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Lyapunov theorem for q-concave Banach spaces

by

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Abstract. A generalization of the Lyapunov convexity theorem is proved for a vector measure with values in a Banach space with unconditional basis, which is q-concave for some $q < \infty$ and does not contain any isomorphic copy of l_2.

1. Introduction. Let X be a Banach space, (Ω, Σ) be a measurable space, where Ω is a set and Σ is a σ-algebra of subsets of Ω. If $m : \Sigma \to X$ is a σ-additive X-valued measure, then the *range* of m is the set $m(\Sigma) = \{m(A) : A \in \Sigma\}$.

The measure m is *non-atomic* if for every set $A \in \Sigma$ with $m(A) \neq 0$, there exist $B \in \Sigma$ with $B \subset A$ such that $m(B) \neq 0$ and $m(A \setminus B) \neq 0$.

According to the famous Lyapunov theorem [4] the range of every $\mathbb{R}^n$-valued non-atomic measure μ is convex. However, this theorem does not generalize directly to the infinite-dimensional case: for every infinite-dimensional Banach space X there is an X-valued non-atomic measure (of bounded variation) whose range is not convex [1, Corollary IX.1.6].

We will call an X-valued measure a *Lyapunov measure* if the closure of its range is convex. And the Banach space X is a *Lyapunov space* if every X-valued non-atomic measure is Lyapunov.

The following result was obtained in [7].

Theorem 1.1 (Uhl). *Let X have Radon-Nikodym property. Then any X-valued measure of bounded variation is Lyapunov.*

For measures with unbounded variation this is no longer true. There exist non-atomic measures with values in Hilbert space which are not Lyapunov. It follows that also spaces containing isomorphic copies of l_2 are not Lyapunov: in particular, all $L_p[0,1]$, $C[0,1]$, l_∞. Nevertheless it was proved in [2] that the sequence spaces c_0 and $l_p, 1 \leq p < \infty, p \neq 2$ are examples of Lyapunov spaces.

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On the other hand, using a compactness argument, it was proven in [6] that in a Banach space with unconditional basis every non-negative (with respect to the order induced by the basis) non-atomic measure is Lyapunov.

Recall that for Banach spaces X, Y and Z, an operator $T : X \to Y$ is said to be *Z-strictly singular* if it is not an isomorphism when restricted to any isomorphic copy of Z in X.

We say that a linear operator $T : L_p(\mu) \to X$ is *narrow* if for every $\epsilon > 0$ and every measurable set $A \subset [0,1]$ there exists $x \in L_p$ with $x^2 = \mathbb{I}_A$ and $\int_{[0,1]} x \, d\mu = 0$ such that $\|Tx\| < \epsilon$ (we call such an x a *mean zero sign*).

In [5, Theorem B] it was shown that for every $1 < p < \infty$ and every Banach space X with an unconditional basis, every l_2-strictly singular operator $T : L_p \to X$ is narrow, where L_p denotes the L_p space on $(0,1)$ with Lebesgue measure.

We will use the following notion of q-concavity [3, 1.d.3].

Let X be a Banach lattice, let V be an arbitrary Banach space and let $1 \le q < \infty$. A linear operator $T : X \to V$ is called *q-concave* if there exists a constant $M < \infty$ such that

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^q\Big)^{1/q} \le M \Big\| \Big(\sum_{i=1}^{n} |x_i|^q\Big)^{1/q} \Big\|$$

for every choice of vectors $\{x_i\}_{i=1}^{n}$ in X.

We say that the space X is *q-concave* if the identity operator on X is q-concave.

2. Main result. The following theorem is a generalization of the result from [2] mentioned above.

THEOREM 2.1. *Let X be a p-concave ($p < \infty$) Banach space with an unconditional basis, which contains no isomorphic copy of l_2. Then X is a Lyapunov space.*

Proof. Assume the contrary: X is not Lyapunov, i.e. there is a non-atomic measure μ with values in X such that the closure of its range is not convex. Then by [2, Lemma 3] there exists $(\Omega, \Sigma, \lambda)$ with a nonnegative measure $\lambda : \Sigma \to \mathbb{R}$ such that for all $A \in \Sigma$ we have $0 \le \lambda(A) \le \text{const}\, \|\mu\|(A)$, and a bounded operator $T : L_\infty(\Omega, \Sigma, \lambda) \to X$ such that

- $T : (L_\infty, w^*) \to (X, w)$ is continuous and $T(\mathbb{I}_A) = \mu(A)$;
- there exists $\epsilon > 0$ such that for any mean zero sign $f \in L_\infty$ (i.e. a function f that takes only the values 1, -1 or 0) we have $\|Tf\| \ge \delta\lambda(\operatorname{supp} f)$.

Since X is p-concave, it follows that the operator T is q-concave for all $q > p$. And by the factorization theorem [3, 1.d.12], T can be factorized through L_q, i.e. $T = ST_1$, where $T_1 : L_\infty(\Omega, \lambda) \to L_q(\Omega, \nu)$ is the formal identity map, positive and continuous, and $S : L_q(\Omega, \nu) \to X$ is bounded. Notice also that since X contains no isomorphic copy of l_2, the operator S is l_2-strictly singular.

We have

$$\|\mu(A)\|_X = \|S(\mathbb{I}_A)\|_X \le \|S\| \cdot \|\mathbb{I}_A\|_{L_q(\nu)} = \|S\| \cdot \nu^{1/q}(A).$$

Thus there exists $C > 0$ such that $0 \le \lambda(A) \le C\nu^{1/q}(A)$ for all $A \subset \Omega$. Then by the Radon–Nikodym theorem we have

$$\lambda(A) = \int_\Omega y(t)\mathbb{I}_A \, d\nu,$$

where $y \in L_1(\Omega, \nu)$ is positive a.e.

Choose $\Omega_0 \subset \Omega$ of positive measure ν so that there are a, b such that $0 < a \le y(t) \le b < \infty$ for all $t \in \Omega_0$. Consider the identity operator $\mathrm{Id} : L_q(\Omega_0, \lambda) \to L_q(\Omega_0, \nu)$ and the operator $S^0 := S \circ \mathrm{Id}$. It follows that for any mean zero sign x on Ω_0 (with respect to λ) such that $|x| = \mathbb{I}_{\Omega_0}$, we have

$$\|S^0(x)\| = \|Sx\| = \|Tx\| > \delta\lambda(\Omega_0).$$

So $S^0 : L_q(\Omega_0, \lambda) \to X$ is l_2-strictly singular but not narrow.

Following the construction in [5, Proposition 3.1], for each $\epsilon > 0$ we can find a tree $\{A_{m,k}\}$ of Ω_0 and an operator $\widetilde{S} : L_q(\Omega_0, \Sigma_1, \lambda) \to X$, where Σ_1 is the σ-algebra generated by $\{A_{m,k}\}$, with the following properties:

- (P1) $\lambda(A_{m,k}) = 2^{-m}\lambda(\Omega_0)$ for all m, k;
- (P2) $\|\widetilde{S}x\| \ge \frac{1}{2}\delta\lambda(\Omega_0)$ for each mean zero sign $x \in L_q(\Omega_0, \Sigma_1, \lambda)$;
- (P3) $\widetilde{S}(h_1') = 0$ and $\widetilde{S}h_n' = (P_{s_n} - P_{s_{n-1}})S^0 h_n'$, where $0 = s_1 < s_2 < \cdots$, the P_n are the basis projections in X, and $\{h_{2^m+k}'\}$ is the Haar system with respect to the tree $\{A_{m,k}\}$, normalized in $L_q(\Omega_0, \Sigma_1, \lambda)$;
- (P4) for all $x \in L_q(\lambda)$ with $\|x\| = 1$, we have $\|\widetilde{S}x\| \le \|S^0 x\| + \epsilon$;
- (P5) for each $x \in L_q(\lambda)$ with $\|x\| = 1$ of the form $x = \sum_{n=L}^N \beta_n h_n'$ we have $\|\widetilde{S}x\| \le \|S^0 x\| + \epsilon_L$ for some sequence $\epsilon_L \to 0$ as $L \to \infty$.

Note that if we consider a map $J : \Sigma_1(\Omega_0) \to \mathfrak{B}(0,1)$ such that

$$J(A_{n,k}) = \Delta_{n,k} = [(k-1)/2^n, k/2^n], \qquad m(J(A_{n,k})) = \lambda^{-1}(\Omega_0)\lambda(A_{n,k}),$$

where $\mathfrak{B}(0,1)$ is the Borel algebra and m is Lebesgue measure, then $L_q(\Omega_0, \Sigma_1, \lambda)$ is isometric to $L_q = L_q((0,1), \mathfrak{B}, m)$. Thus we have an operator $\widetilde{S} : L_q \to X$ of a special structure, not narrow and l_2-strictly singular. This contradicts [5, Theorem B]. ■

It remains an open question whether any Banach space which does not contain isomorphic copies of l_2 is Lyapunov.

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