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Trace inequalities for fractional integrals in grand Lebesgue spaces

by

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Abstract. Criteria guaranteeing the trace inequality for integral transforms of various types with fractional order in (generalized) grand Lebesgue spaces defined, generally speaking, on quasi-metric measure spaces are established. In particular, we derive necessary and sufficient conditions on a measure ν governing the boundedness for fractional maximal and potential operators defined on quasi-metric measure spaces from $L^{p),\theta}(X,\mu)$ to $L^{q),q\theta/p}(X,\nu)$ (trace inequality), where $1<p<q<\infty$, $\theta>0$ and μ satisfies the doubling condition in X. The results are new even for Euclidean spaces. For example, from our general results D. Adams-type necessary and sufficient conditions guaranteeing the trace inequality for fractional maximal functions and potentials defined on so-called s-sets in $\mathbb{R}^n$ follow. Trace inequalities for one-sided potentials, strong fractional maximal functions and potentials with product kernels, fractional maximal functions and potentials defined on the half-space are also proved in terms of Adams-type criteria. Finally, we remark that a Fefferman–Stein-type inequality for Hardy–Littlewood maximal functions and Calderón–Zygmund singular integrals holds in grand Lebesgue spaces.

Introduction. The theory of grand Lebesgue spaces introduced by T. Iwaniec and C. Sbordone [11] is one of the intensively developing directions of modern analysis. These spaces find applications in various fields, for example, in integrability problems of the Jacobian under minimal hypotheses (see [11] for the details).

Structural properties of grand Lebesgue spaces were studied in [6], [2]. In [7] the authors proved that for the boundedness of the Hardy–Littlewood maximal operator in weighted grand Lebesgue spaces $L^{p)}_w$ it is necessary and sufficient that the weight w belongs to the Muckenhoupt class A_p. The same phenomenon was noticed by the present authors [15] for the Hilbert transform. We refer to [14], [13], [20] for one-weight results regarding maximal and singular integrals of various type in these spaces. In [17] the author studied the boundedness of the fractional integral operator in weighted grand

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Lebesgue spaces from the one-weight viewpoint. That result, for example, gives the Sobolev inequality for fractional integrals in grand Lebesgue spaces.

Recall that two-weight characterizations for fractional maximal functions and fractional integrals in the classical Lebesgue spaces defined on quasi-metric measure spaces have already been known (see, e.g., [12] and references cited therein).

Our aim is to establish criteria for the trace inequality for fractional maximal functions and potentials in grand Lebesgue spaces defined, generally speaking, on quasi-metric measure spaces. The conditions derived are of Adams [1] type. It should be stressed that the results are new even for Euclidean spaces.

The paper is organized as follows. In Section 1 we give the definition, some properties and motivations of (generalized) grand Lebesgue spaces. In Section 2 we prove a general theorem regarding a two-weight (two-measure) inequality for linear operators in grand Lebesgue spaces defined on quasi-metric measure spaces. In Section 3, based on the general result, we derive Adams-type necessary and sufficient conditions governing the trace inequalities for fractional maximal functions and integrals defined on spaces of homogeneous type and, as particular cases, formulate the corresponding results for Euclidean spaces. Section 4 is devoted to the same problem for one-sided potentials, while in Section 5 we establish trace inequality criteria for strong fractional maximal functions and potentials with product kernels. In Section 6 we derive Carleson-type necessary and sufficient conditions guaranteeing the trace inequality for fractional maximal functions and potentials defined on the half-space. In Section 7 we remark that Fefferman–Stein-type inequalities hold for grand Lebesgue spaces.

1. Preliminaries. Let $X := (X, d, \mu)$ be a topological space with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a nonnegative real-valued function (*quasi-metric*) d on $X \times X$ satisfying:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) there exists a constant $a_1 > 0$ such that $d(x, y) \le a_1(d(x, z) + d(z, y))$ for all $x, y, z \in X$;
(iii) there exists a constant $a_0 > 0$ such that $d(x, y) \le a_0 d(y, x)$ for all $x, y \in X$.

We assume that the balls $B(x, r) := \{y \in X : d(x, y) < r\}$ are measurable and $0 \le \mu(B(x, r)) < \infty$ for all $x \in X$ and $r > 0$; and that for every neighborhood V of $x \in X$, there exists $r > 0$ such that $B(x, r) \subset V$. Throughout the paper we also suppose that $\mu\{x\} = 0$.

We call the triple (X, d, μ) a *quasi-metric measure space*. If μ satisfies the *doubling condition* $\mu(B(x, 2r)) \le c\mu(B(x, r))$, where the positive constant c is independent of $x \in X$ and $r > 0$, then (X, d, μ) is called a *space of homogeneous type* (SHT). For the definition, examples and properties of SHTs, see, e.g., [21], [3], [4].

Let $1 < p < \infty$ and let φ be a continuous positive function on $(0, p-1)$ satisfying the condition $\lim_{x\to 0+} \varphi(x) = 0$. The *generalized grand Lebesgue spaces* $L^{p),\varphi(\cdot)}(X, \mu)$ is the class of those $f : X \to \mathbb{R}$ for which the norm

$$\|f\|_{L^{p),\varphi(\cdot)}(X,\mu)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varphi(\varepsilon)}{\mu(X)} \int_X |f(x)|^{p-\varepsilon}\, d\mu(x) \right)^{1/(p-\varepsilon)}$$

is finite.

If $\varphi(x) = x^\theta$, where θ is a positive number, then we denote $L^{p),\varphi(\cdot)}(X, \mu)$ by $L^{p),\theta}(X, \mu)$.

It turns out that in the theory of PDEs the generalized grand Lebesgue spaces are an appropriate setting for existence and uniqueness problems, and also regularity problems, for various nonlinear differential equations. The space $L^{p),\theta}$ (defined on domains in $\mathbb{R}^n$) for arbitrary positive θ was introduced in [10], where the authors studied the nonhomogeneous n-harmonic equation $\operatorname{div} A(x, \nabla u) = \mu$. If $\theta = 1$, then $L^{p),\theta}(X, \mu)$ coincides with the Iwaniec–Sbordone space, which we denote by $L^{p)}(X, \mu)$.

The classical Lebesgue space defined with respect to the measure μ on X will be denoted by $L^p(X, \mu)$, where $1 < p < \infty$. It is easy to check that whenever $0 < \varepsilon \le p - 1$ and $\theta_1 < \theta_2$ the following continuous embeddings hold:

$$L^p(X, \mu) \hookrightarrow L^{p),\theta_1}(X, \mu) \hookrightarrow L^{p),\theta_2}(X, \mu) \hookrightarrow L^{p-\varepsilon}(X, \mu).$$

Let $1 < p < \infty$, $0 < \alpha < 1/p$ and q be the Hardy–Littlewood–Sobolev exponent, i.e., $q = \frac{p}{1-\alpha p}$. It is known (see [17], [18]) that the potential operator

$$(J_\alpha f)(x) = \int_0^1 \frac{f(t)}{|x-t|^{1-\alpha}} dt, \qquad x \in [0, 1],$$

is bounded from $L^{p),\theta_1}([0, 1])$ to $L^{q),\theta_2}([0, 1])$ if $\theta_2 \ge q\theta_1/p$. However, this boundedness fails if $\theta_2 < q\theta_1/p$.

Finally we point out that constants (often different in the same series of inequalities) will generally be denoted by c or C. The expression $f(x) \approx g(x)$ means that $c_1 f(x) \le g(x) \le c_2 f(x)$, where the positive constants c_1 and c_2 do not depend on x.

2. General result. To formulate the main result of this section we need to introduce some definitions.

Definition 2.1. Let $1 < p < q < \infty$. Suppose that $M_{p,q}(X,Y)$ is a class of pairs (μ,ν) of finite measures, where (X,d,μ) and (Y,ρ,ν) are quasi-metric measure spaces. We say that a linear operator T belongs to the class $\mathcal{B}(M_{p,q}(X,Y))$ if T is bounded from $L^p(X,\mu)$ to $L^q(Y,\nu)$ for every $(\mu,\nu) \in M_{p,q}(X,Y)$.

Definition 2.2. Let $1 < p < q < \infty$. We say that a class $M_{p,q}(X,Y)$ of couples (μ,ν) of finite measures, where (X,d,μ) and (Y,ρ,ν) are quasi-metric measure spaces, is *allowable* if there exist numbers $\varepsilon_0 \in (0,q-1)$ and $\eta_0 \in (0,p-1)$ such that $(\mu,\nu) \in M_{p,q}(X,Y) \Rightarrow (\mu,\nu) \in M_{p-\eta_0,q-\varepsilon_0}(X,Y)$.

If $X = Y$, then we denote $M_{p,q}(X,Y)$ by $M_{p,q}(X)$.

Let $1 < r < \infty$. We denote by P_r the class of all continuous functions $\phi : [0,r-1) \to (0,\infty)$ satisfying $\lim_{x\to 0}\phi(x) = 0$.

To formulate the main result of this section we need to introduce some auxiliary functions. Let $1 < p < q < \infty$ and let ε_0 and η_0 satisfy the conditions $0 < \varepsilon_0 < q-1$, $0 < \eta_0 < p-1$. Let

$$g(\eta) := \frac{\eta q \varepsilon_0 (p-\eta_0)}{\eta_0(q-\varepsilon_0)(p-\eta) + \eta\varepsilon_0(p-\eta_0)},$$

$$\Psi(x) := \Phi(g(x))^{\frac{p-x}{q-g(x)}}, \tag{2.1}$$

where $\Phi \in P_q$. Observe that $g, \Psi \in P_p$ and $g(\eta_0) = \varepsilon_0$.

Our general theorem reads as follows:

Theorem 2.3. *Let $1 < p < q < \infty$ and let $M_{p,q}(X,Y)$ be an allowable class of pairs of finite measures with the constants ε_0 and η_0. Assume that*

$$T \in \mathcal{B}(M_{p,q}(X,Y)) \cap \mathcal{B}(M_{p-\eta_0,q-\varepsilon_0}(X,Y)).$$

Then T is bounded from $L^{p),\Psi(\cdot)}(X,\mu)$ to $L^{q),\Phi(\cdot)}(Y,\nu)$ for $(\mu,\nu) \in M_{p,q}(X,Y)$, where Ψ and Φ are related by (2.1).

Proof. We use the interpolation argument applied in [14]. Let $(\mu,\nu) \in M_{p,q}(X,Y)$. Choose $\varepsilon \in (\varepsilon_0, q-1)$. It is obvious that $\frac{q-\varepsilon_0}{q-\varepsilon} > 1$. Hence, Hölder's inequality yields

$$\|Tf\|_{L^{q-\varepsilon}(Y,\nu))} \le \Big(\int_Y |Tf(x)|^{q-\varepsilon_0}\, d\nu(x)\Big)^{\frac{1}{q-\varepsilon_0}} \nu(Y)^{\frac{\varepsilon-\varepsilon_0}{(q-\varepsilon_0)(q-\varepsilon)}} \tag{2.2}$$

because $\big(\frac{q-\varepsilon_0}{q-\varepsilon}\big)' = \frac{q-\varepsilon_0}{\varepsilon-\varepsilon_0}$.

Further, since $\varepsilon_0 < \varepsilon < q-1$, we have $0 < \frac{\varepsilon-\varepsilon_0}{(q-\varepsilon_0)(q-\varepsilon)} < \frac{q-1-\varepsilon_0}{q-\varepsilon_0}$.

Consequently, by applying (2.2) we find that

$$\|Tf\|_{L^{q-\varepsilon}(Y,\nu)} \le C\|Tf\|_{L^{q-\varepsilon_0}(Y,\nu)}, \qquad \varepsilon \in (\varepsilon_0, q-1), \tag{2.3}$$

where the positive constant C depends only on ν, q and ε_0.

By assumption,

$$\|Tf\|_{L^q(Y,\nu)} \le c\|f\|_{L^p(X,\mu)}, \qquad \|Tg\|_{L^{q-\varepsilon_0}(Y,\nu)} \le c_0\|g\|_{L^{p-\eta_0}(X,\mu)},$$

where the positive constant c (resp. c_0) does not depend on f (resp. g).

Using now the Riesz–Thorin theorem we conclude that T is bounded from $L^{p-\eta}(X,\mu)$ to $L^{q-\varepsilon}(Y,\nu)$, where

$$\frac{1}{p-\eta} = \frac{t}{p-\eta_0} + \frac{1-t}{p}, \qquad \frac{1}{q-\varepsilon} = \frac{t}{q-\varepsilon_0} + \frac{1-t}{q}, \qquad t \in (0,1).$$

Moreover,

$$\|T\|_{L^{p-\eta}(X,\mu)\to L^{q-\varepsilon}(Y,\nu)} \le \|T\|^{1-t}_{L^p(X,\mu)\to L^q(Y,\nu)}\|T\|^t_{L^{p-\eta_0}(X,\mu)\to L^{q-\varepsilon_0}(Y,\nu)}. \tag{2.4}$$

For given $\Phi \in P_p$, we construct Ψ by (2.1). First observe that (2.3) yields

$$\begin{aligned}
&\sup_{\varepsilon_0<\varepsilon<q-1} \Phi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|Tf\|_{L^{q-\varepsilon}(Y,\nu)} \\
&\qquad \le C\Phi(\varepsilon_0)^{\frac{1}{q-\varepsilon_0}}\|Tf\|_{L^{q-\varepsilon_0}(Y,\nu)}\Big(\sup_{\varepsilon_0<\varepsilon<q-1} \Phi(\varepsilon)^{\frac{1}{q-\varepsilon}}\Big)\Phi(\varepsilon_0)^{-\frac{1}{q-\varepsilon_0}} \\
&\qquad \le C\sup_{0<\varepsilon\le\varepsilon_0} \Phi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|Tf\|_{L^{q-\varepsilon}(Y,\nu)}.
\end{aligned}$$

Hence, (2.4) implies that

$$\begin{aligned}
\|Tf\|_{L^{q),\Phi(\cdot)}(Y,\nu)} &\le C\sup_{0<\varepsilon\le\varepsilon_0} \Phi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|Tf\|_{L^{q-\varepsilon}(Y,\nu)} \\
&\le C\sup_{0<\eta\le\eta_0} \Phi(g(\eta))^{\frac{1}{p-\eta}}\|f\|_{L^{p-\eta}(X,\mu)} \le C\|f\|_{L^{p),\Psi(\cdot)}(X,\mu)}. \ \blacksquare
\end{aligned}$$

3. Fractional maximal functions and potentials. In this section we are interested in the fractional integral operator

$$(T_\alpha f)(x) = \int_X \frac{f(y)}{\mu(B(x,d(x,y)))^{1-\alpha}}\,d\mu(y), \qquad 0<\alpha<1.$$

The fractional maximal function related to T_α is given by

$$(M_\alpha f)(x) = \sup_{B\ni x}\frac{1}{\mu(B)^{1-\alpha}}\int_B |f(y)|\,d\mu(y), \qquad 0<\alpha<1,$$

where the supremum is taken over all balls $B \subset X$ containing x.

The following pointwise inequality is obvious:

$$(T_\alpha f)(x) \ge C_\alpha(M_\alpha f)(x), \qquad f \ge 0.$$

Throughout the paper for $1<p<q<\infty$ and $0<\alpha<1/p$ we shall use the notation

$$A_{p,q,\alpha} := q(1/p-\alpha). \tag{3.1}$$

Also, let D be the function defined by

$$D(\varepsilon) := p - \frac{q-\varepsilon}{A_{p,q,\alpha} + \alpha(q-\varepsilon)}, \qquad \varepsilon \in (0, q-1]. \tag{3.2}$$

It is easy to see that the inverse of D on $(0, \varepsilon_0)$ where ε_0 is sufficiently small is given by

$$B(\eta) = q - \frac{(p-\eta)A_{p,q,\alpha}}{1-\alpha(p-\eta)}.$$

It is clear that

$$\lim_{\varepsilon\to 0} D(\varepsilon) = \lim_{\eta\to 0} B(\eta) = 0$$

and

$$D(\varepsilon) \approx \varepsilon, \qquad \varepsilon \to 0. \tag{3.3}$$

Let us recall an Adams-type [1] trace theorem for fractional integrals. We formulate the result for spaces of homogeneous type (see [8] or [4, Ch. 6]).

THEOREM A. *Let $1 < p < q < \infty$ and $0 < \alpha < 1/p$. Suppose that (X, d, μ) is an SHT and ν is another measure on X. Then the operator T_α is bounded from $L^p(X, \mu)$ to $L^q_\nu(X, \nu)$ if and only if there is a positive constant C such that for all balls B in X,*

$$\nu(B) \le C\mu(B)^{A_{p,q,\alpha}}. \tag{3.4}$$

DEFINITION 3.1. Let $1 < p < q < \infty$ and $0 < \alpha < 1/p$. We say that a pair (μ, ν) of finite measures defined on X belongs to the class $\widetilde{M}_{p,q}(X)$ if condition (3.4) holds.

PROPOSITION A. *Let (X, d, μ) be an SHT. Let $1 < p < q < \infty$ and $0 < \alpha < 1/p$. Suppose that ν is another finite measure on X. Assume that*

$$\liminf_{\mu(B)\to 0} \nu(B)\mu(B)^{A_{p,q,\alpha}} \ne 0.$$

Then the operator M_α (and consequently T_α) is not bounded from $L^{p),\theta_1}(X, \mu)$ to $L^{q),\theta_2}(X, \nu)$ for $0 < \theta_2 < q\theta_1/p$.

Proof. Suppose the contrary: M_α is bounded from $L^{p),\theta_1}(X, \mu)$ to $L^{q),\theta_2}(X, \nu)$. For a ball $B \subset X$ we choose η_B so that

$$\sup_{0<\eta\le p-1} (\eta^{\theta_1}\mu(B))^{\frac{1}{p-\eta}} = (\eta_B^{\theta_1}\mu(B))^{\frac{1}{p-\eta_B}}.$$

Now we claim that (see also [17] for $X = [0, 1]$ and $d\mu = dx$)

$$\lim_{\mu(B)\to 0} \eta_B = 0. \tag{3.5}$$

Suppose that there is a sequence $\{B_n\}$ of balls and a positive number λ such that $\mu(B_n) \to 0$ and $\eta_{B_n} \geq \lambda > 0$ for all $n \in \mathbb{N}$. Pick a ball B_{n_0} so that

$$\frac{\mu(B_{n_0})^{1/\theta_1}(p-1)}{e} < e^{-2p/\lambda}.$$

It is easy to see that for $x \in [\lambda/2, p-1]$,

$$\frac{\mu(B_{n_0})^{1/\theta_1}x}{e} \leq \frac{\mu(B_{n_0})^{1/\theta_1}(p-1)}{e} < e^{-\frac{p}{\lambda/2}} \leq e^{-p/x}.$$

Hence, calculating the derivative of $F(x) = (x^{\theta_1}\mu(B_{n_0}))^{1/(p-x)}$ we see that $F'(x) < 0$ for $x \in [\lambda/2, p-1]$. Consequently, $\eta_{B_{n_0}} < \lambda$. Equality (3.5) is proved.

Further, assume that $\mu(B)$ is small and choose ε_B so that

$$\frac{1}{p-\eta_B} - \frac{A_{p,q,\alpha}}{q-\varepsilon_B} = \alpha.$$

Observe that since $\eta_B \to 0$, we have $\varepsilon_B \to 0$. By taking $f = \chi_B$, we see that

$$\|f\|_{L^{p),\theta_1}(X,\mu)} \leq C\eta_B^{\theta_1/(p-\eta_B)}\mu(B)^{1/(p-\eta_B)},$$

where the positive constant C does not depend on B. On the other hand, there is a positive constant c independent of B such that

$$\|M_\alpha f\|_{L^{q),\theta_2}(X,\nu)} \geq c\varepsilon_B^{\theta_2/(q-\varepsilon_B)}\nu(B)^{1/(q-\varepsilon_B)}\mu(B)^\alpha.$$

Consequently, the boundedness of M_α yields

$$\nu(B)\mu(B)^{(\alpha-1/(p-\eta_B))(q-\varepsilon_B)}[\varepsilon_B^{\theta_2/(q-\varepsilon_B)}\eta_B^{-\theta_1/(p-\eta_B)}]^{q-\varepsilon_B} \leq C. \tag{3.6}$$

Observe now that (3.3) implies

$$\begin{aligned}\varepsilon_B^{\theta_2/(q-\varepsilon_B)}\eta_B^{-\theta_1/(p-\eta_B)} &\approx D(\varepsilon_B)^{\theta_2/(q-\varepsilon_B)}\eta_B^{-\theta_1/(p-\eta_B)}\\ &= \eta_B^{\frac{\theta_2}{A_{p,q,\alpha}(p-\eta_B)}-\frac{\alpha\theta_2}{A_{p,q,\alpha}}-\frac{\theta_1}{p-\eta_B}}\end{aligned}$$

for small ε_B. If η_B tends to 0, then the exponent of the latter expression converges to

$$-\frac{\alpha\theta_2}{A_{p,q,\alpha}} + \frac{\theta_2}{pA_{p,q,\alpha}} - \frac{\theta_1}{p},$$

which is negative due to the condition $\theta_2 < q\theta_1/p$. Hence, the lim inf of the left-hand side of (3.6) is equal to $+\infty$ as $\eta_B \to 0$. ■

This statement gives us the motivation to investigate the boundedness of fractional maximal functions and potentials from $L^{p),\theta_1}(X,\mu)$ to $L^{q),\theta_2}(Y,\nu)$, where $1 < p < q < \infty$ and $\theta_2 \geq q\theta_1/p$. Since $L^{q),q\theta_1/p}(X,\nu) \hookrightarrow L^{q),\theta_2}(Y,\nu)$ for $\theta_2 \geq q\theta_1/p$, it is enough to consider the case $\theta_2 = q\theta_1/p$.

Our main result in this section is the following statement:

Theorem 3.2. *Let $1 < p < q < \infty$ and let $0 < \alpha < 1/p$. Suppose that (X, d, μ) is an SHT and ν is another finite measure on X. Let $\theta > 0$. Then the following conditions are equivalent:*

(i) *the operator T_α is bounded from $L^{p),\theta}(X,\mu)$ to $L^{q),q\theta/p}(X,\nu)$;*
(ii) *the operator M_α is bounded from $L^{p),\theta}(X,\mu)$ to $L^{q),q\theta/p}(X,\nu)$;*
(iii) *condition* (3.4) *holds.*

To prove the main theorem, introduce the function

$$\overline{D}(\varepsilon) := D(\varepsilon)^{A_{p,q,\alpha}+\alpha(q-\varepsilon)},$$

where $A_{p,q,\alpha}$ and D are defined by (3.1) and (3.2) respectively.

By using (3.3) we see that

$$\overline{D}(\varepsilon) \approx \varepsilon^{q/p}. \tag{3.7}$$

Proof of Theorem 3.2. First of all notice that if S_α is T_α or M_α then the boundedness of S_α from $L^{p),\theta}(X,\mu)$ to $L^{q),q\theta/p}(X,\nu)$ is equivalent to the inequality

$$\|S_\alpha f\|_{L^{q),\widetilde{D}(\cdot)}(X,\nu)} \le C\|f\|_{L^{p),\theta}(X,\mu)}, \tag{3.8}$$

where

$$\widetilde{D}(x) := \overline{D}(x^\theta). \tag{3.9}$$

Indeed, it is enough to notice that (3.7) implies

$$\widetilde{D}(x) \approx x^{\theta q/p}, \qquad x \to 0. \tag{3.10}$$

(iii)$\Rightarrow$(i): Observe that for a given ε_0 with $0 < \varepsilon_0 < q-1$, there is $\eta_0 \in (0, p-1)$ such that $(\mu,\nu) \in \widetilde{M}_{p,q}(X) \Rightarrow (\mu,\nu) \in \widetilde{M}_{p-\eta_0,q-\varepsilon_0}(X)$. In fact, η_0 is chosen so that

$$A_{p,q,\alpha} = A_{p-\eta_0,q-\varepsilon_0,\alpha}. \tag{3.11}$$

Hence, the class $\widetilde{M}_{p,q}(X)$ is allowable in the sense of Definition 2.2. By Theorem A we have $T_\alpha \in \mathcal{B}(\widetilde{M}_{p,q}(X)) \cap \mathcal{B}(\widetilde{M}_{p-\eta_0,q-\varepsilon_0}(X))$. Observe also that

$$\frac{1}{p} - \frac{A_{p,q,\alpha}}{q} = \frac{1}{p-\eta_0} - \frac{A_{p,q,\alpha}}{q-\varepsilon_0}.$$

Assuming

$$g(\eta) = q - \frac{(p-\eta)A_{p,q,\alpha}}{1-\alpha(p-\eta)}, \qquad \Phi(x) = x^\theta$$

in Theorem 2.3, we have the desired conclusion. It is easy to see that in this case $\Psi(x) = \widetilde{D}(x)$.

Since the implication (i)$\Rightarrow$(ii) is obvious, it remains to show (ii)$\Rightarrow$(iii). It suffices to see that (3.4) holds for all balls B having sufficiently small measure $\mu(B)$. Let $f=\chi_B$, where $B\subset X$ is a ball. Then

$$\|f\|_{L^{p),\theta}(X,\mu)}\le C\sup_{0<\eta\le p-1}\eta^{\theta/(p-\eta)}\mu(B)^{1/(p-\eta)}=C\eta_B^{\theta/(p-\eta_B)}\mu(B)^{1/(p-\eta_B)},$$

where the positive constant C does not depend on B. Recall that $\eta_B\to 0$ as $\mu(B)\to 0$ (see (3.5)). Thus, η_B is small for sufficiently small $\mu(B)$. Choose ε_B so that

$$\frac{1}{p-\eta_B}-\frac{A_{p,q,\alpha}}{q-\varepsilon_B}=\alpha.$$

Hence,

$$\|M_\alpha f\|_{L^{q),\widetilde{D}(\cdot)}(X,\nu)}\ge c\widetilde{D}(\varepsilon_B)^{1/(q-\varepsilon_B)}\nu(B)^{1/(q-\varepsilon_B)}\mu(B)^{\alpha}.$$

Taking into account (3.8) for $S_\alpha=M_\alpha$, we find that

$$\nu(B)^{1/(q-\varepsilon_B)}\mu(B)^{\alpha}\widetilde{D}(\varepsilon_B)^{1/(q-\varepsilon_B)}\eta_B^{-\theta/(p-\eta_B)}\mu(B)^{-1/(p-\eta_B)}\le C.$$

Observe now that (3.10) implies

$$\widetilde{D}(\varepsilon_B)^{1/(q-\varepsilon_B)}\approx\varepsilon_B^{\theta q/p(q-\varepsilon_B)}\approx D(\varepsilon_B)^{\alpha\theta+A_{p,q,\alpha}/(q-\varepsilon_B)}=\eta_B^{\theta/(p-\eta_B)}.$$

Finally,

$$\nu(B)\le C\mu(B)^{(1/(p-\eta_B)-\alpha)(q-\varepsilon_B)}=C\mu(B)^{A_{p,q,\alpha}}. \ \blacksquare$$

REMARK 3.3. It is easy to see that if the operator T_α is bounded from $L^{p),\theta}(X,\mu)$ to $L^{q),q\theta/p}(X,\nu)$, where μ is a finite measure on X, then ν is also finite on X. Indeed, taking $f\equiv 1$ on X we see that

$$\|T_\alpha f\|_{L^{q),q\theta/p}(X,\nu)}\ge C\sup_{0<\varepsilon\le q-1}\big(\varepsilon^{q\theta/p}\nu(X)\big)^{1/(q-\varepsilon)}=C\big(\varepsilon_0^{q\theta/p}\nu(X)\big)^{1/(q-\varepsilon_0)}.$$

On the other hand,

$$\|f\|_{L^{p),\theta}(X,\mu)}=\sup_{0<\eta\le p-1}\eta^{\theta/(p-\eta)}=\eta_0^{\theta/(p-\eta_0)}.$$

Taking these estimates into account and using the boundedness of T_α, we conclude that $\nu(X)<\infty$.

Let us now formulate the main result of this section for particular cases.

Let Γ be a bounded s-set of $\mathbb{R}^n$ $(0<s\le n)$ in the sense that there is a Borel measure μ on $\mathbb{R}^n$ such that

(i) $\operatorname{supp}\mu=\Gamma$;

(ii) there are positive constants c_1 and c_2 such that for all $x\in\Gamma$ and all $r\in(0,\operatorname{diam}\Gamma)$,

$$c_1r^s\le\mu(\Gamma(x,r))\le c_2r^s, \tag{3.12}$$

where $\Gamma(x,r):=B(x,r)\cap\Gamma$ and $B(x,r)$ is the ball in $\mathbb{R}^n$ with center x and radius r.

It is known (see [22, Theorem 3.4]) that μ is equivalent to the restriction of the Hausdorff s-measure $\mathcal{H}_s$ to Γ; we shall identify μ with $\mathcal{H}_s|_\Gamma$.

For example, connected rectifiable regular curves with respect to the arc-length measure satisfy condition (3.12) for $s = 1$.

Let $0 < \gamma < s$, and let M_γ^Γ and T_γ^Γ be the fractional maximal and potential operators given by

$$(M_\gamma^\Gamma g)(x) = \sup_{r>0} \frac{1}{\mathcal{H}_s(\Gamma(x,r))^{1-\gamma/s}} \int_{\Gamma(x,r)} |g(y)|\, d\mathcal{H}_s(y), \qquad x \in \Gamma,$$

$$(T_\gamma^\Gamma g)(x) = \int_\Gamma \frac{g(y)}{|x-y|^{s-\gamma}}\, d\mathcal{H}_s(y), \qquad x \in \Gamma.$$

It is easy to see that Theorem 3.2 implies the following statement:

Corollary 3.4. *Let $1 < p < q < \infty$ and let $0 < \gamma < s/p$. Suppose that ν is another finite measure on Γ. Let $\theta > 0$. Then the following conditions are equivalent:*

(i) *the operator T_γ^Γ is bounded from $L^{p),\theta}(\Gamma, \mathcal{H}_s)$ to $L^{q),q\theta/p}(\Gamma, \nu)$;*
(ii) *the operator M_γ^Γ is bounded from $L^{p),\theta}(\Gamma, \mathcal{H}_s)$ to $L^{q),q\theta/p}(\Gamma, \nu)$;*
(iii) $\sup_{r>0} \nu(\Gamma(x,r)) r^{q(\gamma - s/p)} < \infty$.

If $\Gamma = \Omega$ is a bounded domain in $\mathbb{R}^n$ and (3.12) is satisfied for $s = n$ (in this case μ is a Lebesgue measure on Ω), then we have the next statement:

Corollary 3.5. *Let $1 < p < q < \infty$ and let $0 < \gamma < n/p$. Suppose that ν is a finite measure on Ω. Let $\theta > 0$. Then the following conditions are equivalent:*

(i) *the operator T_γ^Ω is bounded from $L^{p),\theta}(\Omega, dx)$ to $L^{q),q\theta/p}(\Omega, \nu)$;*
(ii) *the operator M_γ^Ω is bounded from $L^{p),\theta}(\Omega, dx)$ to $L^{q),q\theta/p}(\Omega, \nu)$;*
(iii) $\sup_{r>0} \nu(D(x,r)) r^{q(\gamma - n/p)} < \infty$, *where $D(x,r) := B(x,r) \cap \Omega$.*

4. One-sided potentials. In this section, for $0 < \alpha < 1$, we discuss the trace inequality for one-sided potentials

$$(R_\alpha f)(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \qquad x \in [0,1],$$

$$(W_\alpha f)(x) = \int_x^1 \frac{f(t)}{(t-x)^{1-\alpha}}\, dt, \qquad x \in [0,1].$$

Let us recall the notation $A_{p,q,\alpha} := q(1/p - \alpha)$ for $1 < p < q < \infty$ and $0 < \alpha < 1/p$ (see (3.1)).

The following statement is taken from [4, pp. 131–132].

Theorem B. *Let $1 < p < q < \infty$ and let $0 < \alpha < 1/p$. Suppose that ν is a measure on $[0,1]$. Then the following statements are equivalent:*

(i) *the operator R_α is bounded from $L^p([0,1])$ to $L^q([0,1],\nu)$;*
(ii) *the operator W_α is bounded from $L^p([0,1])$ to $L^q([0,1],\nu)$;*
(iii) *there is a positive constant C such that for all $a \in [0,1]$ and $h \in [0, \min\{a, 1-a\}]$,*

$$\nu([a, a+h)) \le Ch^{A_{p,q,\alpha}}; \tag{4.1}$$

(iv) *there is a positive constant C such that for all $a \in [0,1]$ and $h \in [0, \min\{a, 1-a\}]$,*

$$\nu([a-h, a)) \le Ch^{A_{p,q,\alpha}}. \tag{4.2}$$

Our result is the following statement:

Theorem 4.1. *Let $1 < p < q < \infty$ and let $0 < \alpha < 1/p$. Suppose that ν is a finite measure on $[0,1]$. Let $\theta > 0$. Then the following conditions are equivalent:*

(i) *the operator R_α is bounded from $L^{p),\theta}([0,1])$ to $L^{q),\theta q/p}([0,1],\nu)$;*
(ii) *the operator W_α is bounded from $L^{p),\theta}([0,1])$ to $L^{q),\theta q/p}([0,1],\nu)$;*
(iii) *there is a positive constant C such that for all $x \in [0,1]$ and $h \in [0, \min\{a, 1-a\}]$ condition (4.1) holds.*
(iv) *there is a positive constant C such that for all $a \in [0,1]$ and $h \in [0, \min\{a, 1-a\}]$ condition (4.2) holds.*

Proof. The implications (iii)$\Rightarrow$(i) and (iv)$\Rightarrow$(ii) follow from Theorems B and 2.3 because condition (4.1) (resp. (4.2)) defines an allowable class (see Definition 2.2) of pairs of measures (dx, ν), where dx is the Lebesgue measure. In fact, ε_0 and η_0 are chosen so that equality (3.11) holds. Assuming that $\Phi(x) = x^\theta$, we see that $\Psi(x) = \widetilde{D}(x)$, where $\widetilde{D}$ is given by (3.9). Relation (3.10) completes the proof of these implications (see also the proof of Theorem 3.2 for the details).

Let us see that (i)$\Rightarrow$(iii). As in the proof of Theorem 3.2 it is enough to show that the inequality

$$\|R_\alpha f\|_{L^{q),\widetilde{D}(\cdot)}([0,1],\nu)} \le C\|f\|_{L^{p),\theta}([0,1])} \tag{4.3}$$

implies (4.1). Taking $f = \chi_{[a-h,a)}$ in (4.3), where h is a small positive number, and arguing as in the proof of Theorem 3.2 we derive the desired implication. Analogously, we obtain (ii)$\Rightarrow$(iii) by choosing $f = \chi_{[a,a+h)}$ in the inequality

$$\|W_\alpha f\|_{L^{q),\widetilde{D}(\cdot)}([0,1],\nu)} \le C\|f\|_{L^{p),\theta}([0,1])},$$

where h is sufficiently small. Theorem B yields the equivalence of (iii) and (iv). ∎

5. Strong fractional maximal functions and potentials with product kernels. Let $R_0 := I_0 \times J_0$ be a bounded rectangle in $\mathbb{R}^2$. For $0 < \alpha, \beta < 1$, define the following operators on R_0:

$$(M_{\alpha,\beta} f)(x, y) = \sup_{I\times J \ni (x,y)} \frac{1}{|I|^{1-\alpha}|J|^{1-\beta}} \iint\limits_{I\times J} |f(x, y)|\, dx\, dy,$$

where the supremum is taken over all rectangles $I \times J \subseteq R_0$ containing (x, y), and

$$(I_{\alpha,\beta} f)(x, y) = \iint\limits_{I_0\times J_0} \frac{f(t, \tau)}{|x - t|^{1-\alpha}|y - \tau|^{1-\beta}}\, dt\, d\tau.$$

If $\alpha = \beta$, then $M_{\alpha,\beta}$ and $I_{\alpha,\beta}$ are denoted by $M_{\alpha,\alpha}$ and $I_{\alpha,\alpha}$ respectively.

Let us recall the definition of the well-known Muckhenhoupt A_∞ class with respect to a single variable (see, e.g., [16, p. 182]).

Definition 5.1. Let $R_0 = I_0 \times J_0$ be a rectangle in $\mathbb{R}^2$. We say that an integrable a.e. positive function (weight) u defined on R_0 belongs to the class $A_\infty(I_0)$ with respect to the first variable uniformly in the second one ($u \in A^{(x)}_\infty(I_0)$) if there are positive constants c and δ such that

$$\frac{u_y(E)}{u_y(I)} \le c\left(\frac{|E|}{|I|}\right)^\delta$$

for all $y \in J_0$, all intervals $I \subset I_0$ and all measurable sets $E \subset I$. The class $A^{(y)}_\infty(J_0)$ is defined analogously.

In what follows, the space $L^{q),\theta}(Y, \nu)$ for ν absolutely continuous, $d\nu = u(z)dz$, will be denoted by $L^{q),\theta}(Y, u)$.

Theorem C ([16, Section 4.5]). *Let $R_0 = I_0 \times J_0$ be a rectangle in $\mathbb{R}^2$. Let $1 < p < q < \infty$ and let $0 < \alpha, \beta < 1/p$. Suppose that v is a weight function on R_0. Then the following statements are equivalent:*

(i) *$M_{\alpha,\beta}$ is bounded from $L^p(R_0)$ to $L^q(R_0, v)$;*

(ii) *there is a positive constant C such that for all rectangles $I \times J \subseteq R_0$,*

$$\iint\limits_{I\times J} v(x, y)\, dx\, dy \le C|I|^{q(1/p-\alpha)}|J|^{q(1/p-\beta)}. \tag{5.1}$$

Theorem D ([16, Section 4.5]). *Let $R_0 = I_0 \times J_0$ be a rectangle in $\mathbb{R}^2$. Let $1 < p < q < \infty$ and $0 < \alpha, \beta < 1/p$. Suppose that $v \in A^{(x)}_\infty(I_0) \cup A^{(y)}_\infty(J_0)$. Then the following statements are equivalent:*

(i) *$I_{\alpha,\beta}$ is bounded from $L^p(R_0)$ to $L^q(R_0, v)$;*

(ii) *condition* (5.1) *holds.*

Definition 5.2. Let R_0 be a bounded rectangle in $\mathbb{R}^2$. We say that a pair $(dx, v(x)dx)$, where dx is the Lebesgue measure and v is a weight function on R_0, belongs to the class $\mathcal{M}_{p,q}(R_0)$ ($1<p<q<\infty$, $0<\alpha<1/p$) if there is a positive constant C such that for all rectangles $R\subset R_0$,

$$\iint_R v(x,y)\,dx\,dy \le C|R|^{A_{p,q,\alpha}}, \tag{5.2}$$

where $A_{p,q,\alpha}$ is defined by (3.1).

The main statements of this section are as follows:

THEOREM 5.3. *Let $R_0 = I_0\times J_0$ be a rectangle in $\mathbb{R}^2$. Let $1<p<q<\infty$ and $0<\alpha<1/p$. Suppose that θ is a positive number. Assume that v is a weight function on R_0. Then the following statements are equivalent:*

(i) *$M_{\alpha,\alpha}$ is bounded from $L^{p),\theta}(R_0)$ to $L^{q),q\theta/p}(R_0,v)$;*
(ii) *there is a positive constant C such that for all rectangles $R\subseteq R_0$, condition* (5.2) *holds.*

THEOREM 5.4. *Let $R_0 = I_0\times J_0$ be a rectangle in $\mathbb{R}^2$. Let $1<p<q<\infty$ and $0<\alpha<1/p$. Suppose that $v\in A_\infty^{(x)}(I_0)\cup A_\infty^{(y)}(J_0)$. Then the following statements are equivalent:*

(i) *$I_{\alpha,\alpha}$ is bounded from $L^{p),\theta}(R_0)$ to $L^{q),\theta q/p}(R_0,v)$;*
(ii) *condition* (5.2) *holds.*

Proof of Theorem 5.3. Sufficiency is a direct consequence of Theorems 2.3 and C. Indeed, observe that if $(dx, v(x)dx)\in\mathcal{M}_{p,q}(R_0)$, then $(dx, v(x)dx)$ $\in\mathcal{M}_{p-\eta_0,q-\varepsilon_0}(R_0)$ for some positive numbers ε_0 and η_0 (see also the proof of Theorem 3.2 for the details).

Necessity. We show that (5.2) holds for all rectangles $R\subseteq R_0$ having sufficiently small Lebesgue measure $|R|$. Take $f=\chi_R$. Then

$$\|f\|_{L^{p),\theta}(R_0)} \le C \sup_{0<\eta\le p-1} \eta^{\theta/(p-\eta)}|R|^{1/(p-\eta)} = C\eta_R^{\theta/(p-\eta_R)}|R|^{1/(p-\eta_B)},$$

where the positive constant C does not depend on R. Observe that the inequality similar to (3.5) also holds for rectangles R. Hence η_R is a small positive number when $|R|$ is small. Let us choose ε_R so that

$$\frac{1}{p-\eta_R} - \frac{A_{p,q,\alpha}}{q-\varepsilon_R} = \alpha.$$

Hence,

$$\|M_{\alpha,\alpha}f\|_{L^{q),\widetilde{D}(\cdot)}(Y,v)} \ge c\widetilde{D}(\varepsilon_R)^{1/(q-\varepsilon_R)}\Big(\iint_R v(x,y)\,dx\,dy\Big)^{1/(q-\varepsilon_R)}|R|^{\alpha},$$

where $\widetilde{D}$ is defined by (3.9). Using the two-weight inequality for $M_{\alpha,\alpha}$ we obtain the desired result. ∎

The proof of Theorem 5.4 is similar to that of Theorem 5.3; therefore it is omitted.

6. Potentials on the half-space. Let (X, d, μ) be an SHT with finite measure μ. We introduce the notation

$$\widehat{X} := X \times [0, \infty), \qquad \widehat{B} := B \times [0, 2 \cdot \mathrm{radius}(B)),$$

where B is a ball in X.

In this section, for $0 < \alpha < 1$, we establish trace inequality criteria for the generalized potential operator

$$(\widehat{T}_\alpha f)(x,t) = \int_X f(y)\mu\big(B(x, d(x,y)+t)\big)^{\alpha-1} d\mu(y), \qquad (x,t) \in \widehat{X},$$

in grand Lebesgue spaces.

Together with $\widehat{T}_\alpha$ we are interested in the appropriate fractional maximal operator:

$$(\widehat{M}_\alpha f)(x,t) = \sup \mu(B)^{\alpha-1} \int_B |f(y)|\, d\mu(y),$$

where the supremum is taken over all balls $B \subset X$ containing x and of radius greater than $t/2$.

For the following statement we refer to [9] (see also [4, Section 6.4]).

Theorem E. *Let $0 < \alpha < 1/p$ and let $1 < p < q < \infty$. Then the following statements are equivalent:*

(i) *$\widehat{T}_\alpha$ is bounded from $L^p(X, \mu)$ into $L^q(\widehat{X}, \beta)$;*
(ii) *$\widehat{M}_\alpha$ is bounded from $L^p(X, \mu)$ into $L^q(\widehat{X}, \beta)$;*
(iii) *there exists a constant C such that for all balls $B \subset X$,*

$$\beta(\widehat{B}) \le C\mu(B)^{A_{p,q,\alpha}}, \tag{6.1}$$

where the constant $A_{p,q,\alpha}$ is defined by (3.1).

Our result is the following statement:

Theorem 6.1. *Let $0 < \alpha < 1/p$ and $1 < p < q < \infty$. Suppose that β is a finite measure on $\widehat{B}$. Let $\theta > 0$. Then the following statements are equivalent:*

(i) *$\widehat{T}_\alpha$ is bounded from $L^{p),\theta}(X, \mu)$ into $L^{q),\theta q/p}(\widehat{X}, \beta)$;*
(ii) *$\widehat{M}_\alpha$ is bounded from $L^{p),\theta}(X, \mu)$ into $L^{q),\theta q/p}(\widehat{X}, \beta)$;*
(iii) *condition* (6.1) *holds.*

Proof. The implication (i)⇒(ii) follows from the pointwise inequality

$$\widehat{M}_\alpha(x,t) \le c\widehat{T}_\alpha(x,t), \qquad (x,t) \in \widehat{X}.$$

The fact that (iii)⇒(i) follows from Theorems 2.3 and E because in this case the class of measure pairs (μ, β) satisfying (6.1) is allowable (see Definition 2.2). Arguing as in the proof of Theorem 3.2 we derive the desired result.

Let us see check (ii)$\Rightarrow$(iii). For this it is enough to show that (6.1) holds for all balls B with $\mu(B)$ small. Take $f_B = \chi_B$. Then $(\widehat{M}_\alpha f_B)(x,t) \ge \mu(B)^\alpha$ for $(x,t) \in \widehat{B}$. Now the result follows by substituting f_B in the two-weight inequality for $\widehat{M}_\alpha$ and arguing as in the proof of Theorem 3.2; the details are omitted. ∎

Let Γ be a bounded s-set in $\mathbb{R}^n$ and let $\mathcal{H}_s$ be an appropriate Hausdorff s-measure on Γ (see Section 3 for the details). Let $0 < \gamma < n$.

Define

$$(\widehat{M}_\gamma^\Gamma f)(x,t) = \sup \mathcal{H}_s\big(D\big)^{1-\gamma/s} \int\limits_D |f(y)|\, d\mathcal{H}_s(y),$$

where the supremum is taken over all balls $D \subset \Gamma$ (recall that balls D in Γ have the form $B \cap \Gamma$, where B is a ball in $\mathbb{R}^n$) containing x and of radius greater than $t/2$. Also, let

$$(\widehat{T}_\gamma^\Gamma f)(x,t) = \int\limits_\Gamma f(y)(|x-y|+t)^{\gamma-s}\, d\mathcal{H}_s(y).$$

Corollary 6.2. *Let $0 < \gamma < s/p$ and let $1 < p < q < \infty$. Suppose that β is a finite measure on $\widehat{\Gamma}$. Let $\theta > 0$. Then the following statements are equivalent:*

(i) $\widehat{T}_\gamma^\Gamma$ is bounded from $L^{p),\theta}(\Gamma,\mu)$ into $L^{q),\theta q/p}(\widehat{\Gamma},\beta)$;

(ii) $\widehat{M}_\gamma^\Gamma$ is bounded from $L^{p),\theta}(\Gamma,\mu)$ into $L^{q),\theta q/p}(\widehat{\Gamma},\beta)$;

(iii) there exists a constant C such that for all balls $D \subset \Gamma$,

$$\beta(\widehat{D}) \le C\mu(D)^{q(1/p-\gamma/s)}.$$

7. Remarks on the Fefferman–Stein-type inequality. Let us recall the well-known Fefferman–Stein [5] inequality for the Hardy–Littlewood maximal operator

$$(\mathcal{M}f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int\limits_B |f(y)|\, dy, \qquad x \in \mathbb{R}^n.$$

Theorem F. *Let $1 < p < \infty$. Then there is a positive constant c such that for all f and all nonnegative locally integrable v,*

$$\int\limits_{\mathbb{R}^n} (\mathcal{M}f)(x)^p v(x)\, dx \le c \int\limits_{\mathbb{R}^n} |f(x)|^p (\mathcal{M}v)(x)\, dx.$$

Further, let

$$(Kf)(x) = \int\limits_{\mathbb{R}^n} k(x,y) f(y)\, dy$$

be a Calderón–Zygmund singular operator. Then the following statement holds (see [19]):

Theorem G. *Let $1 < p < \infty$. There is a positive constant c such that for all functions f and all nonnegative locally integrable v,*

$$\int_{\mathbb{R}^n} |(Tf)(x)|^p v(x)\,dx \le \int_{\mathbb{R}^n} |f(x)|^p (\mathcal{M}^{[p]+1} v)(x)\,dx,$$

where $[p]$ is the largest integer less than or equal to p and $\mathcal{M}^k$ is the kth iterate of the operator $\mathcal{M}$. Moreover, the exponent $[p]+1$ is sharp.

Let Ω be a bounded domain in $\mathbb{R}^n$ and let us introduce the maximal operator defined on Ω,

$$\mathcal{M}^{\Omega} g(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{\Omega \cap B} |g(y)|\,dy, \qquad x \in \Omega,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ containing x.

For a weight function V on Ω, set

$$V_{\Omega}(x) := \begin{cases} V(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition B. *Let $1 < p < \infty$ and $\theta > 0$. Suppose that Ω is a bounded open set in $\mathbb{R}^n$. Then:*

(i) *there is a positive constant c such that for all f and all nonnegative integrable V defined on Ω,*

$$\|\mathcal{M}^{\Omega} f\|_{L^{p),\theta}(\Omega,V)} \le c\|f\|_{L^{p),\theta}(\Omega,\mathcal{M}^{\Omega} V)};$$

(ii) *there is a positive constant c such that for all f and all nonnegative integrable V defined on Ω,*

$$\|Tf\|_{L^{p),\theta}(\Omega,V_{\Omega})} \le c\|f\|_{L^{p),\theta}(\Omega,\mathcal{M}^{[p]+1} V_{\Omega})}.$$

Proof. (i) Let us extend f outside Ω by 0.

First observe that in the definition of $\mathcal{M}^{\Omega}$ the supremum can be taken over all balls with $|B| \le a$, where the constant a depends only on $\operatorname{diam}(\Omega)$.

Further, Theorem F yields

$$\|\mathcal{M}^{\Omega} f\|_{L^{p}(\Omega,V_{\Omega})} \le c\|f\|_{L^{p}(\Omega,\mathcal{M}^{\Omega} V_{\Omega})},$$
$$\|\mathcal{M}^{\Omega} f\|_{L^{p-\varepsilon_0}(\Omega,V_{\Omega})} \le c\|f\|_{L^{p-\varepsilon_0}(\Omega,\mathcal{M}^{\Omega} V_{\Omega})},$$

where ε_0 is a small positive number, and the constant c is independent of f and V.

By using the Riesz–Thorin theorem we find that

$$\|\mathcal{M}^{\Omega} f\|_{L^{p-\varepsilon}(\Omega,V)} \le C\|f\|_{L^{p-\varepsilon}(\Omega,\mathcal{M}^{\Omega} V)}, \qquad 0 < \varepsilon < \varepsilon_0,$$

where the positive constant C is independent of f, V and ε.

Hence, since V is integrable on Ω we have (see also the proof of Theorem 2.3 for the details)

$$\|\mathcal{M}^{\Omega}_{\alpha} f\|_{L^{p),\theta}(\Omega,V)} \le C \sup_{0<\varepsilon<\varepsilon_0} \Big(\varepsilon^{\theta} \int_{\Omega} (\mathcal{M}^{\Omega} f(x))^{p-\varepsilon} V(x)\,dx\Big)^{1/(p-\varepsilon)}$$

$$\le C \sup_{0<\varepsilon<\varepsilon_0} \Big(\varepsilon^{\theta} \int_{\Omega} |f(y)|^{p-\varepsilon} (\mathcal{M}^{\Omega} V)(y)\,dy\Big)^{1/(p-\varepsilon)} \le C\|f\|_{L^{p),\theta}(\Omega,\mathcal{M}^{\Omega}V)}.$$

(ii) First observe that

$$\mathcal{M}^{[p-\varepsilon]+1} V_{\Omega}(x) \le \mathcal{M}^{[p]+1} V_{\Omega}(x), \qquad x \in \Omega.$$

Hence, taking Theorem G into account we conclude that

$$\|Tf\|_{L^p(\Omega,V_{\Omega})} \le c\|f\|_{L^p(\Omega,\mathcal{M}^{[p]+1}V_{\Omega})},$$
$$\|Tf\|_{L^{p-\varepsilon_0}(\Omega,V_{\Omega})} \le c\|f\|_{L^{p-\varepsilon_0}(\Omega,\mathcal{M}^{[p]+1}V_{\Omega})},$$

for all f and nonnegative integrable V with support in Ω. Using the same interpolation arguments, we can conclude that

$$\|Tf\|_{L^{p-\varepsilon}(\Omega,V_{\Omega})} \le C\|f\|_{L^{p-\varepsilon}(\Omega,\mathcal{M}^{[p]+1}V_{\Omega})},$$

where the positive constant C is independent of $0 < \varepsilon < \varepsilon_0$, f and V. Arguing as in the case of the maximal operator we get the desired conclusion. ∎

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