is approximatively polyhedral;*
(2) *the characteristic cone V_C is polyhedral in X and $\mathsf{d}_{\mathsf{H}}(C, V_C) < \infty$;*
(3) *the component $\mathcal{H}_C$ contains a polyhedral closed convex set;*
(4) *$\mathcal{H}_C$ contains no positively hiding closed convex set;*
(5) *$\mathcal{H}_C$ is separable;*
(6) $\mathrm{dens}(\mathcal{H}_C) < \mathfrak{c}$.

If X is finite-dimensional, then (1)–(6) *are equivalent to:*

(7) *C is not positively hiding;*
(8) *C is not infinitely hiding.*

Let us recall that the *density* $\mathrm{dens}(X)$ of a topological space X is the smallest cardinality $|D|$ of a dense subset D of X. Topological spaces with at most countable density are called *separable.*

Remark 1. Observe that the closed unit ball $C = \{x \in l_2 : \|x\| \leq 1\}$ in the separable Hilbert space l_2 is positively hiding but not infinitely hiding, so (7) and (8) are not equivalent in infinite-dimensional Banach spaces.

Theorem 1.1 will be proved in Section 7 after long preliminary work in Sections 2–6.

2. Some properties of characteristic cones. This section is of preliminary character and contains some information on convex cones in Banach spaces. All linear (and Banach) spaces considered in this paper are over the field $\mathbb{R}$ of real numbers.

By a *convex cone* in a linear space X we understand a convex subset $C \subset X$ such that $tc \in C$ for any $t \in \bar{\mathbb{R}}_+$ and $c \in C$. Here $\bar{\mathbb{R}}_+ = [0, \infty)$ stands for the closure of the open half-line $\mathbb{R}_+ = (0, \infty)$ in $\mathbb{R}$. For two subsets A, B of X and a real number λ, let $A + B = \{a + b : a \in A,\ b \in B\}$ be the pointwise sum of A and B, and $\lambda A = \{\lambda a : a \in A\}$ be a homothetic copy of A.

Each subset $F \subset X$ generates the cone

$$\operatorname{cone}(F) = \Big\{\sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N} \text{ and } (x_i)_{i=1}^n \in F^n,\ (\lambda_i)_{i=1}^n \in \bar{\mathbb{R}}_+^n\Big\},$$

which contains the convex hull $\operatorname{conv}(F)$ of F.

The following description of polyhedral cones and polyhedral convex sets in finite-dimensional spaces is classical and can be found in [9], [11, Theorems 1.2, 1.3] or [5, §4.3]:

Lemma 2.1. *Let X be a finite-dimensional Banach space.*

(1) *A convex cone $C \subset X$ is polyhedral if and only if*

$$C = \operatorname{cone}(F) \quad \text{for some finite set } F \subset X.$$

(2) *A convex set $C \subset X$ is polyhedral if and only if*

$$C = \operatorname{cone}(F) + \operatorname{conv}(E) \quad \text{for some finite sets } F, E \subset X.$$

We shall be mainly interested in characteristic cones and dual characteristic cones of convex sets in Banach spaces. Let us recall that for a convex subset C of a Banach space X its *characteristic cone* V_C is defined by

$$V_C = \{x \in X : \forall c \in C \ \ c + \bar{\mathbb{R}}_+ x \subset C\} \subset X.$$

By the *dual characteristic cone* of C we understand the convex cone

$$V_C^* = \{x^* \in X^* : \sup x^*(C) < \infty\}$$

in the dual Banach space X^*.

It is clear that $V_C^* = V_{\bar{C}}^*$, where $\bar{C}$ is the closure of C in X. The relation between V_C and $V_{\bar{C}}$ is described in the following simple lemma, whose proof is left to the reader.

Lemma 2.2. *Let C be a convex set in a Banach space X. Then*

(1) *$V_C \subset V_{\bar{C}}$;*
(2) *$V_C = V_{\bar{C}}$ if C is open in X.*

Our next aim is to show that the characteristic cones of two closed convex subsets $A, B \subset X$ with $\mathsf{d_H}(A, B) < \infty$ coincide. For this we shall need:

Lemma 2.3. *For each point c_0 of a convex set C in a Banach space X, each $v \notin V_{\bar{C}}$, and each $\varepsilon \in \mathbb{R}_+$ there is a $t \in \mathbb{R}_+$ such that* $\mathsf{dist}(c_0+tv, C) = \varepsilon$.

Proof. Since $v \notin V_{\bar{C}}$, there is a $t_0 > 0$ such that $c_0 + t_0 v \notin \bar{C}$. Consider the continuous function

$$f : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+, \qquad f : t \mapsto \mathsf{dist}(c_0 + tv, C),$$

and observe that $f(0) = 0$ as $c_0 \in C$. We claim that $\lim_{t\to\infty} f(t) = \infty$. Since $c_0 + t_0 v \notin \bar{C}$, we can apply the Hahn–Banach Theorem to find a linear functional $x^* \in X^*$ with unit norm such that $x^*(c_0 + t_0 v) > \sup x^*(\bar{C}) \geq x^*(c_0)$, which implies that $x^*(v) > 0$. Then for any $t > t_0$ we get

$$\begin{aligned} \mathsf{dist}(tv, C) &= \inf_{c\in C} \|c_0 + tv - c\| \\ &\geq \inf_{c\in C} |x^*(tv) - x^*(c - c_0)| = tx^*(v) - \sup x^*(C - c_0) \end{aligned}$$

and hence $\lim_{t\to\infty} \mathsf{dist}(c_0 + tv, C) = \infty$. By the continuity of f, there is a $t > 0$ with $\mathsf{dist}(c_0 + tv, C) = f(t) = \varepsilon$. ∎

Now we can prove the promised

Lemma 2.4. *Let A, B be closed convex sets in a Banach space X. If $\mathsf{d_H}(A, B) < \infty$, then $V_A = V_B$.*

Proof. We lose no generality assuming that $0 \in A \cap B$. If $V_A \neq V_B$, then we can find a vector $v \in X$ that lies (say) in $V_B \setminus V_A$. By Lemma 2.3, there is a $t > 0$ such that $\mathsf{dist}(tv, A) > \mathsf{d_H}(A, B)$, which is not possible as $tv \in V_B \subset B$. ∎

Observe that for a convex set $C \subset X$ containing zero, the inclusion $V_C \subset C$ implies $V_C^* \subset V_{V_C}^*$.

Lemma 2.5. *For any closed convex set C in a Banach space the dual characteristic cone $V_{V_C}^*$ coincides with the weak* closure $\mathrm{cl}^*(V_C)$ of V_C.*

Proof. We lose no generality assuming that $0 \in C$. Observe that the cone

$$\begin{aligned} V_{V_C}^* &= \{x^* \in X^* : \sup x^*(V_C) < \infty\} \\ &= \{x^* \in X^* : \sup x^*(V_C) = 0\} = \bigcap_{v\in V_C} \{x^* \in X^* : x^*(v) \leq 0\} \end{aligned}$$

is weak* closed in X^*, being an intersection of weak* closed half-spaces in X^*. So, $V_C^* \subset V_{V_C}^*$ implies $\mathrm{cl}^*(V_C^*) \subset V_{V_C}^*$. To prove the reverse inclusion $V_{V_C}^* \subset \mathrm{cl}^*(V_C^*)$, assume that, on the contrary, there is an $x^* \in V_{V_C}^* \setminus \mathrm{cl}^*(V_C^*)$. By the Hahn–Banach Theorem applied to the weak* topology of X^*, there is an $x \in X$ that separates x^* from $\mathrm{cl}^*(V_C^*)$ in the sense that

$$x^*(x) > \sup\{v^*(x) : v^* \in \mathrm{cl}^*(V_C^*)\} \geq \sup\{v^*(x) : v^* \in V_C^*\}.$$

We claim that $v^*(x) \le 0$ for all $v^* \in V_C^*$. Assuming that $v^*(x) > 0$, we can find a $\lambda > 0$ so large that $\lambda v^*(x) > x^*(x)$, which contradicts the choice of x (because $\lambda v^* \in V_C^*$). So, $v^*(x) \le 0$ for all $v^* \in V_C^*$. We claim that $x \in V_C$.

In the opposite case, we could find a $t > 0$ such that $tx \notin C$ (recall that $0 \in C$). Applying the Hahn–Banach Theorem, we find $v^* \in X^*$ such that $v^*(tx) > \sup v^*(C) \ge 0$. Then $v^* \in V_C^*$ and $v^*(x) > 0$, which contradicts the preceding paragraph. Thus $x \in V_C$ and then $x^*(x) > 0$ implies that $\sup x^*(V_C) = \infty$, which contradicts the choice of $x^* \in V_{V_C}^*$. This completes the proof of the inclusion $V_{V_C}^* \subset \mathrm{cl}^*(V_C^*)$. ∎

The following lemma implies that polyhedral convex sets in Banach spaces lie at positive Hausdorff distance from their characteristic cones.

Lemma 2.6. *For a normed space X, linear continuous functionals $f_1, \dots, f_n : X \to \mathbb{R}$, a vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ with non-negative coordinates, and the polyhedral convex set*

$$P_{\mathbf{a}} = \bigcap_{i=1}^{n} f_i^{-1}((-\infty, a_i])$$

we have:

(1) $V_{P_{\mathbf{a}}} = P_{\mathbf{0}}$;
(2) $\mathsf{d_H}(P_{\mathbf{a}}, P_{\mathbf{0}}) \le \mathsf{d_H}(P_{\mathbf{0}}, P_{\mathbf{1}}) \cdot \max_{1\le i\le n} a_i$,

where $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$.

Proof. We consider $\mathbb{R}^n$ as a Banach lattice with coordinatewise operations of minimum and maximum.

(1) The first statement is easy and is left to the reader as an exercise.

(2) To prove the second statement, we first check that $\mathsf{d_H}(P_{\mathbf{0}}, P_{\mathbf{1}}) < \infty$. By Lemma 2.1(2), $P_{\mathbf{1}} = \mathrm{conv}(F) + \mathrm{cone}(E)$ for some finite sets $F, E \subset X$. It follows that $\mathrm{cone}(E)$ coincides with the characteristic cone $P_{\mathbf{0}}$ of $P_{\mathbf{1}}$ and hence $P_{\mathbf{1}} = \mathrm{conv}(F) + P_{\mathbf{0}}$. Then

$$\mathsf{d_H}(P_{\mathbf{1}}, P_{\mathbf{0}}) \le \mathsf{d_H}(\mathrm{conv}(F) + P_{\mathbf{0}}, P_{\mathbf{0}}) \le \mathsf{d_H}(\mathrm{conv}(F), \{0\}) < \infty.$$

Let $a = \max_{1\le i\le n} a_i$. Since the norm of X is homogeneous and $P_{\mathbf{0}} \subset P_{\mathbf{a}} \subset P_{a\mathbf{1}}$, we get the required inequality

$$\mathsf{d_H}(P_{\mathbf{a}}, P_{\mathbf{0}}) \le \mathsf{d_H}(P_{a\mathbf{1}}, P_{\mathbf{0}}) = a \cdot \mathsf{d_H}(P_{\mathbf{1}}, P_{\mathbf{0}}) = \mathsf{d_H}(P_{\mathbf{1}}, P_{\mathbf{0}}) \cdot \max_{1\le i\le n} a_i < \infty. \ ∎$$

3. Recognizing separable components of $\mathrm{Conv_H}(X)$**.** In this section we shall prove some lemmas that will help us to recognize closed convex sets $C \subset X$ with $\mathcal{H}_C$ separable. First we consider the finite-dimensional case. The following lemma was proved by V. Klee [9]. We give an alternative proof based on a Ramsey-theoretic argument.

LEMMA 3.1. *If the component $\mathcal{H}_C$ of a closed convex subset C of a finite-dimensional Banach space X contains a polyhedral convex set, then $\mathcal{H}_C$ contains a countable dense family of polyhedral convex sets.*

Proof. The case $C = X$ is trivial because $\mathcal{H}_C$ then contains a unique convex set X, which is polyhedral as the intersection of the empty family of closed half-spaces. So, we assume that $\mathcal{H}_C$ contains some polyhedral convex set $P \neq X$. We can assume that $0 \in P$. By Lemma 2.4, $\mathsf{d_H}(C, P) < \infty$ implies $V_C = V_P \neq X$.

Write P as a finite intersection of closed half-spaces

$$P = \bigcap_{i=1}^{k} f_i^{-1}((-\infty, a_i])$$

where $f_1, \ldots, f_k : X \to \mathbb{R}$ are linear continuous functionals with unit norm and $a_1, \ldots, a_k$ are real numbers (non-negative as $0 \in P$). According to Lemma 2.6, we can assume that $a_1 = \cdots = a_k = 0$, which implies that P is a polyhedral cone that coincides with its characteristic cone $V_P = V_C$. By Lemma 2.1, $P = \mathrm{cone}(B)$ for some finite subset $B \subset X$.

By assumption, the Banach space X is finite-dimensional and hence separable. So, we can fix a countable dense subset $D \subset X$. Next, for every finite subset $F \subset D$ consider the polyhedral convex set

$$C_F = \mathrm{conv}(F) + V_C = \mathrm{conv}(F) + \mathrm{cone}(B).$$

It remains to check that the countable family

$$\mathcal{C} = \{C_F : F \text{ is a finite subset of } D\}$$

is dense in $\mathcal{H}_C$.

Given $A \in \mathcal{H}_C$ and $\varepsilon > 0$, we shall find a finite subset $F \subset D$ with $\mathsf{d_H}(C_F, A) < 2\varepsilon$. Denote by $\bar{\mathbb{B}}$ the closed unit ball of X. Then clearly $r\bar{\mathbb{B}} = \{r \cdot x : x \in \bar{\mathbb{B}}\} = \{x \in X : \|x\| \leq r\}$ for every $r > 0$.

CLAIM 3.2. *There exists an $r \in \mathbb{R}_+$ so large that the convex set $A_r = (A \cap r\bar{\mathbb{B}}) + P$ is not empty and $\mathsf{d_H}(A_r, A) \leq \varepsilon$.*

Proof. It follows from $\mathsf{d_H}(A, C) < \infty$ that $V_A = V_C = P$ (see Lemma 2.4). Then for each $r \in \mathbb{R}_+$ we get

$$A_r = (A \cap r\bar{\mathbb{B}}) + P \subset A + P = A + V_A = A.$$

Assuming that $\mathsf{d_H}(A_r, A) > \varepsilon$ for all $r \in \mathbb{R}_+$, we can construct an increasing sequence $(r_n)_{n\in\omega}$ of positive real numbers and a sequence $(x_n)_{n\in\omega}$ of points in A such that $\|x_n\| \leq r_n$ and $\mathsf{dist}(x_{n+1}, A_{r_n}) > \varepsilon$ for all $n \in \omega$. Consequently, for every $n < m$ we get

$$(x_m + \varepsilon\bar{\mathbb{B}}) \cap (x_n + P) \subset (x_m + \varepsilon\bar{\mathbb{B}}) \cap (A_{r_n} + P) = \emptyset,$$

which implies $x_m - x_n \notin \varepsilon\bar{\mathbb{B}} + P$.

Recall that $P = \bigcap_{i=1}^k H_i$ where $H_i = f_i^{-1}((-\infty, 0])$ for $i \le k$. Using Lemma 2.6, we can choose $\delta > 0$ such that $\bigcap_{i=1}^k f_i^{-1}((-\infty, \delta]) \subset P + \varepsilon\mathbb{B}$.

It follows that for any $n < m$ we get $x_m - x_n \notin \bigcap_{i=1}^n f_i^{-1}\big((-\infty, \delta]\big)$ and hence there is an $i = i(n, m) \in \{1, \dots, k\}$ such that $x_m - x_n \notin f_i^{-1}\big((-\infty, \delta]\big)$ and thus $f_i(x_m) > f_i(x_n) + \delta$.

The correspondence $i : (n, m) \mapsto i(n, m)$ can be thought of as a finite coloring of the set $[\omega]^2 = \{(n, m) \in \omega^2 : n < m\}$ of pairs of positive integers. The Ramsey Theorem 5 of [6] yields an infinite subset $\Omega \subset \omega$ and $i \in \{1, \dots, k\}$ such that $i(n, m) = i$ and hence $f_i(x_m) > f_i(x_n) + \delta$ for all $n < m$ in Ω. This implies $\sup_{c \in C} f_i(c) \ge \sup_{n \in \Omega} f_i(x_n) = \infty$, which is not possible as $\sup f_i(C) \le (\sup f_i(P)) + \mathsf{d_H}(C, P) = \mathsf{d_H}(C, P) < \infty$. ■

Claim 3.2 yields an $r \in \mathbb{R}_+$ such that $A \cap r\bar{\mathbb{B}} \neq \emptyset$ and $\mathsf{dist}(A_r, A) < \varepsilon$ where $A_r = (A \cap r\bar{\mathbb{B}}) + P$. By [7], the compact convex set $A \cap r\bar{\mathbb{B}}$ can be approximated by a finite subset $F \subset D$ such that $\mathsf{d_H}(\mathrm{conv}(F), A \cap r\bar{\mathbb{B}}) < \varepsilon$. Then the polyhedral convex set $C_F = \mathrm{conv}(F) + P$ satisfies $\mathsf{d_H}(C_F, A_r) < \varepsilon$ and hence $\mathsf{d_H}(C_F, A) \le \mathsf{d_H}(C_F, A_r) + \mathsf{d_H}(A_r, A) < 2\varepsilon$. ■

To generalize Lemma 3.1 to infinite-dimensional Banach spaces X, we now establish some simple properties of maps between spaces of closed convex sets, induced by quotient operators.

Recall that for a Banach space $(X, \|\cdot\|_X)$ and a closed linear subspace $Z \subset X$, the quotient Banach space $Y = X/Z$ is endowed with the norm

$$\|y\|_Y = \inf\{\|x\|_X : x \in q^{-1}(y)\},$$

where $q : X \to Y$, $q : x \mapsto x + Z$, stands for the quotient operator.

The quotient operator $q : X \to Y$ induces an operator $\bar{q} : \mathrm{Conv_H}(X) \to \mathrm{Conv_H}(Y)$ assigning to each closed convex set $C \subset X$ the closure $\bar{q}C$ of its image qC in Y. The following simple lemma is left to the reader as an exercise.

Lemma 3.3. *Let Z be a closed linear subspace of a Banach space X, $Y = X/Z$, and $q : X \to Y$ be the quotient operator.*

(1) *A convex set $C \subset X$ with $Z \subset V_C$ is closed in X if and only if the image qC is closed in Y.*
(2) *A convex set $C \subset X$ with $Z \subset V_C$ is polyhedral in X if and only if its image qC is polyhedral in Y.*
(3) *For any non-empty convex sets $A, B \subset X$ with $Z \subset V_A \cap V_B$ we get $\mathsf{d_H}(A, B) = \mathsf{d_H}(qA, qB)$.*

Now we are able to prove an (infinite-dimensional) generalization of Lemma 3.1, which will be used in the proof of the implications (3)⇒(1, 5) in Theorem 1.1.

Lemma 3.4. *If the component $\mathcal{H}_C$ of a non-empty closed convex subset C of a Banach space X contains a polyhedral convex set, then $\mathcal{H}_C$ contains a countable dense family of polyhedral closed sets, which implies that the space $\mathcal{H}_C$ is separable and the convex set C is approximatively polyhedral.*

Proof. The statement is trivial if $C = X$. So, we assume that $C \neq X$ and $\mathcal{H}_C$ contains a polyhedral convex set P. Replacing P by its shift, we can assume that $0 \in P$. Write

$$P = \bigcap_{i=1}^{k} f_i^{-1}((-\infty, a_i]),$$

where $f_1, \ldots, f_k : X \to \mathbb{R}$ are linear continuous functionals and $a_1, \ldots, a_k$ are non-negative real numbers. It follows from $\mathsf{d}_\mathsf{H}(C, P) < \infty$ that the characteristic cone

$$V_C = V_P = \bigcap_{i=1}^{k} f_i^{-1}((-\infty, 0])$$

is polyhedral and the closed linear subspace

$$Z = -V_C \cap V_C = \bigcap_{i=1}^{k} f_i^{-1}(0)$$

has finite codimension in X.

Then the quotient Banach space $Y = X/Z$ is finite-dimensional. Taking into account that $Z \subset V_P \cap V_C$ and applying Lemma 3.3(1, 3), we conclude that qC and qP are closed convex sets in Y with $\mathsf{d}_\mathsf{H}(qC, qP) < \infty$. Moreover, qP is polyhedral in Y. Since Y is finite-dimensional, we can apply Lemma 3.1 to find a dense countable subset $\mathcal{D}_Y \in \mathcal{H}_{qC}$ that consists of polyhedral convex sets. By Lemma 3.3(2), the countable family $\mathcal{D}_X = \{q^{-1}(D) : D \in \mathcal{D}_X\}$ consists of polyhedral convex subsets of X and by Lemma 3.3(3) it is dense in $\mathcal{H}_C$. ∎

4. Recognizing non-separable components of $\mathrm{Conv}_\mathsf{H}(X)$**.** In this section we develop some tools for recognizing non-separable components of the space $\mathrm{Conv}_\mathsf{H}(X)$.

Lemma 4.1. *Let C be a convex subset of a linear space X and $a, b \in X$ be such that $[a, b] \cap C \neq \emptyset$. Then for any points $x \in \mathrm{conv}(C \cup \{a\})$ and $y \in \mathrm{conv}(X \cup \{b\})$ we have $[x, y] \cap C \neq \emptyset$.*

Proof. The conclusion trivially holds if x or y belongs to C. So, we assume that $x, y \notin C$. It follows that $x = t_x a + (1 - t_x)c_x$ for some $t_x \in (0, 1]$ and $c_x \in C$, and similarly $y = t_y b + (1 - t_y)c_y$ for some $t_y \in (0, 1]$ and $c_y \in C$.

By assumption, $[a, b] \cap C$ contains some point $c = ta + (1 - t)b$ with $t \in [0, 1]$.

The lemma will be proved as soon as we check that $[x,y]$ meets $\mathrm{conv}(\{c,c_x,c_y\})\subset C$, and this will follow as soon as we find $u,\alpha,\alpha_x,\alpha_y\in[0,1]$ such that $\alpha+\alpha_x+\alpha_y=1$ and

$$\alpha c+\alpha_x c_x+\alpha_y c_y=ux+(1-u)y=u(t_xa+(1-t_x)c_x)+(1-u)(t_yb+(1-t_y)c_y).$$

The numbers u and α can be found from the equation

$$ut_xa+(1-u)t_yb=\alpha c=\alpha(ta+(1-t)b),$$

which has a well-defined solution

$$u=\frac{t\cdot t_y}{t\cdot t_y+(1-t)t_x}\quad\text{and}\quad\alpha=\frac{t_xt_y}{t\cdot t_y+(1-t)t_x}.$$

The remaining numbers α_x and α_y are

$$\alpha_x=u(1-t_x),\qquad \alpha_y=(1-u)(1-t_y).\ \blacksquare$$

The following lemma will be used for the proof of the implication (6)⇒(4) of Theorem 1.1.

LEMMA 4.2. *The component $\mathcal{H}_C\subset\mathrm{Conv}_{\mathsf{H}}(X)$ of a closed convex subset C of a Banach space X has $\mathrm{dens}(\mathcal{H}_C)\ge\mathfrak{c}$ provided $\mathcal{H}_C$ contains a positively hiding closed convex subset P of X.*

Proof. Since $\mathcal{H}_C=\mathcal{H}_P$, we lose no generality assuming that C itself is positively hiding, which means that there is an infinite subset $A\subset X\setminus C$ with $\varepsilon=\inf_{a\in A}\mathsf{dist}(a,C)>0$, which is hidden behind C in the sense that for any distinct $a,b\in A$ the segment $[a,b]$ meets C.

Fix any $c_0\in C$ and for every $a\in A$ choose $b_a\in[c_0,a]$ with $\mathsf{dist}(b_a,C)=\varepsilon$. This is possible as $\mathsf{dist}(a,C)\ge\varepsilon$. Lemma 4.1 guarantees that the set $B=\{b_a:a\in A\}$ is infinite and hidden behind C. Moreover, B lies in the 2ε-neighborhood $C+2\varepsilon\mathbb{B}$ of C, where $\mathbb{B}=\{x\in X:\|x\|<1\}$.

Now for any subset $\beta\subset B$ consider the convex set $C_\beta=\mathrm{conv}(C\cup\beta)$. Applying Lemma 4.1 one can show that this set is closed in X and $C_\beta=\bigcup_{b\in\beta}\mathrm{conv}(C\cup\{b\})$. Taking into account that $C\subset C_\beta\subset C+2\varepsilon\mathbb{B}$, we see that $\mathsf{d_H}(C,C_\beta)\le2\varepsilon$ and hence $C_\beta\in\mathcal{H}_C$.

We claim that $\mathsf{d_H}(C_\alpha,C_\beta)\ge\varepsilon$ for any distinct $\alpha,\beta\subset B$. Since $\alpha\ne\beta$, there is (say) a point $b\in\beta\setminus\alpha$. Then $b\in C_\beta$ and $\mathsf{dist}(b,C_\alpha)\ge\varepsilon$. Indeed, assuming that $\mathsf{dist}(b,C_\alpha)<\varepsilon$, we conclude that the open ε-ball $b+\varepsilon\mathbb{B}$ meets $C_\alpha=\mathrm{conv}(C\cup\alpha)=\bigcup_{a\in\alpha}\mathrm{conv}(C\cup\{a\})$ at some x that belongs to $\mathrm{conv}(C\cup\{a\})$ for some $a\in\alpha$. Since the set $B\ni a,b$ is hidden behind C, the segment $[a,b]$ meets C. By Lemma 4.1, the segment $[x,b]$ also meets C, which is not possible as $[x,b]$ lies in the ε-ball $b+\varepsilon\mathbb{B}$, which does not meet C as $\mathsf{dist}(b,C)=\varepsilon$. Thus $\mathsf{dist}(b,C_\alpha)\ge\varepsilon$ and hence $\mathsf{d_H}(C_\beta,C_\alpha)\ge\varepsilon$.

Now we see that $\mathcal{H}_C$ contains the subset $\mathcal{C}=\{C_\beta:\beta\subset B\}$ of cardinality $|\mathcal{C}|\ge2^{|B|}\ge\mathfrak{c}$, consisting of points at mutual distance $\ge\varepsilon$. This implies that $\mathrm{dens}(\mathcal{H}_C)\ge|\mathcal{C}|\ge\mathfrak{c}$. $\blacksquare$

5. Recognizing infinitely hiding convex sets. In this section we develop some tools for recognizing infinitely hiding convex sets. In fact, we shall work with the following relative version of this property.

Let C_0, C be two convex sets in a Banach space X. We shall say that C_0 is *C-infinitely hiding* if C_0 hides some infinite set $A \subset \mathrm{aff}(C_0)$ such that $\sup_{a\in A} \mathsf{dist}(a, C) = \infty$.

It is easy to see that a convex set $C \subset X$ is infinitely hiding if and only if it is C-infinitely hiding.

We start with the following elementary lemma.

Lemma 5.1. *Let $C \ni 0$ be a convex set in a Banach space and $V_{\bar{C}}$ be the characteristic cone of its closure. For a linear subspace $Z \subset X$, the intersection $Z \cap V_{\bar{C}}$ is C-infinitely hiding if the cone $Z \cap V_{\bar{C}}$ is a hiding convex set in Z.*

Proof. Assume that $Z \cap V_{\bar{C}}$ hides some infinite injectively enumerated set $\{a_n\}_{n\in\omega} \subset Z \setminus V_{\bar{C}}$. By Lemma 2.3, for every $n \in \omega$, there is a $t_n > 0$ such that $\mathsf{dist}(t_n a_n, C) > n$. It is clear that for the set $A = \{t_n a_n\}_{n\in\omega}$ we get

$$\sup_{a\in A} \mathsf{dist}(a, C) = \lim_{n\to\infty} \mathsf{dist}(t_n a_n, C) = \infty.$$

It remains to show that for any distinct $n, m \in \omega$ the segment $[t_n a_n, t_m b_m]$ intersects $Z \cap V_{\bar{C}}$.

Since the set $\{a_n, a_m\} \subset A \subset Z$ is hidden behind $Z \cap V_{\bar{C}}$, the segment $[a_n, a_m]$ meets $Z \cap V_{\bar{C}}$ at some $c = \tau a_n + (1-\tau)a_m$ where $\tau \in [0, 1]$. Then for the number

$$u = \frac{\tau t_m}{\tau t_m + (1-\tau)t_n} \in [0, 1]$$

we get

$$\begin{aligned} u t_n a_n + (1-u) t_m a_m &= \frac{t_n t_n}{\tau t_m + (1-\tau)t_n}(\tau a_n + (1-\tau)a_m) \\ &= \frac{t_n t_m}{\tau t_m + (1-\tau)t_n} c \in [t_n a_n, t_m a_m] \cap V_{\bar{C}} \end{aligned}$$

and hence the intersection $Z \cap V_{\bar{C}} \cap [t_n a_n, t_m a_m] \ni u t_n a_n + (1-u) t_m a_m$ is not empty. ∎

By [1], a closed convex subset C of a complete linear metric space X is hiding if and only if C is not polyhedral in its closed linear hull. This characterization combined with Lemma 5.1 implies:

Lemma 5.2. *Let $C \ni 0$ be a convex set in a Banach space and $V_{\bar{C}}$ be the characteristic cone of its closure. For a closed linear subspace $Z \subset X$ the intersection $V = Z \cap V_{\bar{C}}$ is infinitely C-hiding if the cone V is not polyhedral in its closed linear hull $V^{\pm} = \mathrm{cl}(V - V)$.*

This lemma implies its absolute version:

Lemma 5.3. *A closed convex subset C of a Banach space is infinitely hiding if its characteristic cone V_C is not polyhedral in its closed linear hull $V_C^{\pm} = \mathrm{cl}(V_C - V_C)$.*

Next, we derive the infinite hiding property of a convex set from the same property of its projections. We start with the following algebraic fact.

Lemma 5.4. *Let $q : X \to \tilde{X}$ be a linear operator between linear spaces, $E = q^{-1}(0)$ be its kernel, and $C \subset X$ be a convex set such that $V_{C\cap E} - V_{C\cap E} = E$. If the image $\tilde{C} = q(C)$ hides some countable set $\tilde{A} \subset \mathrm{aff}(\tilde{C})$, then C hides some set $A \subset \mathrm{aff}(C)$ with $q(A) = \tilde{A}$.*

Proof. Let $\{\tilde{a}_n : n \in \omega\}$ be an injective enumeration of $\tilde{A}$. By induction, for every $n \in \omega$ we shall choose $a_n \in q^{-1}(\tilde{a}_n) \cap \mathrm{aff}(C)$ so that $[a_n, a_m] \cap C \neq \emptyset$ for any $n < m$, and $a_n \in C$ if $\tilde{a}_n \in \tilde{C}$.

We start by choosing any $a_0 \in q^{-1}(\tilde{a}_0) \cap \mathrm{aff}(C)$. Such a point exists since $q(\mathrm{aff}(C)) = \mathrm{aff}(\tilde{C})$. If $\tilde{a}_0 \in \tilde{C}$, then we can additionally assume that $a_0 \in C$. Assume that for some $n \geq 1$ the points $a_0, \ldots, a_{n-1}$ have been constructed. We need to choose $a_n \in q^{-1}(\tilde{a}_n) \cap \mathrm{aff}(C)$ so that $[a_i, a_n] \cap C \neq \emptyset$ for all $i < n$. If $\tilde{a}_n \in \tilde{C}$, then let $a_n \in C$ be any point with $q(a_n) = \tilde{a}_n$. So, assume that $\tilde{a}_n \notin \tilde{C}$. Let $I_n = \{i \in \omega : i < n,\ \tilde{a}_i \notin \tilde{C}\}$.

Since the set $\tilde{A} \subset \tilde{X}$ is hidden behind $\tilde{C}$, for every $i \in I_n$ the intersection $[\tilde{a}_i, \tilde{a}_n] \cap \tilde{C}$ contains a convex combination $\tilde{c}_i = u_i\tilde{a}_i + (1 - u_i)\tilde{a}_n$ for some $u_i \in (0, 1)$. Since $\tilde{c}_i \in \tilde{C}$, there is a point $c_i \in C$ with $q(c_i) = \tilde{c}_i$. It follows from $\tilde{a}_n = (\tilde{c}_i - u_i\tilde{a}_i)/(1 - u_i)$ that the point $a_i' = (c_i - u_i a_i)/(1 - u_i)$ belongs to $q^{-1}(\tilde{a}_n)$.

As $E = V_{E\cap C} - V_{E\cap C}$, the intersection $\bigcap_{i\in I_n}(a_i' + V_{E\cap C})$ contains some point a_n. Then $u_i a_i + (1 - u_i)a_n \in c_i + V_C \subset C$ and hence $[a_i, a_n] \cap C \neq \emptyset$ for all $i < n$, which completes the inductive step.

This inductive construction gives a countable set $A = \{a_n\}_{n\in\omega}$ that has the required property. ∎

Lemma 5.4 implies its C-infinitely hiding version.

Lemma 5.5. *Let X be a Banach space, E be a closed linear subspace of X, $\tilde{X} = X/E$ and $q : X \to \tilde{X}$ be the quotient operator. Let C_0, C be two convex sets in X and $\tilde{C}_0 = q(C_0)$, $\tilde{C} = q(C)$. Then C_0 is C-infinitely hiding X if $\tilde{C}_0$ is $\tilde{C}$-infinitely hiding and $E = V_{E\cap C_0} - V_{E\cap C_0}$.*

Proof. If $\tilde{C}_0$ is $\tilde{C}$-infinitely hiding, then it hides some infinite set $\tilde{A} \subset \mathrm{aff}(\tilde{C}_0)$ such that $\sup_{\tilde{a}\in\tilde{A}} \mathsf{dist}(\tilde{a}, \tilde{C}) = \infty$. By Lemma 5.4, there is a set $A \subset \mathrm{aff}(C_0)$ with $q(A) = \tilde{A}$, hidden behind C.

Since $\|q\| \leq 1$, for every $a \in A$ and its image $\tilde{a} = q(a)$ we get $\mathsf{dist}(\tilde{a}, \tilde{C}) \leq \mathsf{dist}(a, C)$. Consequently, $\sup_{a\in A} \mathsf{dist}(a, C) \geq \sup_{\tilde{a}\in\tilde{A}} \mathsf{dist}(\tilde{a}, \tilde{C}) = \infty$, which means that C_0 is C-infinitely hiding. ∎

The preceding lemma allows us to derive the C-infinite hiding property of a convex set from that property of its projection. Our next lemma will help us to do the same using the C-infinite hiding property of two-dimensional sections of the convex set.

LEMMA 5.6. *Let C be a closed convex subset of a Banach space X and Z be a two-dimensional linear subspace of X such that the convex set $C \cap Z$ has non-empty interior C_0 in Z, which contains zero. If $\mathsf{d_H}(C_0, V_{C_0}) = \infty$, then C_0 is C-infinitely hiding.*

Proof. Since $\mathsf{d_H}(C_0, V_{C_0}) = \infty$, the open convex subset C_0 of the plane Z is not bounded. Consequently, its characteristic cone $V_{C_0} = V_{\bar{C}_0} = Z \cap V_C$ is unbounded too. Moreover, V_{C_0} is not a plane, nor a half-plane, nor a line (otherwise C_0 would be at finite Hausdorff distance from V_{C_0}). Consequently, we can choose two linearly independent vectors $e_1, e_2 \in Z$ such that V_{C_0} is equal to $\mathrm{cone}(\{e_1\})$ or to $\mathrm{cone}(\{e_1, e_2\})$. Let $e_1^*, e_2^* \in Z^*$ be the coordinate functionals corresponding to the base e_1, e_2 of Z. This means that $z = e_1^*(z)e_1 + e_2^*(z)e_2$ for each $z \in Z$.

If $V_{C_0} = \mathrm{cone}(\{e_1, e_2\})$, then $\mathsf{d_H}(C_0, V_{C_0}) = \infty$ implies that $\inf e_1^*(C_0) = -\infty$ or $\inf e_2^*(C_0) = -\infty$. We lose no generality assuming that $\inf e_2^*(C_0) = -\infty$.

If $V_{C_0} = \mathrm{cone}(\{e_1\})$, then $\mathsf{d_H}(C_0, V_{C_0}) = \infty$ implies that $\inf e_2^*(C_0) = -\infty$ or $\sup e_2^*(C_0) = \infty$. Changing e_2 to $-e_2$ if necessary, we can assume that $\inf e_2^*(C_0) = -\infty$.

So, in both cases we can assume that $\inf e_2^*(C_0) = -\infty$.

By induction, we shall construct a sequence $(a_n)_{n\in\omega}$ of points in $\mathrm{cone}(\{e_1, -e_2\})$ such that for every $n \in \omega$ the following conditions are satisfied:

(1) $\mathsf{dist}(a_n, C) > n$;

(2) $e_1^*(a_n) > e_1^*(a_{n-1}) > 0$, $e_2^*(a_n) < e_2^*(a_{n-1}) < 0$, $\dfrac{|e_2^*(a_n)|}{e_1^*(a_n)} < \dfrac{|e_2^*(a_{n-1})|}{e_1^*(a_{n-1})}$;

(3) $[a_n, a_k] \cap C_0 \neq \emptyset$ for all $k < n$.

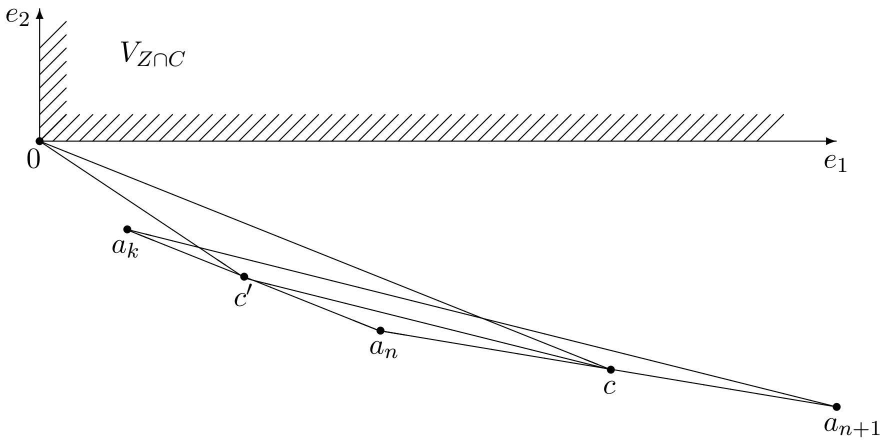
We start by selecting $a_0 \in \mathbb{R}_+ \cdot (e_1 - e_2)$ with $\mathsf{dist}(a_0, C) > 0$. Such a point exists because $e_1 - e_2 \notin V_{C_0} = Z \cap V_C$. Now assume that for some $n \in \mathbb{N}$ we have constructed $a_0, \dots, a_n \in \mathrm{cone}(\{e_1, -e_2\})$ satisfying (1)–(3). It follows from $\inf e_2^*(C_0) = -\infty$ and $e_1 \in V_{C_0} \setminus (-V_{C_0})$ that there exists $c \in C_0$ such that

$$e_1^*(c) > c_1^*(a_n), \qquad e_2^*(c) < e_2^*(a_n) \quad \text{and} \quad \frac{|e_2^*(c)|}{e_1^*(c)} < \frac{|e_2^*(a_n)|}{e_1^*(a_n)}.$$

Now observe that the vector $v = c - a_n$ is not in $V_{C_0} = V_{\bar{C}_0}$. Consequently,

$c+\mathbb{R}_+v \not\subset \bar{C}_0$, which allows us to find $a_{n+1} \in c+\mathbb{R}_+v$ with $\mathsf{dist}(a_{n+1}, C) > n+1$ (using Lemma 2.3). It can be shown that a_{n+1} satisfies condition (2). Since the segment $[a_n, a_{n+1}]$ contains c, it meets $C \cap Z$.

It remains to check that $[a_k, a_{n+1}] \cap C_0 \neq \emptyset$ for every $k < n$. By the inductive assumption, $[a_k, a_n]$ meets C_0 at some c'.



By an elementary plane geometry argument, the segment $[a_k, a_{n+1}]$ meets the triangle $\mathrm{conv}(\{0, c', c\}) \subset C_0$ and hence meets C_0. This completes the inductive step.

We thus obtain an infinite set $A = \{a_n\}_{n\in\omega}$ with $\sup_{a\in A} \mathsf{dist}(a, C) = \infty$ which is hidden behind C_0. This means that C_0 is C-infinitely hiding. ∎

LEMMA 5.7. *Let C be a convex subset of a Banach space X and Z be a finite-dimensional linear subspace of X such that the convex set $C \cap Z$ has non-empty interior C_0 in Z and $0 \in C_0$. If $\mathsf{d_H}(C_0, V_{C_0}) = \infty$, then C_0 is C-infinitely hiding.*

Proof. We use induction on $\dim(Z)$. The conclusion is trivially true if $\dim(Z) \le 1$. Assume that it has been proved for all triples (X, C, Z) with $\dim(Z) < n$. Now suppose $\dim(Z) = n$. Assuming that $\mathsf{d_H}(C_0, V_{C_0}) = \infty$, we need to prove that C_0 is C-infinitely hiding. Assume it is not. Then Lemma 5.6 implies the following fact, which will be used several times in the subsequent proof.

CLAIM 5.8. *For each two-dimensional linear subspace $Z_2 \subset Z$,*

$$\mathsf{d_H}(Z_2 \cap C_0, V_{Z_2\cap C_0}) < \infty.$$

Now consider the characteristic cone $V_{C_0} = V_{\bar{C}_0}$ of the open convex set C_0 in Z.

CLAIM 5.9. *The linear subspace $-V_{C_0} \cap V_{C_0}$ is trivial.*

Proof. Assume that $E = -V_{C_0} \cap V_{C_0} \neq \{0\}$. Consider $\tilde{X} = X/E$, the quotient operator $q : X \to \tilde{X}$, the convex set $\tilde{C} = q(C)$, and the finite-dimensional subspace $\tilde{Z} = q(Z)$ of dimension $\dim(\tilde{Z}) < \dim(Z) = n$. Since $q|Z : Z \to \tilde{Z}$ is open, the convex set $\tilde{C}_0 = q(C_0)$ is open in $\tilde{Z}$ and hence the convex set $\tilde{Z} \cap \tilde{C}$ has non-empty interior in $\tilde{Z}$. Since $E \subset V_C$, the set $\tilde{C}$ is closed in $\tilde{X}$ by Lemma 3.3(1). Now we can see that the triple $(\tilde{X}, \tilde{C}, \tilde{Z})$ satisfies the requirements of Lemma 5.7 with $\dim(\tilde{Z}) < \dim(Z) = n$. So, by the inductive assumption, $\tilde{C}_0$ is $\tilde{C}$-infinitely hiding.

Since $E = V_{E\cap C_0} = V_{E\cap C_0} - V_{E\cap C_0}$, we can apply Lemma 5.4 to conclude that C_0 is C-infinitely hiding in X, contrary to assumption. ■

Claim 5.10. $\dim(V_{C_0}) \geq 2$.

Proof. Assume that $\dim(V_{C_0}) \leq 1$. Since $\mathsf{d_H}(C_0, V_{C_0}) = \infty$, the open convex subset C_0 is unbounded in the finite-dimensional linear space $Z \cap C_0$ and consequently $V_{C_0} \neq \{0\}$. Since $-V_{C_0} \cap V_{C_0} = \{0\}$, we conclude that $V_{C_0} = \bar{\mathbb{R}}_+ e$ for some non-zero $e \in V_{C_0}$. Now consider $E = \mathbb{R}e \subset Z$, $\tilde{X} = X/E$, the finite-dimensional linear subspace $\tilde{Z} = q(Z)$, the convex sets $\tilde{C} = q(C)$, and the open convex set $\tilde{C}_0 = q(C_0)$, which is dense in $\tilde{C}$. Claim 5.8 guarantees that $\tilde{C}_0$ has trivial characteristic cone and hence is bounded in $\tilde{Z}$. This implies that $\mathsf{d_H}(C_0, V_{C_0}) < \infty$, which is the desired contradiction. ■

Since C_0 is not C-infinitely hiding, Lemma 5.2 guarantees that the characteristic cone V_{C_0} is polyhedral in Z and hence

$$V_{C_0} = \bigcap_{i=1}^{k} f_i^{-1}((-\infty, 0])$$

for some linear functionals $f_1, \ldots, f_k : Z \to \mathbb{R}$. We shall assume that the number k in this representation is the smallest possible.

It follows from $\mathsf{d_H}(C_0, V_{C_0}) = \infty$ and Lemma 2.6 that $\sup f_i(C_0) = \infty$ for some $i \leq k$.

Claim 5.11. *The face $f_i^{-1}(0) \cap V_{C_0}$ of V_{C_0} contains a non-zero vector e.*

Proof. By the minimality of k, the cone

$$V = \bigcap \{f_j^{-1}((-\infty, 0]) : 1 \leq j \leq k,\ j \neq i\}$$

is strictly larger than V_{C_0}; choose $x \in V \setminus V_{C_0}$. Then $f_j(x) \leq 0$ for all $j \neq i$, and $f_i(x) > 0$.

Since $\dim(V_{C_0}) \geq 2$, there exists $y \in V_{C_0} \setminus \mathbb{R}x$. The choice of x guarantees that $0 \notin [x, y]$. Since $f_i(y) \leq 0$ and $f_i(x) > 0$, there is an $e \in [x, y]$ with $f_i(e) = 0$. For every $j \neq i$, the inequalities $f_j(x) \leq 0$ and $f_j(y) \leq 0$ imply $f_j(e) \leq 0$. Consequently, e is as required. ■

Consider now the 1-dimensional linear subspace $E = \mathbb{R}e$ of X and let $\tilde{X} = X/E$. Observe that $\tilde{X}$ contains the linear subspace $\tilde{Z} = Z/E$ with

$\dim(\tilde{Z}) = \dim(Z) - 1 < n$. Let $q : X \to \tilde{X}$ be the quotient operator, and $\tilde{C}_0 = q(C_0)$, $\tilde{C} = q(C)$. It follows from $E = q^{-1}(0) \subset Z$ that $\tilde{Z} \cap \tilde{C} = q(Z \cap C)$ and $\tilde{C}_0 = q(C_0)$ coincides with the interior of $q(Z \cap C) = \tilde{Z} \cap \tilde{C}$ in $\tilde{Z}$. So, the triple $(\tilde{X}, \tilde{Z}, \tilde{C})$ satisfies the assumptions of the lemma.

We claim that $\mathsf{d_H}(\tilde{C}_0, V_{\tilde{C}_0}) = \infty$. Since $E \subset f_i^{-1}(0)$, there is a linear functional $\tilde{f}_i : \tilde{Z} \to \mathbb{R}$ such that $f_i = \tilde{f}_i \circ q|Z$.

CLAIM 5.12. $V_{\tilde{C}_0} \subset \tilde{f}_i^{-1}((-\infty, 0])$.

Proof. Assume that $V_{\tilde{C}_0}$ contains some $w \in \tilde{Z}$ with $\tilde{f}_i(w) > 0$. Then $\mathbb{R}_+ w \subset \tilde{Z}$. Pick $v \in q^{-1}(w) \subset Z$ and consider the two-dimensional subspace $Z_2 = \mathrm{lin}(\{v, e\})$. Observe that for every $t \in \mathbb{R}$ we have $f_i(v+te) = \tilde{f}_i(w) > 0$, which implies that $(v + t\mathbb{R}) \cap V_{Z_2 \cap C} = \emptyset$. Then $V_{Z_2 \cap C} \subset \mathbb{R}e - \bar{\mathbb{R}}_+ v$ and $q(V_{Z_2 \cap C}) \subset -\bar{\mathbb{R}}_+ w$. On the other hand, the projection $q(Z_2 \cap C)$ contains the half-line $\mathbb{R}_+ w$, which implies that $\mathsf{d_H}(Z_2 \cap C, V_{Z_2 \cap C}) = \infty$. But this contradicts Claim 5.8. ∎

Taking into account that $\infty = \sup f_i(C_0) = \sup \tilde{f}_i(\tilde{C}_0)$, we conclude that $\mathsf{d_H}(\tilde{C}_0, V_{\tilde{C}_0}) = \infty$ and by the inductive assumption, the open convex set $\tilde{C}_0$ is $\tilde{C}$-infinitely hiding (as $\dim(\tilde{Z}) < \dim(Z) = n$). Since $E = \mathbb{R}_+ e - \mathbb{R}_+ e = V_{E \cap C_0} - V_{E \cap C_0}$, we can apply Lemma 5.4 to conclude that C_0 is C-infinitely hiding in X. This contradiction completes the proof of Lemma 5.7. ∎

Lemma 5.7 implies the final (and main) lemma of this section.

LEMMA 5.13. *A closed convex subset C of a Banach space X is infinitely hiding if $\mathsf{d_H}(A \cap C, A \cap V_C) = \infty$ for some finite-dimensional affine subspace $A \subset X$.*

Proof. It is well-known that $C \cap A$ has non-empty interior C_0 in its affine hull $\mathrm{aff}(C \cap A)$. Moreover, $C \cap A$ coincides with the closure $\bar{C}_0$ of C_0 in A. Shifting the set C if necessary, we can assume that $0 \in C_0$. Then $Z = \mathrm{aff}(C_0)$ is a finite-dimensional linear subspace of X such that $C \cap Z = C \cap A$ has non-empty interior C_0 which contains zero and is dense in $C \cap Z$. It follows that $\mathsf{d_H}(C_0, V_{C_0}) = \mathsf{d_H}(Z \cap C, V_{Z \cap C}) = \infty$ and hence C_0 is C-infinitely hiding by Lemma 5.7, and C is infinitely hiding as $C_0 \subset C$. ∎

6. Approximating by positively hiding convex sets. In this section we search for conditions guaranteeing that a closed convex subset C of a Banach space can be approximated by positively hiding convex subsets of X.

First we construct biorthogonal sequences which are related to convex sets with trivial characteristic cone.

We recall that a sequence $\{(x_n, x_n^*)\}_{n \in \omega} \subset X \times X^*$ is *biorthogonal* if $x_n^*(x_n) = 1$ and $x_n^*(x_k) = 0$ for all $n \neq k$ (see [8, 1.1]).

LEMMA 6.1. *Assume that a closed convex subset C of an infinite-dimensional Banach space X satisfies $V_C = \{0\}$. Then there exists a biorthogonal sequence $\{(x_n, x_n^*)\}_{n\in\omega} \subset X \times V_C^*$ such that $\|x_n^*\| = 1 \le \|x_n\| < 4$ for all $n \in \omega$.*

Proof. Replacing C by a shift of its closed neighborhood, we can assume that C has non-empty interior C_0, which contains zero. Such a replacement does not affect the cones V_C and V_C^* (see Lemma 2.4).

The biorthogonal sequence $\{(x_n, x_n^*)\}_{n\in\omega}$ will be constructed by induction. We start by choosing arbitrary $x_0^* \in V_C^*$ and $x_0 \in X$ with $1 = \|x_0^*\| = x_0^*(x_0) \le \|x_0\| < 4$. Assume that for some $k \in \omega$ a finite biorthogonal sequence $\{(x_n, x_n^*)\}_{n<k} \subset X \times V_C^*$ has been constructed so that $1 = \|x_n^*\| \le \|x_n\| < 4$ for all $n < k$. Let L^* be the linear hull of $\{x_0^*, \dots, x_{k-1}^*\}$ in X^*. In the compact set $L_2^* = \{x^* \in L^* : \|x^*\| \le 2\}$, choose a finite subset $F_2^* \subset L_2^*$ such that for each $x^* \in L_2^*$ there is a $y^* \in F_2^*$ with $\|x^* - y^*\| < 1/8$. For every $f \in F_2^*$ choose an $x_f \in X$ such that $\|x_f\| = 1$ and $f(x_f) > \|f\| - 1/8$.

Let E be the linear hull of the finite set $\{x_i : i < k\} \cup \{x_f : f \in F_2^*\}$ in X. Let $\tilde X = X/E$, $q : X \to \tilde X$ be the quotient operator, and $\tilde C = q(C)$. Since q is open, $\tilde C_0 = q(C_0)$ coincides with the interior of $\tilde C$ in $\tilde X$. We claim that $V_{\tilde C_0} = \{0\}$. Assuming otherwise, pick a non-zero $\tilde v \in V_{\tilde C_0}$, choose any $v \in q^{-1}(\tilde v)$ and consider the finite-dimensional linear subspace $E_v = \mathrm{lin}(E \cup \{v\})$. Then $V_C = \{0\}$ implies that $E_v \cap C_0$ is bounded and hence $q(E_v \cap C_0) = \mathbb{R}\tilde v \cap \tilde C_0$ is also bounded. So, $\bar{\mathbb{R}}_+ \tilde v \not\subset C_0$, which contradicts $\tilde v \in V_{\tilde C_0} \setminus \{0\}$.

As $V_{\tilde C_0} = \{0\}$ implies that $\tilde C \ne \tilde X$, we can find $\tilde x_k^* \in V_{\tilde C_0}^*$ with $\|\tilde x_k^*\| = 1$. Now set $x_k^* = \tilde x_k^* \circ q$ and observe that $\|x_k^*\| = \|\tilde x_k^*\| = 1$ and $x_k^*(x_i) = 0$ for all $i < k$ and $x_k^*(x_f) = 0$ for all $f \in F_2^*$. We claim that $\mathsf{dist}(x_k^*, L^*) > 1/4$. Assuming otherwise, pick $l^* \in L^*$ with $\|x_k^* - l^*\| \le 1/4$. Then $\|l^*\| \le \|x_k^*\| + 1/4$, so $l^* \in L_2^*$ and by the choice of F_2^*, we can find $f \in F_2^*$ such that $\|l^* - f\| < 1/8$. Then $\|x_k^* - f\| \le \|x_k^* - l^*\| + \|l^* - f\| \le 1/4 + 1/8 = 3/8$, $\|f\| \ge \|x_k^*\| - \|x_k^* - f\| \ge 1 - 3/8 = 5/8$ and we obtain a contradiction:

$$\frac{4}{8} = \frac{5}{8} - \frac{1}{8} \le \|f\| - \frac{1}{8} < f(x_f) = |x_k^*(x_f) - f(x_f)| \le \|x_k^* - f\| \cdot \|x_f\| \le \frac{3}{8}.$$

Thus $\mathsf{dist}(x_k^*, L^*) > 1/4$, so the ball $B^* = \{x^* \in X^* : \|x^* - x_k^*\| \le 1/4\}$ does not intersect L^*. By the Banach–Alaoglu Theorem this ball is compact in the weak* topology of X^*. Now the Hahn–Banach Theorem applied to that topology yields an $x_k \in X$ that separates L^* and B^* in the sense that $\sup_{x^*\in L^*} x^*(x_k) < \inf_{x^*\in B^*} x^*(x_k)$. It follows from the linearity of L^* that $x^*(x_k) = 0$ for all $x^* \in L^*$. In particular, $x_i^*(x_k) = 0$ for all $i < k$. Multiplying x_k by a suitable positive constant, we may additionally assume that $x_k^*(x_k) = 1$. Then $\|x_k\| \ge 1$ because x_k^* has unit norm. To finish the inductive step it suffices to check that $\|x_k\| < 4$. Otherwise there exists $x^* \in X^*$ with unit norm such that $x^*(x_k) = \|x_k\| \geq 4$. Then $y^* = x_k^* - \frac{1}{4}x^* \in B^*$ and thus $y^*(x_k) > 0$. On the other hand, $y^*(x_k) = x_k^*(x_k) - \frac{1}{4}x^*(x_k) \leq 1 - \frac{1}{4} \cdot 4 = 0$, which is the desired contradiction. ∎

Lemma 6.2. *Assume that a closed convex subset C of an infinite-dimensional Banach space X has $V_C = \{0\}$. Then for each $\varepsilon > 0$ there is a positively hiding closed convex set $C_\varepsilon \subset X$ with $\mathsf{d_H}(C_\varepsilon, C) \leq \varepsilon$.*

Proof. By Lemma 6.1, there exists a biorthogonal sequence $\{(x_n, x_n^*)\}_{n\in\omega} \subset X \times V_C^*$ such that $1 = \|x_n^*\| \leq \|x_n\| < 4$ for all $n \in \omega$. Then for every $\varepsilon > 0$, a positively hiding convex set C_ε with $\mathsf{d_H}(C_\varepsilon, C) \leq \varepsilon$ can be defined by

$$C_\varepsilon = \left\{x \in \mathrm{cl}(C + \varepsilon\mathbb{B}) : \forall n \in \omega \ \ x_n^*(x) \leq \tfrac{1}{8}\varepsilon + \sup x_n^*(C)\right\}$$

where $\mathbb{B} = \{x \in X : \|x\| < 1\}$. It is clear that $C \subset C_\varepsilon \subset \mathrm{cl}(C + \varepsilon\mathbb{B})$, which implies that $\mathsf{d_H}(C_\varepsilon, C) \leq \varepsilon$. It remains to check that the set C_ε is positively hiding.

For every $n \in \omega$ choose $c_n \in C$ with $x_n^*(c_n) > \sup x_n^*(C) - \frac{1}{16}\varepsilon$ and set $a_n = c_n + \frac{\varepsilon}{4}x_n$. We claim that $\mathsf{dist}(a_n, C_\varepsilon) \geq \frac{1}{16}\varepsilon$. Indeed, for any $c \in C_\varepsilon$, we get $x_n^*(c) \leq \sup x_n^*(C) + \frac{1}{8}\varepsilon$ while

$$x_n^*(a_n) = \frac{\varepsilon}{4}x_n^*(x_n) + x_n^*(c_n) > \frac{\varepsilon}{4} + \sup x_n^*(C) - \frac{\varepsilon}{16} = \frac{3\varepsilon}{16} + \sup x_n^*(C).$$

Consequently, $\|a_n^* - c\| = \|x_n^*\| \cdot \|a_n - c\| \geq x_n^*(a_n) - x^*(c) \geq \frac{3\varepsilon}{16} - \frac{\varepsilon}{8} = \frac{\varepsilon}{16}$.

So, the set $A = \{a_n\}_{n\in\omega}$ has $\inf_{a\in A} \mathsf{dist}(a, C_\varepsilon) \geq \frac{1}{16}\varepsilon$. To show that it is infinite and hidden behind C_ε, it suffices to check that for any distinct n, m the midpoint $\frac{1}{2}a_n + \frac{1}{2}a_m$ of $[a_n, a_m]$ belongs to C_ε. As $a_n, a_m \in \mathrm{cl}(C + \varepsilon\mathbb{B})$, we conclude that $\frac{1}{2}a_n + \frac{1}{2}a_m \in [a_n, a_m] \subset \mathrm{cl}(C + \varepsilon\mathbb{B})$. The inclusion $\frac{1}{2}a_n + \frac{1}{2}a_m \in C_\varepsilon$ will follow from the definition of C_ε as soon as we check that $x_k^*(\frac{1}{2}a_n + \frac{1}{2}a_m) \leq \sup x_k^*(C) + \frac{1}{8}\varepsilon$ for every $k \in \omega$.

If $k \notin \{n, m\}$, then $x_k^*(x_n) = x_k^*(x_m) = 0$ and hence

$$x_k^*(\tfrac{1}{2}a_n + \tfrac{1}{2}a_m) = x_k^*(\tfrac{1}{2}c_n + \tfrac{1}{2}c_m) \leq \sup x_k^*(C).$$

If $k = n$, then

$$x_k^*(\tfrac{1}{2}a_n + \tfrac{1}{2}a_m) = x_n^*(\tfrac{1}{2}c_n + \tfrac{1}{2}c_m) + \tfrac{1}{8}\varepsilon x_n^*(x_n) \leq \sup x_n^*(C) + \tfrac{1}{8}\varepsilon.$$

The case $k = m$ is analogous. ∎

Lemma 6.3. *Assume that for a closed convex subset C of a Banach space X the closed linear subspace $Z = \mathrm{cl}(V_C - V_C)$ has infinite codimension in X. Then for each $\varepsilon > 0$ there is a positively hiding convex set $C_\varepsilon \subset X$ with $\mathsf{d_H}(C_\varepsilon, C) \leq \varepsilon$.*

Proof. Using Lemma 3.3 (by analogy with the proof of Lemma 3.4), we can reduce the proof to the case $-V_C \cap V_C = \{0\}$, which we assume from now on. Replacing C by a shift of its closed neighborhood, we can assume that C has non-empty interior C_0 in X and $0 \in C_0$. If V_C is not polyhedral in $Z = \mathrm{cl}(V_C - V_C)$, then by Lemma 5.3, C is infinitely (and hence positively) hiding and hence we can put $C_\varepsilon = C$. So, we assume that V_C is polyhedral in Z. Since $-V_C \cap V_C = \{0\}$, the polyhedrality of V_C implies that the closed linear space $Z = \mathrm{cl}(V_C - V_C)$ is finite-dimensional and coincides with $V_C - V_C$. Now consider $\tilde{X} = X/Z$, the quotient operator $q : X \to \tilde{X}$, and the convex set $\tilde{C} = q(C)$. Since q is open, the image $\tilde{C}_0 = q(C_0)$ of the interior C_0 of C coincides with the interior of $\tilde{C}$. Now consider the characteristic cone $V_{\tilde{C}_0}$ of the open convex set $\tilde{C}_0$.

If $V_{\tilde{C}_0}$ contains some non-zero vector $\tilde{v}$, then for any $v \in q^{-1}(\tilde{v})$ and for the finite-dimensional linear subspace $E = \mathrm{lin}(Z \cup \{v\})$ the intersection $E \cap C_0$ satisfies $\mathsf{d_H}(E \cap C_0, V_{E\cap C_0}) = \infty$, because $q(V_{E\cap C_0}) \subset q(V_C) = \{0\}$ while $q(E \cap C_0) \supset \bar{\mathbb{R}}_+\tilde{v}$. Then Lemma 5.13 guarantees that C is infinitely (and hence positively) hiding. In this case we can put $C_\varepsilon = C$.

So, it remains to consider the case $V_{\tilde{C}_0} = \{0\}$. In this case, Lemma 6.2 yields a positively hiding closed convex set $\tilde{C}_\varepsilon \subset \tilde{X}$ with $\mathsf{d_H}(\tilde{C}_\varepsilon, \tilde{C}_0) < \varepsilon$. Now the convex set $C_\varepsilon = (C + \varepsilon\mathbb{B}) \cap q^{-1}(\tilde{C}_\varepsilon)$ satisfies $\mathsf{d_H}(C_\varepsilon, C) \le \varepsilon$ and $q(C_\varepsilon) = \tilde{C}_\varepsilon$.

Being positively hiding, $\tilde{C}_\varepsilon$ hides a countably infinite set $\tilde{A} \subset \tilde{X} \setminus \tilde{C}_\varepsilon$ with $\inf_{a\in\tilde{A}} \mathsf{dist}(a, \tilde{C}_\varepsilon) > 0$. As $Z = V_{Z\cap C_0} - V_{Z\cap C_0}$ and $q(C_\varepsilon) = \tilde{C}_\varepsilon$, using Lemma 5.4 we can find an infinite subset $A \subset q^{-1}(\tilde{A})$ hidden behind $C_\varepsilon \subset C$. Since q is not expanding, we have

$$\inf_{a\in A} \mathsf{dist}(a, C) = \inf_{a\in A} \mathsf{dist}(a, C_0) \ge \inf_{\tilde{a}\in\tilde{A}} \mathsf{dist}(\tilde{a}, \tilde{C}_0) > 0.$$

So, C_ε is positively hiding. ∎

Our next approximation lemma will be used in the proof of the implication (4)⇒(2) of Theorem 1.1.

Lemma 6.4. *Let C be a closed convex set in a Banach space X. If $\mathsf{d_H}(C, V_C) = \infty$, then for each $\varepsilon > 0$ there is a positively hiding convex set $\tilde{C} \subset X$ with $\mathsf{d_H}(\tilde{C}, C) \le \varepsilon$.*

Proof. If the closed linear subspace $V_C^{\pm} = \mathrm{cl}(V_C - V_C)$ has infinite codimension in X, then the existence of a positively hiding convex set $C_\varepsilon \subset X$ with $\mathsf{d_H}(C_\varepsilon, C) < \varepsilon$ follows from Lemma 6.3. So, we assume that $V_C^{\pm}$ has finite codimension in X. If V_C is not polyhedral in $V_C^{\pm}$, then C is infinitely (and positively) hiding by Lemma 5.3. In this case we can put $C_\varepsilon = C$. It remains to consider the case of V_C polyhedral in $V_C^{\pm}$. Since $V_C^{\pm}$ has finite codimension in X, V_C is also polyhedral in X and hence the closed linear subspace $V_C^{\mp} = -V_C \cap V_C$ has finite codimension in X.

Then $Y = X/V_C^{\mp}$ is finite-dimensional. Let $q : X \to Y$ be the quotient operator. Lemma 3.3 guarantees that $\mathsf{d_H}(qC, V_{qC}) = \mathsf{d_H}(C, V_C) = \infty$ and then qC is infinitely hiding in Y by Lemma 5.13. By Lemma 5.5, C is infinitely (and hence positively) hiding in X. Letting $C_\varepsilon = C$ finishes the proof. ∎

7. Proof of Theorem 1.1. To prove the first part of Theorem 1.1, it suffices to prove the implications (1)⇒(2)⇒(3)⇒(5)⇒(6)⇒(4)⇒(2) and (3)⇒(1), among which (2)⇒(3) and (5)⇒(6) are trivial.

To prove (1)⇒(2), assume that C is approximatively polyhedral and choose a polyhedral convex set P with $\mathsf{d_H}(C,P) < \infty$. Lemma 2.4 implies that $V_C = V_P$. We have $P = \bigcap_{i=1}^n f_i^{-1}((-\infty, a_i])$ for some $f_1, \dots, f_n : X \to \mathbb{R}$ and some $a_1, \dots, a_n \in \mathbb{R}$. It is easy to check that

$$V_C = V_P = \bigcap_{i=1}^n f_i^{-1}((-\infty, 0]),$$

which means that V_C is polyhedral. By Lemma 2.6, $\mathsf{d_H}(P, V_C) = \mathsf{d_H}(P, V_P) < \infty$. Consequently,

$$\mathsf{d_H}(C, V_C) \le \mathsf{d_H}(C, P) + \mathsf{d_H}(P, V_C) < \infty.$$

The implications (3)⇒(5) and (3)⇒(1) are proved in Lemma 3.4, and (6)⇒(4) in Lemma 4.2. The implication (4)⇒(2) follows from Lemmas 5.3, 6.3 and 6.4.

For the second part of the theorem assuming that X is finite-dimensional, it suffices to check that (4)⇒(7)⇒(8)⇒(2). In fact, the implications (4)⇒(7) ⇒(8) are trivial, while (8)⇒(2) follows from Lemmas 5.3 and 5.13.

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