STUDIA MATHEMATICA 224 (3) (2014)

Heat kernel estimates for critical fractional diffusion operators

by

LONGJIE XIE and XICHENG ZHANG (Wuhan)

Abstract. We construct the heat kernel of the 1/2-order Laplacian perturbed by a first-order gradient term in Hölder spaces and a zero-order potential term in a generalized Kato class, and obtain sharp two-sided estimates as well as a gradient estimate of the heat kernel, where the proof of the lower bound is based on a probabilistic approach.

1. Introduction and main result. For $\alpha \in (0,2)$, let $\Delta^{\alpha/2}$ be the fractional Laplacian in $\mathbb{R}^d$ defined by

$$\Delta^{\alpha/2} f(x) = \lim_{\varepsilon \downarrow 0} \int_{|y| \geq \varepsilon} \frac{f(x+y) - f(x)}{|y|^{d+\alpha}}\, dy.$$

It is well-known that the heat kernel $\rho^{(\alpha)}(t,x)$ of $\Delta^{\alpha/2}$ has the following estimate (e.g. see [10, 8]):

$$\rho^{(\alpha)}(t,x) \asymp \frac{t}{(|x| \vee t^{1/\alpha})^{d+\alpha}}, \tag{1.1}$$

where $\asymp$ means that both sides are comparable up to some positive constants.

In [3], Bogdan and Jakubowski studied the following perturbation of $\Delta^{\alpha/2}$ by a gradient operator:

$$\mathscr{L}_b^{(\alpha)}(x) := \Delta^{\alpha/2} + b(x) \cdot \nabla, \qquad \alpha \in (1,2),$$

where b belongs to Kato's class $\mathscr{K}_d^{\alpha-1}$ defined as follows: for $\gamma > 0$,

$$\mathscr{K}_d^{\gamma} := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) : \lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq \varepsilon} \frac{|f(y)|}{|x-y|^{d-\gamma}}\, dy = 0 \Big\}.$$

Notice that by Hölder's inequality, $L^p(\mathbb{R}^d) \subset \mathscr{K}_d^{\gamma}$ provided $p > d/\gamma$. Sharp two-sided heat kernel estimates for $\mathscr{L}_b^{(\alpha)}$ like the one in (1.1) were ob-

2010 *Mathematics Subject Classification*: Primary 60J35; Secondary 47G20.
Key words and phrases: heat kernel estimate, gradient estimate, critical diffusion operator, Levi's method.

DOI: 10.4064/sm224-3-3

tained in [3]. The reason for limiting α to $(1,2)$ is that the heat kernel $p_1^{(\alpha)}(t,x) = \rho^{(\alpha)}(t,x+t)$ of $\mathscr{L}_1^{(\alpha)}$ is not comparable with $\rho^{(\alpha)}(t,x)$ for $\alpha \in (0,1)$ (see [3]). In [17], Jakubowski and Szczypkowski considered a time-dependent perturbation of $\Delta^{\alpha/2}$. In [15], Jakubowski established a global in time estimate of the heat kernel of $\Delta^{\alpha/2}$ with small singular drifts. In [6], Chen, Kim and Song obtained sharp two-sided estimates for the Dirichlet heat kernel of $\mathscr{L}_b^{(\alpha)}$. Moreover, the Dirichlet heat kernel estimates for nonlocal operators under Feynman–Kac or Schrödinger type perturbations were also considered in [7]. Recently, in [26], Wang and the second named author extended Bogdan and Jakubowski's results to the more general subordinated stable operator on a Riemannian manifold and obtained sharp two-sided estimates as well as a gradient estimate.

However, in the critical case of $\alpha = 1$, the heat kernel estimate for $\mathscr{L}_b^{(1)}$ is an open problem. The critical case is of particular interest in physics and mathematics (see [5, 19, 18, 23, 24] and references therein). We first recall some related results. In [21], Maekawa and Miura obtained upper bounds for the fundamental solutions of general nonlocal diffusions with divergence free drifts. Their proofs are based upon the classical Davies method. In [23] and [24], Silvestre established the Hölder regularity of the critical parabolic operator $\mathscr{L}_b^{(1)}(x)$ with bounded measurable b. In [22], Priola proved the pathwise uniqueness of SDEs with Hölder drifts and driven by Cauchy processes. In [29], the well-posedness of a multidimensional critical Burgers equation was obtained (see [18] for the study of one-dimensional critical Burgers equations).

In this paper we consider the following critical fractional diffusion operator:

$$\mathscr{L}_{t,x} := \mathscr{L}_{t,x}^{a,b,c} := a(t,x)\Delta^{1/2} + b(t,x)\cdot\nabla + c(t,x),$$

where $a,c : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ and $b : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ are measurable functions. We shall prove the following result.

THEOREM 1.1. *Assume that for some $a_0, a_1 > 0$,*

$$a_0 \le a(t,x) \le a_1,$$

and for some $\beta \in (0,1)$,

$$a, b \in \mathbb{H}^\beta, \quad c \in \mathbb{K}_d^1,$$

where $\mathbb{H}^\beta$ (resp. $\mathbb{K}_d^1$) is the Hölder space (resp. the generalized Kato class) defined in Definition 2.2. Then there exists a continuous function $p(t,x;s,y)$ such that:

(i) (C-K equation) *For all $0 \le t < r < s$ and $x, y \in \mathbb{R}^d$, the following Chapman–Kolmogorov equation holds:*

$$\int_{\mathbb{R}^d} p(t,x;r,z)p(r,z;s,y)\,dz = p(t,x;s,y). \tag{1.2}$$

(ii) (Generator) *For any bounded uniformly continuous function f, we have*

$$\lim_{t\uparrow s}\|P_{t,s}f - f\|_\infty = 0, \qquad \lim_{s\downarrow t}\|P_{t,s}f - f\|_\infty = 0, \tag{1.3}$$

where $P_{t,s}f(x) := \int_{\mathbb{R}^d} p(t,x;s,y)f(y)\,dy$. Moreover, if $a,b,c \in C([0,\infty);L^1_{\rm loc}(\mathbb{R}^d))$, then for all $f,g \in C^2_c(\mathbb{R}^d)$,

$$\lim_{t\uparrow s}\frac{1}{s-t}\int_{\mathbb{R}^d} g(x)(P_{t,s}f(x) - f(x))\,dx = \int_{\mathbb{R}^d} g(x)\mathscr{L}_{s,x}f(x)\,dx, \tag{1.4}$$

$$\lim_{s\downarrow t}\frac{1}{s-t}\int_{\mathbb{R}^d} g(x)(P_{t,s}f(x) - f(x))\,dx = \int_{\mathbb{R}^d} g(x)\mathscr{L}_{t,x}f(x)\,dx. \tag{1.5}$$

(iii) (Two-sided estimates) *For any $T > 0$, there exist constants $\kappa_1, \kappa_2 > 0$ such that for all $0 \le t < s \le T$ and $x,y \in \mathbb{R}^d$,*

$$p(t,x;s,y) \le \kappa_1(s-t)(|x-y| + (s-t))^{-d-1}, \tag{1.6}$$

$$p(t,x;s,y) \ge \kappa_2(s-t)(|x-y| + (s-t))^{-d-1}. \tag{1.7}$$

(iv) (Hölder estimate) *Assume that $c \in \mathbb{K}^{1-\gamma}_d$ for some $\gamma \in (0,1)$. Then for any $T > 0$, there exists a constant $\kappa_3 > 0$ such that for all $0 \le t < s \le T$ and $x,x',y \in \mathbb{R}^d$,*

$$\begin{aligned}|p(t,x;s,y) - p(t,x';s,y)| &\le \kappa_3(|x-x'|^\gamma \wedge 1)|s-t|^{1-\gamma}\\ &\quad\times\big\{(|x-y| + (s-t))^{-d-1} + (|x'-y| + (s-t))^{-d-1}\big\}.\end{aligned} \tag{1.8}$$

(v) (Gradient estimate) *If we further assume that $c \in \mathbb{H}^\gamma$ for some $\gamma \in (0,1)$, then for any $T > 0$, there exists a constant $\kappa_4 > 0$ such that for all $0 \le t < s \le T$ and $x,y \in \mathbb{R}^d$,*

$$|\nabla_x p(t,x;s,y)| \le \kappa_4(|x-y| + (s-t))^{-d-1}. \tag{1.9}$$

In order to prove this theorem, we shall use Levi's parametrix method and Duhamel's formula. Compared with the classical case of second-order parabolic equations, the main difficulties are the heavy tail property of Poisson's kernel and the nonlocal property of $\Delta^{1/2}$. We mention that in the case of second-order parabolic equations, the following property of the Gaussian heat kernel plays a key role in Levi's argument (cf. [14, 20]): for any $\beta \in (0,1)$, there is a $C = C(\beta) > 0$ such that

$$t^{-1}|x|^\beta e^{-|x|^2/t} \le t^{\beta/2-1}e^{-|x|^2/(Ct)}, \qquad t > 0,\ x \in \mathbb{R}^d.$$

This means that spatial Hölder regularity can compensate time singularity. However, such an estimate does not hold for Poisson's kernel in view of the heavy tail property. A suitable substitution is an analogue of the so called

3P-inequality (see Lemma 2.1 below). On the other hand, to prove the lower bound (1.7), we shall adopt the probabilistic approach used in [11, 12].

This paper is organized as follows: In Section 2, we prepare some lemmas for later use. In Section 3, by using Levi's method of constructing fundamental solutions, we first construct the heat kernel of $\mathscr{L}^{a,b}_{t,x} = \mathscr{L}^{a,b,0}_{t,x}$. In Section 4, we prove the lower estimate for the heat kernel by a probabilistic argument. In Section 5, we prove Theorem 1.1 by using Duhamel's formula.

We conclude this section by introducing the following conventions: The letter C with or without subscripts will denote a positive constant, whose value is not important and may change in different places. We write $f(x) \preceq g(x)$ to mean that there exists a constant $C_0 > 0$ such that $f(x) \le C_0 g(x)$ for all x; and $f(x) \asymp g(x)$ to mean that there exist $C_1, C_2 > 0$ such that $C_1 g(x) \le f(x) \le C_2 g(x)$ for all x.

2. Preliminaries

2.1. Basic estimates. For $\gamma, \beta \in \mathbb{R}$, we introduce the following function on $\mathbb{R}_+ \times \mathbb{R}^d$:

$$\varrho^\beta_\gamma(t,x) := t^\gamma\{|x|^\beta \wedge 1\}(|x|^2+t^2)^{-(d+1)/2} \asymp t^\gamma\{|x|^\beta \wedge 1\}(|x|+t)^{-d-1}. \tag{2.1}$$

By simple calculations, there exists a constant $C_d > 0$ such that for all $\beta \in [0,1/2]$ and $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \varrho^\beta_\gamma(t,x)\,dx \le C_d t^{\gamma+\beta-1}. \tag{2.2}$$

Indeed,

$$\int_{\mathbb{R}^d} \frac{|x|^\beta}{(|x|+t)^{d+1}}\,dx = \int_{\mathbb{R}^d} \frac{|ty|^\beta}{(|ty|+t)^{d+1}} t^d\,dy$$
$$= t^{\beta-1}\int_{\mathbb{R}^d} \frac{|x|^\beta}{(|x|+1)^{d+1}}\,dx = Ct^{\beta-1},$$

which implies (2.2). Notice that the following 3P-inequality holds (cf. [3, Lemma 2.1]):

$$\varrho^0_1(t,x)\varrho^0_1(s,y) \preceq (\varrho^0_1(t,x)+\varrho^0_1(s,y))\varrho^0_1(t+s,x+y). \tag{2.3}$$

For $t < s$ and $x, y \in \mathbb{R}^d$, set

$$\varrho^\beta_\gamma(t,x;s,y) := \varrho^\beta_\gamma(s-t,y-x).$$

Let $\mathcal{B}(\gamma,\beta)$ be the usual Beta function defined by

$$\mathcal{B}(\gamma,\beta) := \int_0^1 (1-s)^{\gamma-1}s^{\beta-1}\,ds, \qquad \gamma,\beta > 0.$$

The following lemma is an analogue of the 3P-inequality, which will play a crucial role in what follows.

LEMMA 2.1. *Let $\beta_1,\beta_2\in[0,1/4]$ and $\gamma_1,\gamma_2\in\mathbb{R}$. There exists a constant $C_d>0$ only depending on d such that for all $0\le t<r<s<\infty$ and $x,y\in\mathbb{R}^d$,*

$$(2.4)\qquad \int_{\mathbb{R}^d}\varrho^{\beta_1}_{\gamma_1}(t,x;r,z)\varrho^{\beta_2}_{\gamma_2}(r,z;s,y)\,dz \le C_d\big\{(r-t)^{\gamma_1+\beta_1+\beta_2-1}(s-r)^{\gamma_2}\varrho^0_0(t,x;s,y) + (r-t)^{\gamma_1+\beta_1-1}(s-r)^{\gamma_2}\varrho^{\beta_2}_0(t,x;s,y) + (r-t)^{\gamma_1}(s-r)^{\gamma_2+\beta_1+\beta_2-1}\varrho^0_0(t,x;s,y) + (r-t)^{\gamma_1}(s-r)^{\gamma_2+\beta_2-1}\varrho^{\beta_1}_0(t,x;s,y)\big\},$$

and if $\gamma_1>-\beta_1$ and $\gamma_2>-\beta_2$, then

$$(2.5)\qquad \int_t^s\int_{\mathbb{R}^d}\varrho^{\beta_1}_{\gamma_1}(t,x;r,z)\varrho^{\beta_2}_{\gamma_2}(r,z;s,y)\,dz\,dr \le C_d\big\{\varrho^0_{\gamma_1+\gamma_2+\beta_1+\beta_2}(t,x;s,y)\mathcal{B}(\gamma_1+\beta_1+\beta_2,1+\gamma_2) + \varrho^{\beta_2}_{\gamma_1+\gamma_2+\beta_1}(t,x;s,y)\mathcal{B}(\gamma_1+\beta_1,1+\gamma_2) + \varrho^0_{\gamma_1+\gamma_2+\beta_1+\beta_2}(t,x;s,y)\mathcal{B}(\gamma_2+\beta_1+\beta_2,1+\gamma_1) + \varrho^{\beta_1}_{\gamma_1+\gamma_2+\beta_2}(t,x;s,y)\mathcal{B}(\gamma_2+\beta_2,1+\gamma_1)\big\}.$$

Moreover, there exist $p>1$ and a constant $C>0$ such that for all $0\le t<s<\infty$ and $x\ne y\in\mathbb{R}^d$,

$$(2.6)\qquad \int_t^s\Big(\int_{\mathbb{R}^d}\varrho^{\beta_1}_{\gamma_1}(t,x;r,z)\varrho^{\beta_2}_{\gamma_2}(r,z;s,y)\,dz\Big)^p\,dr\le\frac{C}{|x-y|^{(d+1)p}}.$$

Proof. First of all, in view of

$$(|x-y|^2+|s-t|^2)^{(d+1)/2} \le 2^d\big\{(|x-z|^2+|r-t|^2)^{(d+1)/2}+(|z-y|^2+|s-r|^2)^{(d+1)/2}\big\},$$

we have

$$(2.7)\qquad \varrho^0_0(t,x;r,z)\varrho^0_0(r,z;s,y)\le 2^d(\varrho^0_0(t,x;r,z)+\varrho^0_0(r,z;s,y))\varrho^0_0(t,x;s,y).$$

Noticing that $(a+b)^\beta\le a^\beta+b^\beta$ for $\beta\in(0,1)$ implies

$$(|x-z|^{\beta_1}\wedge1)(|z-y|^{\beta_2}\wedge1)\le(|x-z|^{\beta_1}\wedge1)((|x-z|^{\beta_2}+|x-y|^{\beta_2})\wedge1) \le |x-z|^{\beta_1+\beta_2}\wedge1+(|x-z|^{\beta_1}\wedge1)(|x-y|^{\beta_2}\wedge1),$$

$$(|x-z|^{\beta_1}\wedge1)(|z-y|^{\beta_2}\wedge1)\le((|z-y|^{\beta_1}+|x-y|^{\beta_1})\wedge1)(|z-y|^{\beta_2}\wedge1) \le |z-y|^{\beta_1+\beta_2}\wedge1+(|z-y|^{\beta_2}\wedge1)(|x-y|^{\beta_1}\wedge1),$$

so that we have

$$
\begin{aligned}
&\varrho_{\gamma_1}^{\beta_1}(t,x;r,z)\varrho_{\gamma_2}^{\beta_2}(r,z;s,y)\\
&\quad = |r-t|^{\gamma_1}|s-r|^{\gamma_2}(|x-z|^{\beta_1}\wedge 1)(|z-y|^{\beta_2}\wedge 1)\varrho_0^0(t,x;r,z)\varrho_0^0(r,z;s,y)\\
&\quad \preceq |r-t|^{\gamma_1}|s-r|^{\gamma_2}\big((|x-z|^{\beta_1}\wedge 1)(|x-y|^{\beta_2}\wedge 1)+|x-z|^{\beta_1+\beta_2}\wedge 1\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \varrho_0^0(t,x;r,z)\varrho_0^0(t,x;s,y)\\
&\qquad + |r-t|^{\gamma_1}|s-r|^{\gamma_2}\big((|z-y|^{\beta_2}\wedge 1)(|x-y|^{\beta_1}\wedge 1)+|z-y|^{\beta_1+\beta_2}\wedge 1\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \varrho_0^0(r,z;s,y)\varrho_0^0(t,x;s,y)\\
&\quad \preceq |s-r|^{\gamma_2}\big(\varrho_{\gamma_1}^{\beta_1+\beta_2}(t,x;r,z)\varrho_0^0(t,x;s,y)+\varrho_{\gamma_1}^{\beta_1}(t,x;r,z)\varrho_0^{\beta_2}(t,x;s,y)\big)\\
&\qquad + |r-t|^{\gamma_1}\big(\varrho_{\gamma_2}^{\beta_1+\beta_2}(r,z;s,y)\varrho_0^0(t,x;s,y)+\varrho_{\gamma_2}^{\beta_2}(r,z;s,y)\varrho_0^{\beta_1}(t,x;s,y)\big).
\end{aligned}
$$

Estimate (2.4) follows from (2.2), and estimate (2.5) follows by observing that for $\gamma,\beta>0$,

$$
\int_t^s (r-t)^{\gamma-1}(s-r)^{\beta-1}\,dr = (s-t)^{\gamma+\beta-1}\mathcal{B}(\gamma,\beta). \tag{2.8}
$$

Finally, (2.6) follows from (2.4) and (2.8). ∎

2.2. Hölder space and Kato class. We introduce the following classes of functions.

Definition 2.2. For $\beta\in(0,1]$, define the *Hölder space* by

$$
\mathbb{H}^\beta := \Big\{ f\in\mathscr{B}(\mathbb{R}\times\mathbb{R}^d) : \|f\|_{\mathbb{H}^\beta} := \sup_{t\in\mathbb{R}}\sup_{x\in\mathbb{R}^d}|f(t,x)| + \sup_{t\in\mathbb{R}}\sup_{x\neq y\in\mathbb{R}^d}\frac{|f(t,x)-f(t,y)|}{|x-y|^\beta}<\infty\Big\}.
$$

For $\gamma>0$, define the *generalized Kato class* by

$$
\mathbb{K}_d^\gamma := \Big\{ f\in L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^d) : \lim_{\varepsilon\downarrow 0} K^\gamma(\varepsilon)=0\Big\},
$$

where

$$
K^\gamma(\varepsilon) := \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d}\int_0^\varepsilon\int_{\mathbb{R}^d}\varrho_\gamma^0(s,y)|f(t\pm s,x-y)|\,dy\,ds, \qquad \varepsilon>0.
$$

A function f on $\mathbb{R}_+\times\mathbb{R}^d$ will be automatically extended to $\mathbb{R}\times\mathbb{R}^d$ by letting $f(t,\cdot)=0$ for $t\le 0$. The following proposition gives a characterization for $\mathbb{K}_d^\gamma$ (see [1, 28, 26] for more discussion).

Proposition 2.3. *For $\gamma>0$ and $p,q\in[1,\infty]$ with $d/p+1/q<\gamma$, we have*

$$
L^q(\mathbb{R};L^p(\mathbb{R}^d))\subset\mathbb{K}_d^\gamma,
$$

and for $\gamma\in(0,d)$,

$$
\mathscr{K}_d^\gamma\subset\mathbb{K}_d^\gamma.
$$

Proof. Noticing that

$$\int_0^\varepsilon\int_{\mathbb{R}^d} \varrho^0_\gamma(s,y)|f(t\pm s,x-y)|\,dy\,ds=\int_0^\varepsilon s^{\gamma-1}\int_{\mathbb{R}^d}\varrho^0_1(s,y)|f(t\pm s,x-y)|\,dy\,ds,$$

by Hölder's inequality, for the first inclusion, it is enough to prove

$$\lim_{\varepsilon\downarrow 0} I(\varepsilon):=\lim_{\varepsilon\downarrow 0}\int_0^\varepsilon\Big(\int_{\mathbb{R}^d}\varrho^0_1(s,y)^{p^*}\,dy\Big)^{q^*/p^*}s^{(\gamma-1)q^*}\,ds=0, \tag{2.9}$$

where $q^*:=q/(q-1)$ and $p^*:=p/(p-1)$. As in the proof of (2.2), we have

$$\int_{\mathbb{R}^d}\varrho^0_1(s,y)^{p^*}\,dy\preceq s^{d-dp^*},$$

and since $dq^*/p^*-dq^*+(\gamma-1)q^*>-1$ because $d/p+1/q<\gamma$, we obtain

$$I(\varepsilon)\preceq\int_0^\varepsilon s^{dq^*/p^*-dq^*+(\gamma-1)q^*}\,ds\preceq\varepsilon^{1+dq^*/p^*-dq^*+(\gamma-1)q^*},$$

and thus (2.9) holds.

Next we prove the second inclusion. Assume $f\in\mathscr{K}^\gamma_d$. By definitions,

$$\sup_{x\in\mathbb{R}^d}\int_0^\varepsilon\int_{\mathbb{R}^d}\varrho^0_\gamma(s,y)|f(x-y)|\,dy\,ds\le I_1(\varepsilon)+I_2(\varepsilon),$$

where

$$I_1(\varepsilon):=\sup_{x\in\mathbb{R}^d}\int_0^\varepsilon\int_{|y|\le\varepsilon}\frac{s^\gamma|f(x-y)|}{(|y|+s)^{d+1}}\,dy\,ds,$$

$$I_2(\varepsilon):=\sup_{x\in\mathbb{R}^d}\int_0^\varepsilon\int_{|y|>\varepsilon}\frac{s^\gamma|f(x-y)|}{(|y|+s)^{d+1}}\,dy\,ds.$$

For $I_1(\varepsilon)$, in view of $\gamma<d$, we have

$$\begin{aligned}I_1(\varepsilon)&\le\sup_{x\in\mathbb{R}^d}\int_{|y|\le\varepsilon}|f(x-y)|\Big(\int_{|y|}^\varepsilon s^{\gamma-d-1}\,ds+|y|^{-d-1}\int_0^{|y|}s^\gamma\,ds\Big)\,dy\\&\le\sup_{x\in\mathbb{R}^d}\int_{|y|\le\varepsilon}|f(x-y)|\Big(\frac{|y|^{-d+\gamma}}{d-\gamma}+\frac{|y|^{-d+\gamma}}{\gamma+1}\Big)\,dy\to 0,\qquad\varepsilon\downarrow 0.\end{aligned}$$

For $I_2(\varepsilon)$, we have

$$I_2(\varepsilon)\le\sup_{x\in\mathbb{R}^d}\int_{|y|>\varepsilon}\frac{|f(x-y)|}{|y|^{d+1}}\,dy\int_0^\varepsilon s^\gamma\,ds=\frac{1}{\gamma+1}\sup_{x\in\mathbb{R}^d}\int_{|y|>\varepsilon}\frac{\varepsilon^{\gamma+1}|f(x-y)|}{|y|^{d+1}}\,dy,$$

which converges to zero as $\varepsilon\downarrow 0$ by [3, Lemma 11]. ∎

2.3. Estimates of the freezing kernel. Let $\rho(t,x)$ be the heat kernel of the Cauchy operator $\Delta^{1/2}$, i.e.,

$$\partial_t \rho(t,x) = \Delta^{1/2}\rho(t,x). \tag{2.10}$$

It is well-known that

$$\begin{aligned}\rho(t,x) &= \pi^{-(d+1)/2}\Gamma\left(\frac{d+1}{2}\right)(|x|^2+t^2)^{-(d+1)/2}t \\ &= \pi^{-(d+1)/2}\Gamma\left(\frac{d+1}{2}\right)\varrho_1^0(t,x),\end{aligned}$$

which is also called the Poisson kernel (cf. [25]), where Γ is the usual Gamma function. By elementary calculations, one has

$$|\nabla_x \rho(t,x)| \preceq t(|x|+t)^{-d-2}, \qquad |\partial_t \rho(t,x)| \preceq (|x|+t)^{-d-1}, \tag{2.11}$$

$$|\nabla_x^2 \rho(t,x)| + |\nabla_x \partial_t \rho(t,x)| \preceq (|x|+t)^{-d-2}, \tag{2.12}$$

$$|\nabla_x^3 \rho(t,x)| + |\nabla_x^2 \partial_t \rho(t,x)| \preceq (|x|+t)^{-d-3}, \tag{2.13}$$

where for $k \in \mathbb{N}$, ∇^k denotes the kth-order gradient operator.

Let $a : [0,\infty) \times \mathbb{R}^d \to (0,\infty)$ and $b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be bounded measurable functions. We define

$$p_0(t,x;s,y) := \rho\Big(\int_t^s a(r,y)\,dr, x-y+\int_t^s b(r,y)\,dr\Big),$$

and

$$\mathscr{L}_{t,y}^{a,b} := a(t,y)\Delta_x^{1/2} + b(t,y)\cdot\nabla_x. \tag{2.14}$$

By (2.10) and the Lebesgue differentiation theorem, for all $x,y \in \mathbb{R}^d$ and almost all $t<s$, we have

$$\partial_t p_0(t,x;s,y) + \mathscr{L}_{t,y}^{a,b} p_0(t,\cdot;s,y)(x) = 0. \tag{2.15}$$

We prepare the following important estimates for later use.

Lemma 2.4. *Suppose that for some $a_0, a_1, b_1 > 0$,*

$$a_0 \le a(r,y) \le a_1, \qquad |b(r,y)| \le b_1. \tag{2.16}$$

Then

$$p_0(t,x;s,y) \asymp \varrho_1^0(t,x;s,y), \tag{2.17}$$

and

$$|\Delta_x^{1/2} p_0(t,x;s,y)| \preceq \big(|x-y|+|s-t|\big)^{-d-1}, \tag{2.18}$$

$$|\nabla_x p_0(t,x;s,y)| \preceq |s-t|\big(|x-y|+|s-t|\big)^{-d-2}, \tag{2.19}$$

$$|\partial_t p_0(t,x;s,y)| \preceq \big(|x-y|+|s-t|\big)^{-d-1}, \tag{2.20}$$

$$(2.21)\qquad |\nabla_x \Delta_x^{1/2} p_0(t,x;s,y)| \preceq \big(|x-y|+|s-t|\big)^{-d-2},$$

$$(2.22)\qquad |\nabla_x^2 p_0(t,x;s,y)| \preceq \big(|x-y|+|s-t|\big)^{-d-2}.$$

Moreover, if we further assume that $a,b\in\mathbb{H}^\beta$ for some $\beta\in(0,1)$, then

$$(2.23)\qquad \Big|\int_{\mathbb{R}^d} \nabla_x p_0(t,x;s,y)\,dy\Big| \preceq (s-t)^{\beta-1},$$

$$(2.24)\qquad \Big|\int_{\mathbb{R}^d} \Delta_x^{1/2} p_0(t,x;s,y)\,dy\Big| \preceq (s-t)^{\beta-1},$$

$$(2.25)\qquad \Big|\int_{\mathbb{R}^d} \partial_t p_0(t,x;s,y)\,dy\Big| \preceq (s-t)^{\beta-1},$$

$$(2.26)\qquad \Big(\lim_{s\downarrow t}\Big) \lim_{t\uparrow s} \sup_{x\in\mathbb{R}^d} \Big|\int_{\mathbb{R}^d} p_0(t,x;s,y)\,dy - 1\Big| = 0,$$

and for all $w\in\mathbb{R}^d$ and $\gamma\in[0,\beta]$,

$$(2.27)\qquad \Big|\int_{\mathbb{R}^d} \big(\nabla_x p_0(t,x+w;s,y) - \nabla_x p_0(t,x;s,y)\big)\,dy\Big| \preceq |w|^\gamma (s-t)^{\beta-\gamma-1}.$$

Proof. For simplicity of notation, we write

$$F_t^s(y) := \int_t^s a(r,y)\,dr, \qquad G_t^s(y) := \int_t^s b(r,y)\,dr.$$

(1) By (2.16), we have

$$(2.28)\qquad F_t^s(y) \asymp s-t, \qquad G_t^s(y) \preceq s-t, \qquad y\in\mathbb{R}^d,$$

and for any $|w|\preceq|s-t|$,

$$(2.29)\qquad |x+w-y+G_t^s(y)| + |s-t| \asymp |x-y| + |s-t|.$$

Estimate (2.17) follows by definition. For (2.18), by (2.10) we have

$$\begin{aligned}\Delta_x^{1/2} p_0(t,x;s,y) &= (\Delta_x^{1/2}\rho)\big(F_t^s(y), x-y+G_t^s(y)\big)\\ &= (\partial_t\rho)\big(F_t^s(y), x-y+G_t^s(y)\big).\end{aligned}$$

Estimate (2.18) follows from (2.11). Similarly, (2.19)–(2.22) follow from (2.11), (2.12) and (2.15).

(2) Define

$$\begin{aligned}\xi(t,x;s,y;z) &:= \rho\Big(\int_t^s a(r,z)\,dr, x-y+\int_t^s b(r,z)\,dr\Big)\\ &= \rho\big(F_t^s(z), x-y+G_t^s(z)\big).\end{aligned}$$

Clearly, for any $t < s$ and $x, z \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \xi(t,x;s,y;z)\,dy = \int_{\mathbb{R}^d} \rho(F_t^s(z),y)\,dy = 1$$

and

$$\int_{\mathbb{R}^d} \nabla_x \xi(t,x;s,y;z)\,dy = 0, \qquad \int_{\mathbb{R}^d} \Delta_x^{1/2}\xi(t,x;s,y;z)\,dy = 0.$$

Thus, to prove (2.23), it suffices to show that

$$\Big|\int_{\mathbb{R}^d} \big(\nabla_x p_0(t,x;s,y) - \nabla_x \xi(t,x;s,y;z)\big)\,dy\Big|_{z=x} \preceq (s-t)^{\beta-1}. \tag{2.30}$$

Since $a, b \in \mathbb{H}^\beta$, by the definitions of p_0 and ξ, one has

$$\begin{aligned}
&|\nabla_x p_0(t,x;s,y) - \nabla_x \xi(t,x;s,y;z)|_{z=x}| \qquad (2.31)\\
&= |(\nabla_x\rho)\big(F_t^s(y), x-y+G_t^s(y)\big) - (\nabla_x\rho)\big(F_t^s(x), x-y+G_t^s(x)\big)|\\
&\preceq \|a\|_{\mathbb{H}^\beta}(|x-y|^\beta \wedge 1)|s-t|\\
&\qquad\times \int_0^1 |\nabla_x\partial_t\rho|\big(\theta F_t^s(y) + (1-\theta)F_t^s(x), x-y+G_t^s(y)\big)\,d\theta\\
&\quad + \|b\|_{\mathbb{H}^\beta}(|x-y|^\beta \wedge 1)|s-t|\\
&\qquad\times \int_0^1 |\nabla_x^2\rho|\big(F_t^s(x), x-y+\theta G_t^s(y) + (1-\theta)G_t^s(x)\big)\,d\theta\\
&\overset{(2.12),(2.28),(2.29)}{\preceq} \frac{(|x-y|^\beta \wedge 1)|s-t|}{(|x-y|+|s-t|)^{d+2}} \le \frac{|x-y|^\beta \wedge 1}{(|x-y|+|s-t|)^{d+1}},
\end{aligned}$$

which gives (2.30) by (2.2).

Similarly, we can prove

$$\Big|\int_{\mathbb{R}^d} \big(\Delta_x^{1/2} p_0(t,x;s,y) - \Delta_x^{1/2}\xi(t,x;s,y;z)\big)\,dy\Big|_{z=x} \preceq (s-t)^{\beta-1},$$

$$\Big|\int_{\mathbb{R}^d} \big(p_0(t,x;s,y) - \xi(t,x;s,y;z)\big)\,dy\Big|_{z=x} \preceq (s-t)^{\beta}.$$

Thus, (2.24) and (2.26) follow.

(3) Next, we prove (2.25). By (2.15), (2.18), (2.19), (2.23) and (2.24), we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} \partial_t p_0(t,x;s,y)\,dy\Big|\\
&\qquad = \Big|\int_{\mathbb{R}^d} \big(a(t,y)\Delta_x^{1/2} p_0(t,x;s,y) + b(t,y)\cdot\nabla_x p_0(t,x;s,y)\big)dy\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le |a(t,x)|\Big|\int_{\mathbb{R}^d} \Delta_x^{1/2} p_0(t,x;s,y)\,dy\Big| + |b(t,x)|\Big|\int_{\mathbb{R}^d} \nabla_x p_0(t,x;s,y)\,dy\Big| \\
&\quad + \int_{\mathbb{R}^d} |a(t,y)-a(t,x)|\cdot|\Delta_x^{1/2} p_0(t,x;s,y)|\,dy \\
&\quad + \int_{\mathbb{R}^d} |b(t,y)-b(t,x)|\cdot|\nabla_x p_0(t,x;s,y)|\,dy \\
&\preceq (s-t)^{\beta-1} + \int_{\mathbb{R}^d} \varrho_0^\beta(t,x;s,y)\,dy \overset{(2.2)}{\preceq} (s-t)^{\beta-1}.
\end{aligned}
$$

(4) Lastly, we prove (2.27). If $|w| \le |s-t|$, then

$$
\begin{aligned}
&|(\nabla_x p_0(t,x+w;s,y) - \nabla_x \xi(t,x+w;s,y;z)|_{z=x}) \\
&\qquad\qquad - (\nabla_x p_0(t,x;s,y) - \nabla_x \xi(t,x;s,y;z)|_{z=x})| \\
&\quad = \Big| w\cdot \int_0^1 \Big[(\nabla_x^2 \rho)\big(F_t^s(y), x+\theta w - y + G_t^s(y)\big) \\
&\qquad\qquad - (\nabla_x^2 \rho)\big(F_t^s(x), x+\theta w - y + F_t^s(x)\big)\Big]\,d\theta\Big| \\
&\quad \preceq |w| \frac{(s-t)(|x-y|^\beta \wedge 1)}{(|x-y|+(s-t))^{d+3}} \preceq |w|^\gamma (s-t)^{\beta-\gamma} \varrho_0^0(t,x;s;y),
\end{aligned}
$$

where we have used the same argument as in proving (2.31). Integrating both sides with respect to y and using (2.2), we obtain (2.27) for $|w| \le |s-t|$. If $|w| > |s-t|$, (2.27) follows from (2.23). ∎

3. Heat kernel of $\mathscr{L}_{t,x}^{a,b} := a(t,x)\Delta^{1/2} + b(t,x)\cdot\nabla$.

We look for the heat kernel of $\mathscr{L}_{t,x}^{a,b}$ in the following form:

$$
p_{a,b}(t,x;s,y) = p_0(t,x;s,y) + \int_t^s \int_{\mathbb{R}^d} p_0(t,x;r,z) q(r,z;s,y)\,dz\,dr. \tag{3.1}
$$

Levi's classical argument (see [20, 14]) suggests that $q(t,x;s,y)$ must satisfy the following integro-differential equation:

$$
q(t,x;s,y) = q_0(t,x;s,y) + \int_t^s \int_{\mathbb{R}^d} q_0(t,x;r,z) q(r,z;s,y)\,dz\,dr, \tag{3.2}
$$

where

$$
\begin{aligned}
q_0(t,x;s,y) &:= (a(t,x)-a(t,y))\Delta_x^{1/2} p_0(t,x;s,y) \\
&\quad + (b(t,x)-b(t,y))\cdot\nabla_x p_0(t,x;s,y).
\end{aligned} \tag{3.3}
$$

For $r \in (t,s)$, set

$$\phi_{s,y}(t,x,r) := \int_{\mathbb{R}^d} p_0(t,x;r,z)q(r,z;s,y)\,dz,$$

and

$$\varphi_{s,y}(t,x) := \int_t^s \phi_{s,y}(t,x,r)\,dr = \int_t^s\int_{\mathbb{R}^d} p_0(t,x;r,z)q(r,z;s,y)\,dz\,dr.$$

In this section, we shall work on the time interval $[0,1]$, and always assume

$$0 \le t < s \le 1, \qquad x \ne y \in \mathbb{R}^d,$$

and for some $\beta \in (0,1)$,

$$a, b \in \mathbb{H}^\beta. \tag{3.4}$$

3.1. Solving the integro-differential equation (3.2). Our first task is thus to solve the integral-differential equation (3.2). Let us recursively define

$$q_n(t,x;s,y) := \int_t^s\int_{\mathbb{R}^d} q_0(t,x;r,z)q_{n-1}(r,z;s,y)\,dz\,dr, \qquad n \in \mathbb{N}. \tag{3.5}$$

LEMMA 3.1. *For $\beta \in (0,1/4]$, there exists a constant $C_d > 0$ such that for all $n \in \mathbb{N}$,*

$$|q_n(t,x;s,y)| \le \frac{(C_d\Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)}\big(\varrho^0_{(n+1)\beta}(t,x;s,y) + \varrho^\beta_{n\beta}(t,x;s,y)\big). \tag{3.6}$$

Proof. First of all, by (3.4) and Lemma 2.4, we have

$$|q_0(t,x;s,y)| \le C_d\varrho^\beta_0(t,x;s,y).$$

Notice that $\mathcal{B}(\gamma,\beta)$ is symmetric, and nonincreasing with respect to each of γ and β.

For $n=1$, by Lemma 2.1, we have

$$|q_1| \le C_d\mathcal{B}(2\beta,1)\varrho^0_{2\beta} + C_d\mathcal{B}(\beta,1)\varrho^\beta_\beta \le C_d\mathcal{B}(\beta,\beta)\{\varrho^0_{2\beta} + \varrho^\beta_\beta\}.$$

Suppose now that

$$|q_n| \le \gamma_n\{\varrho^0_{(n+1)\beta} + \varrho^\beta_{n\beta}\},$$

where $\gamma_n > 0$ will be determined below. By Lemma 2.1 we have

$$\begin{aligned}
|q_{n+1}| &\le C_d\gamma_n\big\{\mathcal{B}(\beta,1+(n+1)\beta) + \mathcal{B}((n+2)\beta,1) + \mathcal{B}(2\beta,1+n\beta)\big\}\varrho^0_{(n+2)\beta}\\
&\qquad + C_d\gamma_n\big\{\mathcal{B}((n+1)\beta,1) + \mathcal{B}(\beta,1+n\beta)\big\}\varrho^\beta_{(n+1)\beta}\\
&\le C_d\gamma_n\mathcal{B}(\beta,(n+1)\beta)\big\{\varrho^0_{(n+2)\beta} + \varrho^\beta_{(n+1)\beta}\big\}\\
&=: \gamma_{n+1}\big\{\varrho^0_{(n+2)\beta} + \varrho^\beta_{(n+1)\beta}\big\},
\end{aligned}$$

where

$$\gamma_{n+1} = C_d \gamma_n \mathcal{B}(\beta, (n+1)\beta).$$

Hence, from $\mathcal{B}(\gamma,\beta) = \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}$, we obtain

$$\gamma_n = C_d^{n+1}\mathcal{B}(\beta,\beta)\mathcal{B}(\beta,2\beta)\cdots\mathcal{B}(\beta,n\beta) = \frac{(C_d\Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)},$$

which gives (3.6). ∎

We also need the following Hölder continuity of q_n with respect to x.

Lemma 3.2. *For all $n \ge 0$, $\beta \in (0,1/4]$ and $\gamma \in (0,\beta)$, we have*

$$|q_n(t,x;s,y) - q_n(t,x';s,y)| \preceq \frac{(C_d\Gamma(\beta))^{n+1}}{\Gamma(n\beta+\gamma)}(|x-x'|^{\beta-\gamma}\wedge 1) \times \big\{(\varrho^0_{\gamma+n\beta} + \varrho^\beta_{\gamma+(n-1)\beta})(t,x;s,y) + (\varrho^0_{\gamma+n\beta} + \varrho^\beta_{\gamma+(n-1)\beta})(t,x';s,y)\big\}.$$

Proof. Let us first prove

$$|q_0(t,x;s,y) - q_0(t,x';s,y)| \preceq (|x-x'|^{\beta-\gamma}\wedge 1)\big\{(\varrho^0_\gamma + \varrho^\beta_{\gamma-\beta})(t,x;s,y) + (\varrho^0_\gamma + \varrho^\beta_{\gamma-\beta})(t,x';s,y)\big\}. \tag{3.7}$$

In the case of $|x-x'| > 1$, we have

$$|q_0(t,x;s,y)| \preceq (\varrho^0_\beta + \varrho^\beta_0)(t,x;s,y) \le (\varrho^0_\gamma + \varrho^\beta_{\gamma-\beta})(t,x;s,y)$$

and

$$|q_0(t,x';s,y)| \preceq (\varrho^0_\beta + \varrho^\beta_0)(t,x';s,y) \le (\varrho^0_\gamma + \varrho^\beta_{\gamma-\beta})(t,x';s,y).$$

In the case of $1 \ge |x-x'| > |s-t|$, by (2.18) and (2.19) we have

$$|q_0(t,x;s,y)| \preceq \varrho^\beta_0(t,x;s,y) = (s-t)^{\beta-\gamma}\varrho^\beta_{\gamma-\beta}(t,x;s,y) \preceq |x-x'|^{\beta-\gamma}\varrho^\beta_{\gamma-\beta}(t,x;s,y),$$

and also

$$|q_0(t,x';s,y)| \preceq |x-x'|^{\beta-\gamma}\varrho^\beta_{\gamma-\beta}(t,x';s,y).$$

Suppose now that

$$|x-x'| \le |s-t|. \tag{3.8}$$

We can write

$$\begin{aligned}
&|q_0(t,x;s,y) - q_0(t,x';s,y)| \\
&\quad\le |a(t,x)-a(t,y)|\cdot|\Delta^{1/2}_x p_0(t,x;s,y) - \Delta^{1/2}_{x'} p_0(t,x';s,y)| \\
&\qquad + |a(t,x)-a(t,x')|\cdot|\Delta^{1/2}_{x'} p_0(t,x';s,y)| \\
&\qquad + |b(t,x)-b(t,y)|\cdot|\nabla_x p_0(t,x;s,y) - \nabla_{x'} p_0(t,x';s,y)| \\
&\qquad + |b(t,x)-b(t,x')|\cdot|\nabla_{x'} p_0(t,x';s,y)| \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For I_1, by (2.21) and the mean value theorem, for some $\theta \in [0,1]$ we have

$$I_1 \preceq \{|x-y|^\beta \wedge 1\}|x-x'|(|x+\theta(x'-x)-y|+|s-t|)^{-d-2}.$$

By (3.8), we have

$$|x-y|+|s-t| \le |x+\theta(x'-x)-y|+2|s-t|.$$

Hence,

$$\begin{aligned} I_1 &\preceq \{|x-y|^\beta \wedge 1\}|x-x'|(|x-y|+|s-t|)^{-d-2} \\ &\preceq |x-x'|^{\beta-\gamma}\frac{|s-t|^{1+\gamma-\beta}\{|x-y|^\beta \wedge 1\}}{|x-y|+|s-t|}(|x-y|+|s-t|)^{-d-1} \\ &\preceq |x-x'|^{\beta-\gamma}|s-t|^\gamma(|x-y|+|s-t|)^{-d-1} = |x-x'|^{\beta-\gamma}\varrho^0_\gamma(t,x;s,y). \end{aligned}$$

By (2.19),

$$I_2 \preceq |x-x'|^\beta(|x'-y|+|s-t|)^{-d-1} \preceq |x-x'|^{\beta-\gamma}\varrho^0_\gamma(t,x';s,y).$$

Similarly,

$$I_3 \preceq |x-x'|^{\beta-\gamma}\varrho^0_\gamma(t,x;s,y), \qquad I_4 \preceq |x-x'|^{\beta-\gamma}\varrho^0_\gamma(t,x';s,y).$$

Combining the above calculations, we obtain (3.7).

Now, by (3.5), (3.7) and Lemma 3.1, for $n \in \mathbb{N}$ we have

$$\begin{aligned} &|q_n(t,x;s,y)-q_n(t,x';s,y)| \\ &\quad\preceq \int_t^s\int_{\mathbb{R}^d} |q_0(t,x;r,z)-q_0(t,x';r,z)|q_{n-1}(r,z;s,y)\,dz\,dr \\ &\quad\preceq \frac{(C_d\Gamma(\beta))^n}{\Gamma(n\beta)}(|x-x'|^{\beta-\gamma}\wedge 1) \\ &\qquad\qquad \times \int_t^s\int_{\mathbb{R}^d} \{(\varrho^0_\gamma+\varrho^\beta_{\gamma-\beta})(t,x;r,z)+(\varrho^0_\gamma+\varrho^\beta_{\gamma-\beta})(t,x';r,z)\} \\ &\qquad\qquad\qquad \times \big\{\varrho^0_{n\beta}(r,z;s,y)+\varrho^\beta_{(n-1)\beta}(r,z;s,y)\big\}\,dz\,dr, \end{aligned}$$

which yields the result by Lemma 2.1. ■

Basing on the above two lemmas, we obtain

THEOREM 3.3. *The function $q(t,x;s,y) := \sum_{n=0}^\infty q_n(t,x;s,y)$ solves the integro-differential equation* (3.2). *Moreover, for $\beta \in (0,1/4]$,*

$$|q(t,x;s,y)| \preceq \varrho^\beta_0(t,x;s,y)+\varrho^0_\beta(t,x;s,y), \tag{3.9}$$

and for any $\gamma \in (0,\beta)$,

$$\begin{aligned} &|q(t,x;s,y)-q(t,x';s,y)| \\ &\quad\preceq (|x-x'|^{\beta-\gamma}\wedge 1)\big\{(\varrho^0_\gamma+\varrho^\beta_{\gamma-\beta})(t,x;s,y)+(\varrho^0_\gamma+\varrho^\beta_{\gamma-\beta})(t,x';s,y)\big\}. \end{aligned} \tag{3.10}$$

Proof. By Lemma 3.1, one sees that

$$\sum_{n=0}^{\infty}|q_n(t,x;s,y)| \le \sum_{n=0}^{\infty}\frac{(C_d\Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)}\big(\varrho^0_{(n+1)\beta}(t,x;s,y)+\varrho^\beta_{n\beta}(t,x;s,y)\big)$$
$$\le \Big\{\sum_{n=0}^{\infty}\frac{(C_d\Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)}\Big\}\big(\varrho^0_{\beta}(t,x;s,y)+\varrho^\beta_{0}(t,x;s,y)\big).$$

Since the series is convergent, we obtain (3.9). Similarly, estimate (3.10) follows from Lemma 3.2. Moreover, by (3.5) we have

$$\sum_{n=0}^{m+1} q_n(t,x;s,y) = q_0(t,x;s,y) + \int_t^s\int_{\mathbb{R}^d} q_0(t,x;r,z)\sum_{n=0}^{m} q_n(r,z;s,y)\,dz\,dr,$$

which yields (3.2) by letting $m\to\infty$ on both sides. ∎

3.2. Smoothness of $\varphi_{s,y}(t,x)$. Below we study the smoothness of the function $(t,x)\mapsto\varphi_{s,y}(t,x)$. Notice that by (2.17), (3.9) and (2.4),

$$|\phi_{s,y}(t,x,r)| \le \int_{\mathbb{R}^d} p_0(t,x;r,z)|q(r,z;s,y)|\,dz \tag{3.11}$$
$$\preceq \int_{\mathbb{R}^d}\varrho^0_1(t,x;r,z)(\varrho^0_\beta+\varrho^\beta_0)(r,z;s,y)\,dz$$
$$\preceq \big((r-t)^\beta+(s-r)^\beta+(r-t)(s-r)^{\beta-1}\big)\varrho^0_0(t,x;s,y)+\varrho^\beta_0(t,x;s,y).$$

Lemma 3.4. *For all $x\ne y\in\mathbb{R}^d$ and almost all $t<s$, we have*

$$\partial_t\varphi_{s,y}(t,x) = -q(t,x;s,y) - \int_t^s\int_{\mathbb{R}^d}\mathscr{L}^{a,b}_{t,z}p_0(t,\cdot;r,z)(x)q(r,z;s,y)\,dz\,dr. \tag{3.12}$$

Proof.

Claim 1. *For $r\in(t,s)$, we have*

$$\partial_t\phi_{s,y}(t,x,r) = \int_{\mathbb{R}^d}\partial_t p_0(t,x;r,z)q(r,z;s,y)\,dz. \tag{3.13}$$

Proof of Claim 1. Write

$$\frac{\phi_{s,y}(t+\varepsilon,x,r)-\phi_{s,y}(t,x,r)}{\varepsilon}$$
$$=\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\big(p_0(t+\varepsilon,x;r,z)-p_0(t,x;r,z)\big)q(r,z;s,y)\,dz$$
$$=\int_{\mathbb{R}^d}\Big(\int_0^1\partial_t p_0(t+\theta\varepsilon,x;r,z)\,d\theta\Big)q(r,z;s,y)\,dz.$$

By (2.18) and (2.19), for $|\varepsilon| < (r-t)/2$ we have

$$|\partial_t p_0(t+\theta\varepsilon, x; r, z)| \preceq (|x-z| + t + \theta\varepsilon - r)^{-d-1} \preceq (|x-z| + (r-t))^{-d-1},$$

which together with (3.9) yields

$$|\partial_t p_0(t+\theta\varepsilon, x; r, z) q(r, z; s, y)| \preceq \varrho_0^0(t, x; r, z)(\varrho_\beta^0 + \varrho_0^\beta)(r, z; s, y) =: g(z).$$

By (2.4), one sees that

$$\int_{\mathbb{R}^d} g(z)\, dz < \infty.$$

Hence, by the dominated convergence theorem,

$$\lim_{\varepsilon \to 0} \frac{\phi_{s,y}(t+\varepsilon, x, r) - \phi_{s,y}(t, x, r)}{\varepsilon} = \int_{\mathbb{R}^d} \partial_t p_0(t, x; r, z) q(r, z; s, y)\, dz,$$

and (3.13) is proven.

Claim 2. *For $x \neq y$, we have*

$$\int_t^s \int_{r'}^s |\partial_{r'} \phi_{s,y}(r', x, r)|\, dr\, dr' < \infty. \tag{3.14}$$

Proof of Claim 2. By (3.13),

$$\begin{aligned} |\partial_{r'} \phi_{s,y}(r', x, r)| &\leq \int_{\mathbb{R}^d} |\partial_{r'} p_0(r', x; r, z)| \cdot |q(r, z; s, y) - q(r, x; s, y)|\, dz \\ &\quad + |q(r, x; s, y)| \Big| \int_{\mathbb{R}^d} \partial_{r'} p_0(r', x; r, z)\, dz \Big| \\ &=: Q_{s,y}^{(1)}(r', x, r) + Q_{s,y}^{(2)}(r', x, r). \end{aligned} \tag{3.15}$$

For $Q_{s,y}^{(1)}(r', x, r)$, by (2.20) and (3.10), we have

$$\begin{aligned} &\int_t^s \int_{r'}^s Q_{s,y}^{(1)}(r', x, r)\, dr\, dr' \\ &\preceq \int_t^s \int_{r'}^s \int_{\mathbb{R}^d} \varrho_0^{\beta-\gamma}(r', x; r, z)(\varrho_\gamma^0 + \varrho_{\gamma-\beta}^\beta)(r, x; s, y)\, dz\, dr\, dr' \\ &\quad + \int_t^s \int_{r'}^s \int_{\mathbb{R}^d} \varrho_0^{\beta-\gamma}(r', x; r, z)(\varrho_\gamma^0 + \varrho_{\gamma-\beta}^\beta)(r, z; s, y)\, dz\, dr\, dr' \\ &\preceq \int_t^s \int_{r'}^s (r-r')^{\beta-\gamma-1}(\varrho_\gamma^0 + \varrho_{\gamma-\beta}^\beta)(r, x; s, y)\, dr\, dr' \\ &\quad + \int_t^s (\varrho_\beta^0 + \varrho_0^\beta + \varrho_\gamma^{\beta-\gamma})(r', x; s, y)\, dr' \end{aligned} \tag{3.16}$$

$$\preceq \frac{1}{|x-y|^{d+1}} \int_t^s \int_{r'}^s (r-r')^{\beta-\gamma-1}((s-r)^\gamma + (s-r)^{\gamma-\beta})\, dr\, dr'$$

$$+ \frac{1}{|x-y|^{d+1}} \int_t^s ((s-r')^\gamma + 1 + (s-r')^\beta)\, dr' < \infty.$$

For $Q^{(2)}_{s,y}(r', x, r)$, by (2.25) and (3.9) we have

$$\int_t^s \int_{r'}^s Q^{(2)}_{s,y}(r', x, r)\, dr\, dr' \preceq \int_t^s \int_{r'}^s (\varrho^0_\beta + \varrho^\beta_0)(r, x; s, y)(r-r')^{\beta-1}\, dr\, dr' < \infty. \tag{3.17}$$

Combining (3.15)–(3.17), we obtain (3.14).

Claim 3. *For fixed r, x, s, y, we have*

$$\lim_{t\uparrow r} \phi_{s,y}(t, x, r) = q(r, x; s, y). \tag{3.18}$$

Proof of Claim 3. By (2.26), it suffices to prove that

$$\lim_{t\uparrow r} \Big| \int_{\mathbb{R}^d} p_0(t, x; r, z)(q(r, z; s, y) - q(r, x; s, y))\, dz \Big| = 0.$$

Notice that for any $\delta > 0$,

$$\Big| \int_{\mathbb{R}^d} p_0(t, x; r, z)(q(r, z; s, y) - q(r, x; s, y))\, dz \Big|$$

$$\le \int_{|x-z|\le\delta} p_0(t, x; r, z)|q(r, z; s, y) - q(r, x; s, y)|\, dz$$

$$+ \int_{|x-z|>\delta} p_0(t, x; r, z)|q(r, z; s, y) - q(r, x; s, y)|\, dz$$

$$=: J_1(\delta, t, r) + J_2(\delta, t, r).$$

For any $\varepsilon > 0$, by (3.10), there exists a $\delta = \delta(r, x, s, y) > 0$ such that for all $|x - z| \le \delta$,

$$|q(r, z; s, y) - q(r, x; s, y)| \le \varepsilon.$$

Thus,

$$J_1(\delta, t, r) \le \varepsilon \int_{|x-z|\le\delta} p_0(t, x; r, z)\, dz \le \varepsilon \int_{\mathbb{R}^d} p_0(t, x; r, z)\, dz$$

$$\preceq \varepsilon \int_{\mathbb{R}^d} \varrho^0_1(t, x; r, z)\, dz \le \varepsilon.$$

On the other hand, we have

$$J_2(\delta,t,r) \overset{(2.17)}{\preceq} (r-t)\int_{|x-z|>\delta} \frac{|q(r,z;s,y)|+|q(r,x;s,y)|}{|x-z|^{d+1}}\,dz$$
$$\le (r-t)\Big(\delta^{-d-1}\int_{\mathbb{R}^d}|q(r,z;s,y)|\,dz + |q(r,x;s,y)|\int_{|z|>\delta}|z|^{-d-1}\,dz\Big),$$

which, by (3.9) and (2.2), converges to zero as $t\uparrow r$. Thus (3.18) is proved.

Now, by integration by parts and (3.18), we have

$$\int_t^r \partial_{r'}\phi_{s,y}(r',x,r)\,dr' = q(r,x;s,y) - \phi_{s,y}(t,x,r).$$

Integrating both sides with respect to r from t to s, and then using (3.14) and Fubini's theorem, we obtain

$$\int_t^s q(r,x;s,y)\,dr - \varphi_{s,y}(t,x)$$
$$= \int_t^s\int_t^r \partial_{r'}\phi_{s,y}(r',x,r)\,dr'\,dr \overset{(3.14)}{=} \int_t^s\int_{r'}^s \partial_{r'}\phi_{s,y}(r',x,r)\,dr\,dr'$$
$$\overset{(3.13),(2.15)}{=} -\int_t^s\int_{r'}^s\int_{\mathbb{R}^d} \mathscr{L}^{a,b}_{r',z}p_0(r',\cdot;r,z)(x)q(r,z;s,y)\,dz\,dr\,dr',$$

which in turn implies (3.12) by the Lebesgue differentiation theorem. ■

Lemma 3.5. *For all $t<s$ and $x\neq y$, we have*

$$\nabla_x\varphi_{s,y}(t,x) = \int_t^s\int_{\mathbb{R}^d}\nabla_x p_0(t,x;r,z)q(r,z;s,y)\,dz\,dr, \tag{3.19}$$

$$\Delta_x^{1/2}\varphi_{s,y}(t,x) = \int_t^s\int_{\mathbb{R}^d}\Delta_x^{1/2}p_0(t,x;r,z)q(r,z;s,y)\,dz\,dr, \tag{3.20}$$

where the integrals are understood in the sense of iterated integrals. Moreover,

$$t\mapsto \nabla_x\varphi_{s,y}(t,x),\ \Delta_x^{1/2}\varphi_{s,y}(t,x) \text{ are continuous.} \tag{3.21}$$

Proof. First of all, for fixed $t<r<s$, since

$$(x,z)\mapsto p_0(t,x;r,z)\in C_b^\infty(\mathbb{R}^d\times\mathbb{R}^d)$$

and

$$z\mapsto q(r,z;s,y)\in C_b(\mathbb{R}^d),$$

by Lemma 2.4, it is easy to see that

$$\nabla_x \phi_{s,y}(t,x,r) = \int_{\mathbb{R}^d} \nabla_x p_0(t,x;r,z) q(r,z;s,y)\, dz, \tag{3.22}$$

$$\Delta_x^{1/2} \phi_{s,y}(t,x,r) = \int_{\mathbb{R}^d} \Delta_x^{1/2} p_0(t,x;r,z) q(r,z;s,y)\, dz. \tag{3.23}$$

(1) We first prove that for any $t < s$ and $x \neq y$, there exists a $p > 1$ such that

$$\sup_{|w| \le |x-y|/2} I(p,w) < \infty, \text{ where } I(p,w) := \int_t^s |\nabla_x \phi_{s,y}(t,x+w;r)|^p\, dr. \tag{3.24}$$

Indeed, by (3.22) and (2.23) we have

$$\begin{aligned}
I(p,w) &\preceq \int_t^s \Big| \int_{\mathbb{R}^d} \nabla_x p_0(t,x+w;r,z)(q(r,z;s,y) - q(r,x+w;s,y))\, dz \Big|^p dr \\
&\quad + \int_t^s \Big| \int_{\mathbb{R}^d} \nabla_x p_0(t,x+w;r,z)\, dz \Big|^p |q(r,x+w;s,y)|^p\, dr \\
&\overset{(2.19),(3.10),(2.23)}{\preceq} \int_t^s \Big(\int_{\mathbb{R}^d} \varrho_0^{\beta-\gamma}(t,x+w;r,z)(\varrho_\gamma^0 + \varrho_{\gamma-\beta}^\beta)(r,z;s,y)\, dz \Big)^p dr \\
&\quad + \int_t^s \Big(\int_{\mathbb{R}^d} \varrho_0^{\beta-\gamma}(t,x+w;r,z)(\varrho_\gamma^0 + \varrho_{\gamma-\beta}^\beta)(r,x+w;s,y)\, dz \Big)^p dr \\
&\quad + \int_t^s (r-t)^{p(\beta-1)} (\varrho_\beta^0 + \varrho_0^\beta)^p (r,x+w;s,y)\, dr \\
&=: I_1(p,w) + I_2(p,w) + I_3(p,w).
\end{aligned}$$

For $I_1(p,w)$, it follows from (2.6) that for some $p > 1$,

$$\sup_{|w| \le |x-y|/2} I_1(p,w) < \infty.$$

For $I_2(p,w)$, by (2.1) and (2.2), for all $|w| \le |x-y|/2$ we have

$$\begin{aligned}
&I_2(p,w) \\
&\preceq \int_t^s \Big(\int_{\mathbb{R}^d} \varrho_0^{\beta-\gamma}(t,x+w;r,z)\, dz \Big)^p \Big(\frac{(s-r)^\gamma}{|x+w-y|^{d+1}} + \frac{(s-r)^{\gamma-\beta}}{|x+w-y|^{d+1}} \Big)^p dr \\
&\preceq \int_t^s (r-t)^{p(\beta-\gamma-1)} \Big(\frac{1}{|x-y|^{d+1}} + \frac{(s-r)^{\gamma-\beta}}{|x-y|^{d+1}} \Big)^p dr < \infty,
\end{aligned}$$

provided $p < \frac{1}{1+\gamma-\beta} \wedge \frac{1}{\beta-\gamma}$. Similarly, for $p < \frac{1}{1-\beta}$,

$$\sup_{|w|\le|x-y|/2} I_3(p,w) < \infty. \tag{3.25}$$

Thus, we obtain (3.24).

Now, for $e_i = (0,\dots,1,\dots,0) \in \mathbb{R}^d$, we can write

$$\frac{\varphi_{s,y}(t,x+\varepsilon e_i) - \varphi_{s,y}(t,x)}{\varepsilon} = \int_t^s\int_0^1 \partial_{x_i}\phi_{s,y}(t,x+\theta\varepsilon e_i,r)\,d\theta\,dr.$$

By (3.24) one can take limits to get

$$\begin{aligned}\partial_{x_i}\varphi(t,x) &= \lim_{\varepsilon\to0}\frac{\varphi(t,x+\varepsilon e_i)-\varphi(t,x)}{\varepsilon}\\ &= \int_t^s\int_0^1 \lim_{\varepsilon\to0}\partial_{x_i}\phi(t,x+\theta\varepsilon e_i,r)\,d\theta\,dr = \int_t^s \partial_{x_i}\phi(t,x,r)\,dr,\end{aligned}$$

and (3.19) is proven.

(2) Next, we prove (3.20). Recalling the definition of $\phi_{s,y}$, we have

$$\begin{aligned}&\nabla_x\phi_{s,y}(t,x+w,r) - \nabla_x\phi_{s,y}(t,x,r)\\ &\quad= \int_{\mathbb{R}^d}(\nabla_x p_0(t,x+w;r,z) - \nabla_x p_0(t,x;r,z))q(r,z;s,y)\,dz\\ &\quad= \int_{\mathbb{R}^d}\big(\nabla_x p_0(t,x+w;r,z)(q(r,z;s,y)-q(r,x+w;s,y))\\ &\qquad\qquad - \nabla_x p_0(t,x;r,z)(q(r,z;s,y)-q(r,x;s,y))\big)\,dz\\ &\qquad + \Big\{q(r,x+w;s,y)\int_{\mathbb{R}^d}\nabla_x p_0(t,x+w;r,z)\,dz\\ &\qquad\qquad - q(r,x;s,y)\int_{\mathbb{R}^d}\nabla_x p_0(t,x;r,z)\,dz\Big\}\\ &\quad=: \int_{\mathbb{R}^d} Q(t,x;r,z;s,y;w)\,dz + R(r,t,x;s,y;w).\end{aligned}$$

We now prove that for any $\gamma\in(0,\beta)$ and $\sigma\in(0,\beta-\gamma)$,

$$\begin{aligned}|Q(t,x;r,z;s,y;w)| &\preceq |w|^\sigma\varrho^{\beta-\gamma}_{-\sigma}(t,x+w;r,z)\\ &\quad\times\big((\varrho^\beta_{\gamma-\beta}+\varrho^0_\gamma)(r,x+w;s,y)+(\varrho^\beta_{\gamma-\beta}+\varrho^0_\gamma)(r,z;s,y)\big)\\ &+|w|^\sigma\varrho^{\beta-\gamma}_{-\sigma}(t,x;r,z)\big((\varrho^\beta_{\gamma-\beta}+\varrho^0_\gamma)(r,x;s,y)+(\varrho^\beta_{\gamma-\beta}+\varrho^0_\gamma)(r,z;s,y)\big)\\ &+|w|^\sigma\varrho^0_{\beta-\gamma-\sigma}(t,x;r,z)\big((\varrho^\beta_{\gamma-\beta}+\varrho^0_\gamma)(r,x+w;s,y)+(\varrho^\beta_{\gamma-\beta}+\varrho^0_\gamma)(r,z;s,y)\big)\\ &+|w|^\sigma\varrho^0_{\beta-\gamma-\sigma}(t,x;r,z)+|w|^\sigma\varrho^0_{\beta-\gamma-\sigma}(t,x;r,z)\\ &\quad\times\big((\varrho^\beta_{\gamma-\beta}+\varrho^0_\gamma)(r,x+w;s,y)+(\varrho^\beta_{\gamma-\beta}+\varrho^0_\gamma)(r,x;s,y)\big),\end{aligned} \tag{3.26}$$

and for $w \in \mathbb{R}^d$,

$$(3.27)\quad |R(r,t,x;s,y;w)| \preceq |w|^{\beta-\gamma}(r-t)^{\beta-1}\big\{(\varrho^0_\gamma + \varrho^\beta_{\gamma-\beta})(r,x+w;s,y) + (\varrho^0_\gamma + \varrho^\beta_{\gamma-\beta})(r,x;s,y)\big\} + |w|^\gamma (r-t)^{\beta-\gamma-1}(\varrho^0_\beta + \varrho^\beta_0)(r,x;s,y).$$

First, we assume $|w| > |r-t|$. By (3.10), we have

$$\begin{aligned} &|\nabla_x p_0(t,x;r,z)(q(r,z;s,y) - q(r,x;s,y))| \\ &\preceq \varrho^0_0(t,x;r,z)(|x-z|^{\beta-\gamma} \wedge 1)\big((\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,x;s,y) + (\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,z;s,y)\big) \\ &\preceq |w|^\sigma \varrho^{\beta-\gamma}_{-\sigma}(t,x;r,z)\big((\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,x;s,y) + (\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,z;s,y)\big), \end{aligned}$$

and also

$$\begin{aligned} &|\nabla_x p_0(t,x+w;r,z)(q(r,z;s,y) - q(r,x+w;s,y))| \\ &\quad\preceq |w|^\sigma \varrho^{\beta-\gamma}_{-\sigma}(t,x+w;r,z) \\ &\qquad\qquad \times \big((\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,x+w;s,y) + (\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,z;s,y)\big). \end{aligned}$$

Next, we assume $|w| \le |r-t|$. Noticing that

$$|x+w-z| \le |x-z| + |w| \le |x-z| + |r-t|$$

and

$$|x-z| \le |x+w-z| + |w| \le |x+w-z| + |r-t|,$$

we deduce that for any $\theta_0 \in (0,1)$,

$$|w| \cdot |\nabla^2_x p_0(t,x+\theta_0 w;r,z)| \cdot |x+w-z|^{\beta-\gamma} \preceq |w|^\sigma \varrho^0_{\beta-\gamma-\sigma}(t,x;r,z).$$

Hence, for some $\theta_0 \in (0,1)$,

$$\begin{aligned} &|(\nabla_x p_0(t,x+w;r,z) - \nabla_x p_0(t,x;r,z))(q(r,z;s,y) - q(r,x+w;s,y))| \\ &\quad\preceq |w| \cdot |\nabla^2_x p_0(t,x+\theta_0 w;r,z)| \cdot |x+w-z|^{\beta-\gamma} \\ &\qquad\qquad \times \big((\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,x+w;s,y) + (\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,z;s,y)\big) \\ &\quad\preceq |w|^\sigma \varrho^0_{\beta-\gamma-\sigma}(t,x;r,z) \\ &\qquad\qquad \times \big((\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,x+w;s,y) + (\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,z;s,y)\big). \end{aligned}$$

Similarly,

$$\begin{aligned} &|\nabla_x p_0(t,x;r,z)(q(r,x;s,y) - q(r,x+w;s,y))| \\ &\preceq |w|^\sigma \varrho^0_{\beta-\gamma-\sigma}(t,x;r,z)\big((\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,x+w;s,y) + (\varrho^\beta_{\gamma-\beta} + \varrho^0_\gamma)(r,x;s,y)\big). \end{aligned}$$

Combining the above, we obtain (3.26). Finally, (3.27) follows from Lemma 2.4 and Theorem 3.3.

(3) Now, we can prove that for any $t < s$ and $x \neq y$, there exists a $p > 1$ such that

$$\sup_{\varepsilon \leq |x-y|/2} J(p, \varepsilon) < \infty, \tag{3.28}$$

where

$$J(p, \varepsilon) := \int_t^s \Big| \int_{|w| \geq \varepsilon} \frac{\phi_{s,y}(t, x + w, r) - \phi_{s,y}(t, x, r)}{|w|^{d+1}} \, dw \Big|^p \, dr.$$

Indeed, notice that

$$\begin{aligned} &J(p, \varepsilon) \\ &\preceq \int_t^s \Big| \int_{\varepsilon \leq |w| \leq |x-y|/2} \frac{\phi_{s,y}(t, x + w, r) - \phi_{s,y}(t, x, r) - w \cdot \nabla_x \phi_{s,y}(t, x, r)}{|w|^{d+1}} \, dw \Big|^p \, dr \\ &\quad + \int_t^s \Big| \int_{|w| > |x-y|/2} \frac{\phi_{s,y}(t, x + w, r) - \phi_{s,y}(t, x, r)}{|w|^{d+1}} \, dw \Big|^p \, dr \\ &=: J_1(p, \varepsilon) + J_2(p). \end{aligned}$$

For $J_1(p, \varepsilon)$, observe that

$$\begin{aligned} &J_1(p, \varepsilon) \\ &= \int_t^s \Big| \int_{\varepsilon \leq |w| \leq |x-y|/2} \frac{w}{|w|^{d+1}} \Big(\int_0^1 \big(\nabla_x \phi_{s,y}(t, x + \theta w, r) - \nabla_x \phi_{s,y}(t, x, r) \big) \, d\theta \Big) dw \Big|^p dr \\ &\preceq \int_t^s \Big(\int_{\varepsilon \leq |w| \leq |x-y|/2} \int_0^1 \frac{\int_{\mathbb{R}^d} |Q(t, x; r, z; s, y; \theta w)| \, dz}{|w|^d} \, d\theta \, dw \Big)^p \, dr \\ &\quad + \int_t^s \Big(\int_{\varepsilon \leq |w| \leq |x-y|/2} \int_0^1 \frac{R(r, t, x; s, y; \theta w)}{|w|^d} \, d\theta \, dw \Big)^p \, dr. \end{aligned}$$

Applying (3.26), (3.27) and making use of (2.6), as in proving (3.35), we find that for some $p > 1$,

$$\sup_{\varepsilon \leq |x-y|/2} J_1(p, \varepsilon) < \infty.$$

For $J_2(p)$, from (3.11) we deduce that for some $p > 1$,

$$J_2(p) \leq \int_t^s \Big| \int_{|w| > |x-y|/2} \frac{|\phi_{s,y}(t, x + w, r)| + |\phi_{s,y}(t, x, r)|}{|w|^{d+1}} \, dw \Big|^p \, dr < \infty.$$

Thus, (3.28) is proven.

(4) By (3.28) and Fubini's theorem, we have

$$\begin{aligned}\Delta_x^{1/2}\varphi_{s,y}(t,x) &= \lim_{\varepsilon\downarrow 0}\int\limits_{|w|\ge\varepsilon}\int\limits_t^s \frac{\phi_{s,y}(t,x+w,r)-\phi_{s,y}(t,x,r)}{|w|^{d+1}}\,dr\,dw\\ &= \lim_{\varepsilon\downarrow 0}\int\limits_t^s\int\limits_{|w|\ge\varepsilon} \frac{\phi_{s,y}(t,x+w,r)-\phi_{s,y}(t,x,r)}{|w|^{d+1}}\,dw\,dr\\ &= \int\limits_t^s\lim_{\varepsilon\downarrow 0}\int\limits_{|w|\ge\varepsilon} \frac{\phi_{s,y}(t,x+w,r)-\phi_{s,y}(t,x,r)}{|w|^{d+1}}\,dw\,dr\\ &= \int\limits_t^s \Delta_x^{1/2}\phi_{s,y}(t,x,r)\,dr,\end{aligned}$$

which together with (3.23) yields (3.20).

(5) Lastly, (3.21) follows from (3.19), (3.20) and an easy limiting procedure. ■

3.3. Heat kernel of $\mathscr{L}^{a,b}_{t,x}$. We need the following maximum principle (cf. [30, Theorem 2.3]).

THEOREM 3.6 (Maximal principle). *For $T>0$, let $u\in C_b([0,T]\times\mathbb{R}^d)$ be such that for almost all $t\in[0,T]$ and all $x\in\mathbb{R}^d$,*

$$\partial_t u(t,x)+\mathscr{L}^{a,b}_{t,x}u(t,x)=0. \tag{3.29}$$

Assume that

$$\lim_{t\uparrow T}\|u(t)-u(T)\|_\infty=0,\qquad \sup_{t\in[0,s]}\|\nabla u(t)\|_\infty<\infty,\qquad s\in[0,T), \tag{3.30}$$

and

(3.31) *for each $x\in\mathbb{R}^d$, $t\mapsto\Delta^{1/2}u(t,x),\nabla u(t,x)$ are continuous on $[0,T)$.*

Then for each $t\in[0,T)$,

$$\sup_{x\in\mathbb{R}^d}u(t,x)\le\sup_{x\in\mathbb{R}^d}u(T,x).$$

In particular, there is a unique solution to equation (3.29) *with a given final value at time T in the class of $u\in C_b([0,T]\times\mathbb{R}^d)$ satisfying* (3.30) *and* (3.31).

Proof. Without loss of generality, we may assume that u is nonnegative. Otherwise, we can subtract from u its infimum. By the assumption, it suffices to prove that for any $t<s<T$,

$$\sup_{x\in\mathbb{R}^d}u(t,x)\le\sup_{x\in\mathbb{R}^d}u(s,x). \tag{3.32}$$

Below we fix $s \in (0,T)$. Let $\chi : \mathbb{R}^d \to [0,1]$ be a smooth function with $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| > 2$. For $R > 0$, define the following cutoff function:

$$\chi_R(x) := \chi(x/R).$$

For $R, \delta > 0$, consider

$$u_R^\delta(t,x) := u(t,x)\chi_R(x) + (t-s)\delta.$$

Then

$$\partial_t u_R^\delta(t,x) + \mathscr{L}_{t,x}^{a,b} u_R^\delta(t,x) = g_R^\delta(t,x) + \delta, \tag{3.33}$$

where

$$\begin{aligned} g_R^\delta(t,x) := {} & a(t,x)\big(\Delta^{1/2}(u\chi_R)(t,x) - \Delta^{1/2}u(t,x)\chi_R(x)\big) \\ & + b(t,x)\cdot\nabla\chi_R(x)u(t,x). \end{aligned} \tag{3.34}$$

Our aim is to prove that for each $\delta > 0$, there exists an $R_0 \ge 1$ such that for all $t \in [0,s)$ and $R > R_0$,

$$\sup_{x\in\mathbb{R}^d} u_R^\delta(t,x) \le \sup_{x\in\mathbb{R}^d} u_R^\delta(s,x). \tag{3.35}$$

If this is proven, then letting $R \to \infty$ and $\delta \to 0$ and noticing that $\sup_{x\in\mathbb{R}^d} u_R^\delta(s,x) \le \sup_{x\in\mathbb{R}^d} u(s,x)$, we obtain (3.32).

We first prove that for each $s < T$, there exists a constant $C_s > 0$ such that

$$\sup_{t\in[0,s]} \|g_R^\delta(t)\|_\infty \le \frac{C_s}{R^{1/2}}. \tag{3.36}$$

Indeed, by definition, we have

$$\begin{aligned} &|\Delta^{1/2}(u\chi_R)(t,x) - \Delta^{1/2}u(t,x)\chi_R(x)| \\ &\quad \le \int_{\mathbb{R}^d} |u(t,x+z)-u(t,x)|\,|\chi_R(x+z)-\chi_R(x)|\,\frac{dz}{|z|^{d+1}} \\ &\qquad + |u(t,x)|\cdot|\Delta^{1/2}\chi_R(x)| \\ &\quad \le 2\|u(t)\|_\infty(2\|\chi_R\|_\infty)^{1/2}\|\nabla\chi_R\|_\infty^{1/2}\int_{|z|>1}\frac{dz}{|z|^{d+1/2}} \\ &\qquad + \|\nabla u(t)\|_\infty\|\nabla\chi_R\|_\infty\int_{|z|\le 1}\frac{dz}{|z|^{d-1}} + \|u(t)\|_\infty\|\Delta^{1/2}\chi_R\|_\infty \\ &\quad \preceq \|u(t)\|_\infty\|\chi\|_\infty^{1/2}\frac{\|\nabla\chi\|_\infty^{1/2}}{R^{1/2}} + \|\nabla u(t)\|_\infty\frac{\|\nabla\chi\|_\infty}{R} + \|u(t)\|_\infty\frac{\|\Delta^{1/2}\chi\|_\infty}{R}, \end{aligned}$$

which gives (3.36) by (3.34), (3.30) and $a, b \in \mathbb{H}^\beta$.

We now use a contradiction argument to prove (3.35). Fix

$$R > (C_s/\delta)^2. \tag{3.37}$$

Suppose that (3.35) does not hold. Since $t \mapsto \sup_{x\in\mathbb{R}^d} u_R^\delta(t,x)$ is continuous on $[0,s]$, there must exist $t_0 \in [0,s)$ such that

$$\sup_{(t,x)\in[0,s)\times\mathbb{R}^d} u_R^\delta(t,x) = \sup_{t\in[0,s)}\Big(\sup_{x\in\mathbb{R}^d} u_R^\delta(t,x)\Big) = \sup_{x\in\mathbb{R}^d} u_R^\delta(t_0,x)$$

and further, for some $x_0 \in \mathbb{R}^d$,

$$\sup_{(t,x)\in[0,s)\times\mathbb{R}^d} u_R^\delta(t,x) = \sup_{x\in\mathbb{R}^d} u_R^\delta(t_0,x) = u_R^\delta(t_0,x_0).$$

In particular,

$$\nabla u_R^\delta(t_0,x_0) = 0, \tag{3.38}$$

and

$$\Delta^{1/2} u_R^\delta(t_0,x_0) = \lim_{\varepsilon\downarrow 0}\int_{|z|\ge\varepsilon}\big(u_R^\delta(t_0,x_0+z) - u_R^\delta(t_0,x_0)\big)|z|^{-d-1}\,dz \le 0. \tag{3.39}$$

Moreover, by (3.33), for any $h \in (0, s-t_0)$, we have

$$\begin{aligned}
0 &\ge \frac{u_R^\delta(t_0+h,x_0) - u_R^\delta(t_0,x_0)}{h}\\
&= -\frac{1}{h}\int_{t_0}^{t_0+h} \mathscr{L}^{a,b}_{r,x_0} u_R^\delta(r,x_0)\,dr + \frac{1}{h}\int_{t_0}^{t_0+h} g_R^\delta(r,x_0)\,dr + \delta.
\end{aligned}$$

Since

$$t \mapsto \Delta^{1/2} u_R^\delta(t,x_0), \nabla u_R^\delta(t,x_0) \text{ are continuous,}$$

letting $h \to 0$, by (3.38), (3.39) and (3.36), we obtain

$$0 \ge -\varlimsup_{h\downarrow 0}\bigg(\frac{1}{h}\int_{t_0}^{t_0+h} a(r,x_0)\,dr\bigg)\Delta^{1/2} u_R^\delta(t_0,x_0) - \frac{C_s}{R^{1/2}} + \delta \ge -\frac{C_s}{R^{1/2}} + \delta,$$

which contradicts (3.37). ■

Now, we prove the following main result of this section.

THEOREM 3.7. *Assume that $a, b \in \mathbb{H}^\beta$ for some $\beta \in (0,1)$ and satisfy* (2.16). *Then there exists a unique transition probability density function $p_{a,b}(t,x;s,y)$ such that:*

(i) *For all $x \ne y \in \mathbb{R}^d$ and almost all $t < s$,*

$$\partial_t p_{a,b}(t,x;s,y) + \mathscr{L}^{a,b}_{t,x} p_{a,b}(t,\cdot;s,y)(x) = 0. \tag{3.40}$$

(ii) *For all $0 \le t < s \le 1$ and $x, y \in \mathbb{R}^d$,*

$$p_{a,b}(t,x;s,y) \preceq \varrho_1^0(t,x;s,y). \tag{3.41}$$

(iii) *For any* $\gamma \in (0,1)$,

$$(3.42)\quad |p_{a,b}(t,x;s,y) - p_{a,b}(t,x';s,y)| \preceq (|x-x'|^{\gamma} \wedge 1)\{\varrho^0_{1-\gamma}(t,x;s,y) + \varrho^0_{1-\gamma}(t,x';s,y)\},$$

$$(3.43)\quad |\nabla_x p_{a,b}(t,x;s,y)| + |\Delta^{1/2}_x p_{a,b}(t,x;s,y)| \preceq \varrho^0_0(t,x;s,y).$$

(iv) *For any bounded uniformly continuous function* f *on* $\mathbb{R}^d$,

$$(3.44)\quad \big(\lim_{s\downarrow t}\big) \lim_{t\uparrow s} \sup_{x\in\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} p_{a,b}(t,x;s,y) f(y)\,dy - f(x)\Big| = 0,$$

$$(3.45)\quad \big(\lim_{s\downarrow t}\big) \lim_{t\uparrow s} \sup_{y\in\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} p_{a,b}(t,x;s,y) f(x)\,dx - f(y)\Big| = 0.$$

(iv) *For all* $f \in C^\infty_b(\mathbb{R}^d)$ *and* $t < s$,

$$(3.46)\quad P^{a,b}_{t,s} f(x) = f(x) + \int_t^s P^{a,b}_{t,r}(\mathscr{L}^{a,b}_{r,\cdot} f)(x)\,dr, \qquad \textit{where}$$

$$(3.47)\quad P^{a,b}_{t,s} f(x) := \int_{\mathbb{R}^d} p_{a,b}(t,x;s,y) f(y)\,dy.$$

Proof. Without loss of generality, we may assume $\beta \in (0, 1/4]$. It suffices to verify that $p_{a,b}$ defined by (3.1) has all the required properties.

(1) First, we prove (3.40). By (3.1), for all $x \neq y \in \mathbb{R}^d$ and almost all $t < s$, we have

$$\begin{aligned}
&\partial_t p_{a,b}(t,x;s,y)\\
&\overset{(3.12)}{=} \partial_t p_0(t,x;s,y) - q(t,x;s,y) - \int_t^s \int_{\mathbb{R}^d} \mathscr{L}^{a,b}_{t,y} p_0(t,\cdot;r,z)(x) q(r,z;s,y)\,dz\,dr\\
&\overset{(2.15)}{=} -\mathscr{L}^{a,b}_{t,y} p_0(t,\cdot;s,y)(x) - q_0(t,x;s,y) - \int_t^s \int_{\mathbb{R}^d} q_0(t,x;r,z) q(r,z;s,y)\,dz\,dr\\
&\quad - \int_t^s \int_{\mathbb{R}^d} \mathscr{L}^{a,b}_{t,y} p_0(t,\cdot;r,z)(x) q(r,z;s,y)\,dz\,dr.
\end{aligned}$$

Recalling that

$$(3.48)\quad q_0(t,x;s,y) = \big(\mathscr{L}^{a,b}_{t,x} - \mathscr{L}^{a,b}_{t,y}\big) p_0(t,\cdot;s,y)(x),$$

we further have

$$\begin{aligned}
\partial_t p_{a,b}(t,x;s,y) &= -\mathscr{L}^{a,b}_{t,x} p_0(t,\cdot;s,y)(x)\\
&\quad - \int_t^s \int_{\mathbb{R}^d} \mathscr{L}^{a,b}_{t,x} p_0(t,\cdot;r,z)(x) q(r,z;s,y)\,dz\,dr,
\end{aligned}$$

which together with (3.19) and (3.20) yields (3.40).

(2) Recalling that $t,s\in(0,1)$, by (3.11) one has

$$\int_t^s\int_{\mathbb{R}^d} p_0(t,x;r,z)|q(r,z;s,y)|\,dz\,dr \preceq \varrho^0_{1+\beta}(t,x;s,y)+\varrho^\beta_1(t,x;s,y)\le \varrho^0_1(t,x;s,y), \tag{3.49}$$

which in turn gives (3.41) by (3.1) and (2.17).

(3) As in proving (3.7), for any $\gamma\in(0,1)$ we have

$$|p_0(t,x;s,y)-p_0(t,x';s,y)| \preceq (|x-x'|^\gamma\wedge 1)\big(\varrho^0_{1-\gamma}(t,x;s,y)+\varrho^0_{1-\gamma}(t,x';s,y)\big).$$

Thus, by (3.9) and Lemma 2.1, we have

$$\begin{aligned}
&\int_t^s\int_{\mathbb{R}^d} |p_0(t,x;r,z)-p_0(t,x';r,z)|\,|q(r,z;s,y)|\,dz\,dr\\
&\preceq (|x-x'|^\gamma\wedge 1)\\
&\quad\times\int_t^s\int_{\mathbb{R}^d}\big(\varrho^0_{1-\gamma}(t,x;r,z)+\varrho^0_{1-\gamma}(t,x';r,z)\big)(\varrho^0_\beta+\varrho^\beta_0)(r,z;s,y)\,dz\,dr\\
&\preceq (|x-x'|^\gamma\wedge 1)\big((\varrho^0_{1+\beta-\gamma}+\varrho^\beta_{1-\gamma})(t,x;s,y)+(\varrho^0_{1+\beta-\gamma}+\varrho^\beta_{1-\gamma})(t,x';s,y)\big)\\
&\preceq (|x-x'|^\gamma\wedge 1)\big(\varrho^0_{1-\gamma}(t,x;s,y)+\varrho^0_{1-\gamma}(t,x';s,y)\big),
\end{aligned}$$

which together with (3.1) yields (3.42).

Recall the definition of $\varphi_{s,y}(t,x)$. By (3.19), we can write

$$\begin{aligned}
\nabla_x\varphi_{s,y}(t,x) &= \int_t^{(t+s)/2}\int_{\mathbb{R}^d}\nabla_x p_0(t,x;r,z)\big(q(r,z;s,y)-q(r,x;s,y)\big)\,dz\,dr\\
&\quad+\int_t^{(t+s)/2}\Big(\int_{\mathbb{R}^d}\nabla_x p_0(t,x;r,z)\,dz\Big)q(r,x;s,y)\,dr\\
&\quad+\int_{(t+s)/2}^{s}\int_{\mathbb{R}^d}\nabla_x p_0(t,x;r,z)q(r,z;s,y)\,dz\,dr\\
&=: Q_1(t,x;s,y)+Q_2(t,x;s,y)+Q_3(t,x;s,y).
\end{aligned}$$

For $Q_1(t,x;s,y)$, by (2.19), (3.10) and Lemma 2.1, we have

$$\begin{aligned}
&|Q_1(t,x;s,y)|\\
&\preceq \int_t^{(t+s)/2}\int_{\mathbb{R}^d}\varrho^{\beta-\gamma}_0(t,x;r,z)\\
&\qquad\times\big\{(\varrho^0_\gamma+\varrho^\beta_{\gamma-\beta})(r,x;s,y)+(\varrho^0_\gamma+\varrho^\beta_{\gamma-\beta})(r,z;s,y)\big\}\,dz\,dr
\end{aligned}$$

$$\leq \int_t^{(t+s)/2} \Big(\int_{\mathbb{R}^d} \varrho_0^{\beta-\gamma}(t,x;r,z)\,dz\Big)(\varrho_\gamma^0 + \varrho_{\gamma-\beta}^\beta)(r,x;s,y)\,dr$$

$$+ \int_t^s \int_{\mathbb{R}^d} \varrho_0^{\beta-\gamma}(t,x;r,z)(\varrho_\gamma^0 + \varrho_{\gamma-\beta}^\beta)(r,z;s,y)\,dz\,dr$$

$$\overset{(2.2),(2.5)}{\preceq} \Big(\int_t^{(t+s)/2} (r-t)^{\beta-\gamma-1}(1+(s-r)^{\gamma-\beta})\varrho_0^0(r,x;s,y)\,dr\Big)$$

$$+ (\varrho_\beta^0 + \varrho_0^\beta + \varrho_\gamma^{\beta-\gamma})(t,x;s,y) \preceq \varrho_0^0(t,x;s,y).$$

For $Q_2(t,x;s,y)$, we have

$$|Q_2(t,x;s,y)| \overset{(2.23),(3.9)}{\preceq} \int_t^{(t+s)/2} (r-t)^{\beta-1}\big\{\varrho_0^\beta(r,x;s,y) + \varrho_\beta^0(r,x;s,y)\big\}\,dr \preceq \varrho_0^0(t,x;s,y).$$

For $Q_3(t,x;s,y)$, we have

$$|Q_3(t,x;s,y)| \overset{(2.19),(3.9)}{\preceq} \int_{(t+s)/2}^s \int_{\mathbb{R}^d} \varrho_0^0(t,x;r,z)\big\{\varrho_0^\beta(r,z;s,y) + \varrho_\beta^0(r,z;s,y)\big\}\,dz\,dr \overset{(2.4),(2.2)}{\preceq} \varrho_0^0(t,x;s,y).$$

Combining the above, we obtain

$$|\nabla_x \varphi_{s,y}(t,x)| \preceq \varrho_0^0(t,x;s,y). \tag{3.50}$$

Similarly,

$$|\Delta_x^{1/2} \varphi_{s,y}(t,x)| \preceq \varrho_0^0(t,x;s,y). \tag{3.51}$$

Then, (3.43) follows from (3.1), (2.18), (2.19) and (3.50), (3.51).

(4) We now prove (3.44). As in proving (3.18), we can show that for any bounded uniformly continuous function f,

$$\Big(\lim_{s\downarrow t}\Big) \lim_{t\uparrow s} \sup_{x\in\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} p_0(t,x;s,y)f(y)\,dy - f(x)\Big| = 0.$$

Moreover, by (3.11), we also have

$$\Big|\int_{\mathbb{R}^d}\int_t^s\int_{\mathbb{R}^d} p_0(t,x;r,z)q(r,z;s,y)f(y)\,dz\,dr\,dy\Big|$$
$$\preceq \int_{\mathbb{R}^d}\big(\varrho^0_{1+\beta}(t,x;s,y)+\varrho^\beta_1(t,x;s,y)\big)\,dy$$
$$\overset{(2.2)}{\preceq} |s-t|^\beta \to 0, \qquad t\uparrow s \text{ or } s\downarrow t.$$

Thus, (3.44) is proven by (3.1). The limit (3.45) can be deduced similarly.

(5) For $f\in C_b^\infty(\mathbb{R}^d)$, if we set $u_s^f(t,x):=\int_{\mathbb{R}^d}p_{a,b}(t,x;s,y)f(y)\,dy$, then by (3.40) and (3.44),

$$\partial_t u_s^f(t,x)+\mathscr{L}^{a,b}_{t,x}u_s^f(t,x)=0, \qquad \lim_{t\uparrow s}\|u_s^f(t)-f\|_\infty=0,$$

and by (3.43) and (2.2),

$$\|\nabla u_s^f(t)\|_\infty \preceq \|f\|_\infty(s-t)^{-1}.$$

Moreover, the continuity of $t\mapsto \Delta^{1/2}u_s^f(t,x), \nabla u_s^f(t,x)$ on $[0,s)$ follows from (3.21). Thus, by uniqueness (see Theorem 3.6), it follows that

$$\int_{\mathbb{R}^d} p_{a,b}(t,x;s,y)\,dy = u_s^1(t,x)\equiv 1, \qquad t<s,\ x\in\mathbb{R}^d. \tag{3.52}$$

Moreover, if $f\le 0$, then

$$u_s^f(t,x)\le 0,$$

which implies that

$$p_{a,b}(t,x;s,y)\ge 0.$$

(6) The following C-K equation holds: for all $t<r<s$ and $x,y\in\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p_{a,b}(t,x;r,z)p_{a,b}(r,z;s,y)\,dz = p_{a,b}(t,x;s,y). \tag{3.53}$$

To prove this, it suffices to show that for any $f\in C_b^\infty(\mathbb{R}^d)$,

$$P^{a,b}_{t,s}f(x)=P^{a,b}_{t,r}P^{a,b}_{r,s}f(x), \tag{3.54}$$

where $P^{a,b}_{t,s}f(x)$ is defined by (3.47). This can be proven as above by using the maximum principle (i.e. uniqueness). In particular, $\{p_{a,b}(t,x;s,y)\}$ is a family of transition probability density functions. The uniqueness of $p_{a,b}$ can also be deduced from the maximum principle.

(7) Set

$$u_s(t,x):=f(x)+\int_t^s P^{a,b}_{t,r}(\mathscr{L}^{a,b}_{r,\cdot}f)(x)\,dr.$$

As in Subsection 3.2, we can prove that for almost all $t < s$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}\partial_t u_s(t,x) &= \partial_t\Big(\int_t^s P^{a,b}_{t,r}(\mathscr{L}^{a,b}_{r,\cdot} f)(x)\,dr\Big)\\ &= -\mathscr{L}^{a,b}_{t,x} f(x) + \int_t^s \partial_t P^{a,b}_{t,r}(\mathscr{L}^{a,b}_{r,\cdot} f)(x)\,dr\\ &= -\mathscr{L}^{a,b}_{t,x} f(x) - \int_t^s \mathscr{L}^{a,b}_{t,x} P^{a,b}_{t,r}(\mathscr{L}^{a,b}_{r,\cdot} f)(x)\,dr\\ &= -\mathscr{L}^{a,b}_{t,x} f(x) - \mathscr{L}^{a,b}_{t,x}\Big(\int_t^s P^{a,b}_{t,r}(\mathscr{L}^{a,b}_{r,\cdot} f)(x)\,dr\Big)\\ &= -\mathscr{L}^{a,b}_{t,x} u_s(t,x).\end{aligned}$$

Moreover, by (3.44), we have

$$\lim_{t\uparrow s} \|u_s(t) - f\|_\infty = 0.$$

As in step (5), using Theorem 3.6, we obtain

$$P^{a,b}_{t,s} f(x) = u_s(t,x). \blacksquare$$

4. Proof of the lower bound of $p_{a,b}(t,x;s,y)$. By Theorem 3.7, we know that

$$\{p_{a,b}(t,x;s,y) : 0 \le t < s < \infty,\ x,y \in \mathbb{R}^d\}$$

is a family of transition probability density functions. By (3.44) and (3.46), it also determines a family of strong Markov processes

$$\big(\Omega, \mathscr{F}, (\mathbb{P}_{t,x})_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}; (X_s)_{s\ge 0}\big).$$

For any $f \in C^2_b(\mathbb{R}^d)$, it follows from (3.46) and the Markov property of X that under $\mathbb{P}_{t,x}$, with respect to the filtration $\mathscr{F}_s := \sigma\{X_t, t \le s\}$,

$$M^f_s := f(X_s) - f(X_t) - \int_t^s \mathscr{L}^{a,b}_r f(X_r)\,dr \text{ is a martingale.} \tag{4.1}$$

In other words, $\mathbb{P}_{t,x}$ solves the martingale problem for $(\mathscr{L}^{a,b}_r, C^2_b(\mathbb{R}^d))$ (cf. [13]).

Let

$$J(r,x,y) := \frac{a(r,x)}{|x-y|^{d+1}}.$$

We now determine the Lévy system for X. The proof of the following result is similar to one in [6]. However, our process is time inhomogeneous, so we give the details.

LEMMA 4.1. *Suppose that A and B are two disjoint open sets in $\mathbb{R}^d$. Then*

$$\sum_{t<r\le s} 1_{\{X_{r-}\in A,\, X_r\in B\}} - \int_t^s 1_A(X_r)\int_B J(r,X_r,z)\,dz\,dr$$

is a $\mathbb{P}_{t,x}$-martingale for every $t\ge 0$ and $x\in\mathbb{R}^d$.

Proof. First of all, by letting $f(x)=x_i$ in (4.1), one sees that $(X_s)_{s\ge t}$ is a semi-martingale under $\mathbb{P}_{t,x}$. Let $f\in C_b^2(\mathbb{R}^d)$ with $f=0$ on A and $f=1$ on B. By Itô's formula, we have

$$f(X_s)-f(X_t)=\sum_{i=1}^d\int_t^s \partial_i f(X_{r-})\,dX_r^i+\sum_{t<r\le s}\beta_r(f) + \frac{1}{2}\sum_{i,j=1}^d\int_t^s \partial_{ij}^2 f(X_{r-})\,d\langle (X^i)^c,(X^j)^c\rangle_r, \tag{4.2}$$

where

$$\beta_r(f):=f(X_r)-f(X_{r-})-\sum_{i=1}^d \partial_i f(X_{r-})(X_r^i-X_{r-}^i).$$

Let M_s^f be defined by (4.1). Then

$$N_s:=\int_t^s 1_A(X_{r-})\,dM_r^f \text{ is a } \mathbb{P}_{t,x}\text{-martingale.}$$

By (4.1) and (4.2), we can write

$$\begin{aligned}N_s&=\sum_{i=1}^d\int_t^s 1_A(X_{r-})\partial_i f(X_{r-})\,dX_r^i+\sum_{t<r\le s}1_A(X_{r-})\beta_r(f)\\&\quad+\frac{1}{2}\sum_{i,j=1}^d\int_t^s 1_A(X_{r-})\partial_{ij}^2 f(X_{r-})\,d\langle (X^i)^c,(X^j)^c\rangle_r\\&\quad-\int_t^s 1_A(X_r)\mathscr{L}_r^{a,b} f(X_r)\,dr.\end{aligned}$$

Since $f(x)=\partial_i f(x)=\partial_{ij}^2 f(x)=0$ for $x\in A$, we further have

$$\begin{aligned}N_s&=\sum_{t<r\le s}1_A(X_{r-})f(X_r)-\int_t^s 1_A(X_r)a(r,X_r)\Delta^{1/2}f(X_r)\,dr\\&=\sum_{t<r\le s}1_A(X_{r-})f(X_r)-\int_t^s 1_A(X_r)\int_{\mathbb{R}^d} f(z)J(r,X_r,z)\,dz\,dr.\end{aligned}$$

Letting $f_n\to 1_B$, we obtain the desired result. ∎

In particular, Lemma 4.1 implies that

$$\mathbb{E}_{t,x}\Big[\sum_{t<r\le s} 1_A(X_{r-})1_B(X_r)\Big] = \mathbb{E}_{t,x}\Big[\int_t^s\int_{\mathbb{R}^d} 1_A(X_r)1_B(z)J(r,X_r,z)\,dz\,dr\Big].$$

Let f be a nonnegative measurable function f on $\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d$ that vanishes along the diagonal. By a routine measure-theoretic argument, we get

$$\mathbb{E}_{t,x}\Big[\sum_{t<r\le s} f(r,X_{r-},X_r)\Big] = \mathbb{E}_{t,x}\Big[\int_t^s\int_{\mathbb{R}^d} f(r,X_r,z)J(r,X_r,z)\,dz\,dr\Big].$$

Finally, we can follow the same method as in [9] to get

Lemma 4.2. *Let f be a nonnegative measurable function on $\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d$ that vanishes along the diagonal. Then for every stopping time T (with respect to the filtration of X), we have*

$$\mathbb{E}_{t,x}\Big[\sum_{t<r\le T} f(r,X_{r-},X_r)\Big] = \mathbb{E}_{t,x}\Big[\int_t^T\int_{\mathbb{R}^d} f(r,X_r,z)J(r,X_r,z)\,dz\,dr\Big]. \tag{4.3}$$

For any Borel set A, let

$$\sigma_A^t := \inf\{s\ge t: X_s\in A\},\qquad \tau_A^t := \inf\{s\ge t: X_s\notin A\},$$

be the hitting and exit time, respectively, of A. We need the following two lemmas.

Lemma 4.3. *There exists a constant $\lambda_0\in(0,1/2)$ such that for all $\delta>0$,*

$$\sup_{t,x}\mathbb{P}_{t,x}(\tau^t_{B(x,\delta)}\le t+\lambda_0\delta)\le 1/2. \tag{4.4}$$

Proof. Let f be a nonnegative smooth function on $\mathbb{R}^d$ with

$$f(0)=0\quad\text{and}\quad f(y)=1\quad\text{for } |y|>1.$$

For fixed $\delta>0$, $t\ge 0$ and $x\in\mathbb{R}^d$, set

$$f_x^\delta(y) := \delta f((x-y)/\delta).$$

Since $\mathbb{P}_{t,x}$ solves the martingale problem and $f_x^\delta(x)=0$, by (4.1) we have

$$\mathbb{E}_{t,x}(f_x^\delta(X_{(t+\lambda_0\delta)\wedge\tau^t_{B(x,\delta)}})) = \mathbb{E}_{t,x}\Big(\int_t^{(t+\lambda_0\delta)\wedge\tau^t_{B(x,\delta)}} \mathscr{L}_r^{a,b}f_x^\delta(X_r)\,dr\Big). \tag{4.5}$$

On the other hand, by the definition of $\mathscr{L}_r^{a,b}$ and (3.4), we have

$$\begin{aligned}|\mathscr{L}_r^{a,b}f_x^\delta(y)| \le \|a\|_\infty &\int_{|z|\le\delta}(f_x^\delta(y+z)-f_x^\delta(y)-z\cdot\nabla f_x^\delta(y))|z|^{-d-1}\,dz\\ &+\|a\|_\infty\int_{|z|>\delta}(f_x^\delta(y+z)-f_x^\delta(y))|z|^{-d-1}\,dz+\|\nabla f_x^\delta\|_\infty\|b\|_\infty\end{aligned}$$

$$\le \|a\|_\infty\Big[\|\nabla^2 f_x^\delta\|_\infty \int_{|z|\le\delta} |z|^{1-d}\,dz + 2\|f_x^\delta\|_\infty \int_{|z|>\delta} |z|^{-d-1}\,dz\Big] + \|\nabla f_x^\delta\|_\infty\|b\|_\infty$$
$$= \|a\|_\infty\Big[\|\nabla^2 f\|_\infty \int_{|z|\le 1} |z|^{1-d}\,dz + 2\|f\|_\infty \int_{|z|>1} |z|^{-d-1}\,dz\Big] + \|\nabla f\|_\infty\|b\|_\infty =: c_0.$$

Hence, by (4.5), we obtain

$$\delta\mathbb{P}_{t,x}(\tau^t_{B(x,\delta)} \le t+\lambda_0\delta) \le \mathbb{E}_{t,x}(f_x^\delta(X_{(t+\lambda_0\delta)\wedge\tau^t_{B(x,\delta)}})) \le c_0\lambda_0\delta,$$

which gives (4.4) by choosing $\lambda_0 = 1/(2(c_0+1))$. ∎

Lemma 4.4. *Let λ_0 be as in Lemma 4.3. For all $\lambda \in (0,\lambda_0]$, there exists $c_1 = c_1(\lambda) > 0$ such that for all $\delta > 0$, $t \ge 0$ and $x, y \in \mathbb{R}^d$ with $|x-y| \ge 3\delta$,*

$$\mathbb{P}_{t,x}(\sigma^t_{B(y,\delta)} \le t+\lambda\delta) \ge \frac{c_1\delta^{d+1}}{|x-y|^{d+1}}. \tag{4.6}$$

Proof. In view of $|x-y|\ge 3\delta$, we have

$$X_s \notin B(y,\delta) \subset B(x,\delta)^c, \qquad s < \tau^t_{B(x,\delta)},$$

and

$$1_{X_{(t+\lambda\delta)\wedge\tau^t_{B(x,\delta)}} \in B(y,\delta)} = \sum_{s\le (t+\lambda\delta)\wedge\tau^t_{B(x,\delta)}} 1_{X_s\in B(y,\delta)}.$$

Thus, by (4.3), we have

$$\begin{aligned}\mathbb{P}_{t,x}(\sigma^t_{B(y,\delta)} \le t+\lambda\delta) &\ge \mathbb{P}_{t,x}(X_{(t+\lambda\delta)\wedge\tau^t_{B(x,\delta)}} \in B(y,\delta))\\ &= \mathbb{E}_{t,x}\int_t^{(t+\lambda\delta)\wedge\tau^t_{B(x,\delta)}}\int_{B(y,\delta)} J(r,X_r,z)\,dz\,dr\\ &\overset{(2.16)}{\ge} \mathbb{E}_{t,x}\int_t^{(t+\lambda\delta)\wedge\tau^t_{B(x,\delta)}}\int_{B(y,\delta)} \frac{a_0}{|z-X_r|^{d+1}}\,dz\,dr.\end{aligned}$$

Since $|x-y|\ge 3\delta$, for all $z\in B(y,\delta)$ and $X_r \in B(x,\delta)$ we have

$$|z-X_r| \le |z-y| + |x-y| + |X_r - x| \le 3|x-y|.$$

Thus,

$$\begin{aligned}\mathbb{P}_{t,x}(\sigma^t_{B(y,\delta)} \le t+\lambda\delta) &\ge \mathbb{E}_{t,x}\Big(\int_t^{(t+\lambda\delta)\wedge\tau^t_{B(x,\delta)}} dr\Big)\int_{B(y,\delta)} \frac{a_0}{(3|x-y|)^{d+1}}\,dz\\ &\ge \lambda\delta\mathbb{P}_{t,x}(\tau^t_{B(x,\delta)} > t+\lambda\delta)\frac{a_0\mathrm{Vol}(B(y,\delta))}{(3|x-y|)^{d+1}},\end{aligned}$$

which gives the desired result by (4.4). ∎

THEOREM 4.5. *In the situation of Theorem* 3.7, *we have*

$$p_{a,b}(t,x;s,y) \succeq \varrho_1^0(t,x;s,y). \tag{4.7}$$

Proof. First of all, by (3.1), (2.17) and (3.49), there are $c_1, c_2 > 0$ such that

$$p_{a,b}(t,x;s,y) \geq c_1\varrho_1^0(t,x;s,y) - c_2\varrho_{1+\beta}^0(t,x;s,y) - c_2\varrho_1^\beta(t,x;s,y).$$

In particular, if $s-t \leq (c_1/(4c_2))^{1/\beta}$ and $|x-y| \leq s-t$, then

$$p_{a,b}(t,x;s,y) \geq \frac{1}{2}c_1\varrho_1^0(t,x;s,y) \succeq (s-t)^{-d}. \tag{4.8}$$

Thus, for some $c_3 > 0$ and all $0 \leq t < s \leq 1$ and $|x-y| \leq s-t$,

$$p_{a,b}(t,x;s,y) \succeq (s-t)^{-d} \geq c_3\varrho_1^0(t,x;s,y). \tag{4.9}$$

In fact, if

$$s-t \leq 2(c_1/(4c_2))^{1/\beta} \quad \text{and} \quad |x-y| \leq s-t,$$

then by the C-K equation (3.54), we have

$$\begin{aligned} p_{a,b}(t,x;s,y) &= \int_{\mathbb{R}^d} p_{a,b}\big(t,x;\tfrac{t+s}{2},z\big)p_{a,b}\big(\tfrac{t+s}{2},z;s,y\big)\,dz \\ &\geq \int_{B((x+y)/2,(s-t)/2)} p_{a,b}\big(t,x;\tfrac{t+s}{2},z\big)p_{a,b}\big(\tfrac{t+s}{2},z;s,y\big)\,dz \\ &\overset{(4.8)}{\succeq} (s-t)^{-2d}\operatorname{Vol}\big(B\big(\tfrac{x+y}{2},\tfrac{s-t}{2}\big)\big) \succeq (s-t)^{-d}. \end{aligned}$$

Using the above estimate repeatedly, we obtain (4.9).

Now, we assume

$$|x-y| \geq s-t =: 3\delta.$$

Let λ_0 be as in Lemma 4.3. By the strong Markov property of X,

$$\begin{aligned} &\mathbb{P}_{t,x}(X_{t+2\lambda_0\delta} \in B(y,2\delta)) \\ &\quad\geq \mathbb{P}_{t,x}\Big(\sigma := \sigma^t_{B(y,\delta)} \leq t+\lambda_0\delta;\ \sup_{s\in[\sigma,\sigma+\lambda_0\delta]}|X_s - X_\sigma| < \delta\Big) \\ &\quad= \mathbb{E}_{t,x}\Big(\mathbb{P}_{r,z}\Big(\sup_{s\in[r,r+\lambda_0\delta]}|X_s - z| < \delta\Big)\Big|_{(r,z)=(\sigma,X_\sigma)};\sigma^t_{B(y,\delta)} \leq t+\lambda_0\delta\Big) \\ &\quad\geq \inf_{r,z}\mathbb{P}_{r,z}(\tau^r_{B(z,\delta)} > r+\lambda_0\delta)\mathbb{P}_{t,x}(\sigma^t_{B(y,\delta)} \leq t+\lambda_0\delta) \\ &\quad\overset{(4.4)}{\geq} \frac{1}{2}\mathbb{P}_{t,x}(\sigma^t_{B(y,\delta)} \leq t+\lambda_0\delta) \overset{(4.6)}{\geq} \frac{c_1\delta^{d+1}}{2|x-y|^{d+1}}. \end{aligned}$$

Hence, by (4.9), we have

$$\begin{aligned} p_{a,b}(t,x;s,y) &\geq \int\limits_{B(y,2\delta)} p_{a,b}(t,x;t+2\lambda_0\delta,z)p_{a,b}(t+2\lambda_0\delta,z;s,y)\,dz \\ &\geq \inf_{z\in B(y,2\delta)} p_{a,b}(t+2\lambda_0\delta,z;s,y)\mathbb{P}_{t,x}(X_{t+2\lambda_0\delta}\in B(y,2\delta)) \\ &\succeq (s-t)^{-d}\cdot\frac{\delta^{d+1}}{|x-y|^{d+1}} \succeq \varrho_1^0(t,x;s,y), \end{aligned}$$

which together with (4.9) yields the desired lower bound. ∎

5. Proof of Theorem 1.1. By Duhamel's formula, we construct the heat kernel $p(t,x;s,y)$ of $\mathscr{L}_{t,x}$ by solving the following integral equation:

$$p(t,x;s,y) = p_{a,b}(t,x;s,y) + \int_t^s\int_{\mathbb{R}^d} p_{a,b}(t,x;r,z)c(r,z)p(r,z;s,y)\,dz\,dr. \tag{5.1}$$

For $0\le t<s$ and $x,y\in\mathbb{R}^d$, set $\Theta_0(t,x;s,y) := p_{a,b}(t,x;s,y)$, and define recursively, for $n\in\mathbb{N}$,

$$\Theta_n(t,x;s,y) := \int_t^s\int_{\mathbb{R}^d} p_{a,b}(t,x;r,z)c(r,z)\Theta_{n-1}(r,z;s,y)\,dz\,dr. \tag{5.2}$$

For $\gamma\in(0,1]$ and $c\in\mathbb{K}_d^\gamma$, define

$$\ell_\gamma^c(\varepsilon) := \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d}\int_0^\varepsilon\int_{\mathbb{R}^d}\varrho_\gamma^0(s,z)\big(|c(s-t,x-z)|+|c(t+s,x+z)|\big)\,dz\,ds.$$

LEMMA 5.1. *If $c\in\mathbb{K}_d^1$, then there exists a constant $\Lambda>0$ such that for all $n\in\mathbb{N}$,*

$$|\Theta_n(t,x;s,y)| \le \{\Lambda\ell_1^c(s-t)\}^n\varrho_1^0(t,x;s,y). \tag{5.3}$$

If $c\in\mathbb{K}_d^{1-\gamma}$ for some $\gamma\in(0,1)$, then there exists a constant $C_1>0$ such that for any $n\in\mathbb{N}$,

$$\begin{aligned} |\Theta_n(t,x;s,y)-\Theta_n(t,x';s,y)| &\le C_1(|x-x'|^\gamma\wedge 1)\{\Lambda\ell_1^c(s-t)\}^{n-1}\ell_{1-\gamma}^c(s-t) \\ &\quad\times(\varrho_1^0(t,x;s,y)+\varrho_1^0(t,x';s,y)). \end{aligned} \tag{5.4}$$

If $c\in\mathbb{H}^\gamma$ for some $\gamma\in(0,1)$, then there exists a constant $C_2>0$ such that for any $n\in\mathbb{N}$,

$$|\nabla_x\Theta_n(t,x;s,y)| \le C_2\{\Lambda\|c\|_\infty(s-t)\}^n\varrho_0^0(t,x;s,y). \tag{5.5}$$

Proof. **(1)** First of all, by (3.41), for some $C_0>0$,

$$p_{a,b}(t,x;s,y)\le C_0\varrho_1^0(t,x;s,y).$$

Now, we use induction to prove (5.3). Suppose that (5.3) is true for $n \in \mathbb{N}$. Then

$$
\begin{aligned}
|\Theta_{n+1}(t,x;s,y)| &\le \int_t^s \int_{\mathbb{R}^d} p_{a,b}(t,x;r,z)|c(r,z)| \cdot |\Theta_n(r,z;s,y)|\,dz\,dr \\
&\le C_0\{\Lambda \ell_1^c(s-t)\}^n \int_t^s \int_{\mathbb{R}^d} \varrho_1^0(t,x;r,z)\varrho_1^0(r,z;s,y)|c(r,z)|\,dz\,dr \\
&\overset{(2.3)}{\le} \Lambda\{\Lambda \ell_1^c(s-t)\}^n \int_t^s \int_{\mathbb{R}^d} (\varrho_1^0(t,x;r,z) + \varrho_1^0(r,z;s,y))|c(r,z)|\,dz\,dr\,\varrho_1^0(t,x;s,y) \\
&\le \{\Lambda \ell_1^c(s-t)\}^{n+1} \varrho_1^0(t,x;s,y).
\end{aligned}
$$

(2) By (5.2) and (3.42), we have

$$
\begin{aligned}
&|\Theta_n(t,x;s,y) - \Theta_n(t,x';s,y)| \\
&\preceq (|x-x'|^\gamma \wedge 1) \int_t^s \int_{\mathbb{R}^d} \big(\varrho_{1-\gamma}^0(t,x;r,z) + \varrho_{1-\gamma}^0(t,x';r,z)\big) \\
&\qquad\qquad \times |c(r,z)| \cdot |\Theta_{n-1}(r,z;s,y)|\,dz\,dr \\
&\overset{(5.3)}{\preceq} (|x-x'|^\gamma \wedge 1)\{\Lambda \ell_1^c(s-t)\}^{n-1} \\
&\qquad \times \int_t^s \int_{\mathbb{R}^d} \big(\varrho_{1-\gamma}^0(t,x;r,z) + \varrho_{1-\gamma}^0(t,x';r,z)\big)|c(r,z)|\varrho_1^0(r,z;s,y)|\,dz\,dr \\
&\overset{(2.7)}{\preceq} (|x-x'|^\gamma \wedge 1)\{\Lambda \ell_1^c(s-t)\}^{n-1} \\
&\times \Big\{ \int_t^s \int_{\mathbb{R}^d} (r-t)^{1-\gamma}(s-r)\big(\varrho_0^0(t,x;r,z) + \varrho_0^0(r,z;s,y)\big)|c(r,z)|\,dz\,dr\,\rho_0^0(t,x;s,y) \\
&+ \int_t^s \int_{\mathbb{R}^d} (r-t)^{1-\gamma}(s-r)\big(\varrho_0^0(t,x';r,z) + \varrho_0^0(r,z;s,y)\big)|c(r,z)|\,dz\,dr\,\rho_0^0(t,x';s,y) \Big\} \\
&\le C_1(|x-x'|^\gamma \wedge 1)\{\Lambda \ell_1^c(s-t)\}^{n-1} \\
&\qquad \times \Big\{ \int_t^s \int_{\mathbb{R}^d} \big(\varrho_{1-\gamma}^0(t,x;r,z) + \varrho_{1-\gamma}^0(r,z;s,y)\big)|c(r,z)|\,dz\,dr\,\rho_1^0(t,x;s,y) \\
&+ \int_t^s \int_{\mathbb{R}^d} \big(\varrho_{1-\gamma}^0(t,x';r,z) + \varrho_{1-\gamma}^0(r,z;s,y)\big)|c(r,z)|\,dz\,dr\,\rho_1^0(t,x';s,y) \Big\} \\
&\le C_1(|x-x'|^\gamma \wedge 1)\{\Lambda \ell_1^c(s-t)\}^{n-1} \ell_{1-\gamma}^c(s-t)(\varrho_1^0(t,x;s,y) + \varrho_1^0(t,x';s,y)),
\end{aligned}
$$

and (5.4) holds.

(3) If c is bounded, then by definition and (2.2), it is easy to see that for some $C_1 > 0$,

$$\ell_\gamma^c(\varepsilon) \le C_1\|c\|_\infty \varepsilon^\gamma, \qquad \varepsilon > 0. \tag{5.6}$$

As in Lemma 3.5, one can prove

$$\nabla_x \Theta_n(t,x;s,y) = \int_t^s \int_{\mathbb{R}^d} \nabla_x p_{a,b}(t,x;r,z) c(r,z) \Theta_{n-1}(r,z;s,y)\,dz\,dr.$$

By (3.52), we can write

$$\begin{aligned}
&\nabla_x \Theta_n(t,x;s,y)\\
&\quad= \int_t^{(t+s)/2} \int_{\mathbb{R}^d} \nabla_x p_{a,b}(t,x;r,z)\\
&\qquad\qquad \times \big(c(r,z)\Theta_{n-1}(r,z;s,y) - c(r,x)\Theta_{n-1}(r,x;s,y)\big)\,dz\,dr\\
&\qquad+ \int_{(t+s)/2}^s \int_{\mathbb{R}^d} \nabla_x p_{a,b}(t,x;r,z) c(r,z)\Theta_{n-1}(r,z;s,y)\,dz\,dr\\
&\quad= \int_t^{(t+s)/2} \int_{\mathbb{R}^d} \nabla_x p_{a,b}(t,x;r,z) c(r,z)\big(\Theta_{n-1}(r,z;s,y) - \Theta_{n-1}(r,x;s,y)\big)\,dz\,dr\\
&\qquad+ \int_t^{(t+s)/2} \Big(\int_{\mathbb{R}^d} \nabla_x p_{a,b}(t,x;r,z)(c(r,z)-c(r,x))\,dz\Big)\Theta_{n-1}(r,x;s,y)\,dr\\
&\qquad+ \int_{(t+s)/2}^s \int_{\mathbb{R}^d} \nabla_x p_{a,b}(t,x;r,z) c(r,z)\Theta_{n-1}(r,z;s,y)\,dz\,dr\\
&\quad=: Q_1(t,x;s,y) + Q_2(t,x;s,y) + Q_3(t,x;s,y).
\end{aligned}$$

For $Q_1(t,x;s,y)$, by (3.43), (5.6) and (5.4), we have

$$\begin{aligned}
Q_1(t,x;s,y) &\preceq \{\Lambda\|c\|_\infty(s-t)\}^{n-1} \int_t^{(t+s)/2} \int_{\mathbb{R}^d} \varrho_0^\gamma(t,x;r,z)\varrho_{1-\gamma}^0(r,z;s,y)\,dz\,dr\\
&\quad+ \{\Lambda\|c\|_\infty(s-t)\}^{n-1} \int_t^{(t+s)/2} \Big(\int_{\mathbb{R}^d} \varrho_0^\gamma(t,x;r,z)\,dz\Big)\varrho_{1-\gamma}^0(r,x;s,y)\,dr\\
&\overset{(2.4),(2.2)}{\preceq} \{\Lambda\|c\|_\infty(s-t)\}^n \varrho_0^0(t,x;s,y).
\end{aligned}$$

For $Q_2(t,x;s,y)$, by (5.6) and (5.3), we have

$$\begin{aligned} Q_2(t,x;s,y) &\preceq \{\varLambda\|c\|_\infty(s-t)\}^{n-1}\int_t^{(t+s)/2}\Big(\int_{\mathbb{R}^d}\varrho_0^\gamma(t,x;r,z)\,dz\Big)\varrho_1^0(r,x;s,y)\,dr \\ &\preceq \{\varLambda\|c\|_\infty(s-t)\}^{n-1}\Big(\int_t^{(t+s)/2}(r-t)^{\gamma-1}(s-r)\,dr\Big)\varrho_0^0(t,x;s,y) \\ &\preceq \{\varLambda\|c\|_\infty(s-t)\}^{n}\varrho_0^0(t,x;s,y). \end{aligned}$$

For $Q_3(t,x;s,y)$, we have

$$\begin{aligned} Q_3(t,x;s,y) &\preceq \{\varLambda\|c\|_\infty(s-t)\}^{n-1}\int_{(t+s)/2}^{s}\int_{\mathbb{R}^d}\varrho_0^0(t,x;r,z)\varrho_1^0(r,z;s,y)\,dz\,dr \\ &\preceq \{\varLambda\|c\|_\infty(s-t)\}^{n-1}\Big(\int_{(t+s)/2}^{s}((s-r)(r-t)^{-1}+1)\,dr\Big)\varrho_0^0(t,x;s,y) \\ &\preceq \{\varLambda\|c\|_\infty(s-t)\}^{n}\varrho_0^0(t,x;s,y). \end{aligned}$$

Combining the above, we obtain (5.5). ■

Now we are in a position to give

Proof of Theorem 1.1. By the standard time shift technique, it suffices to prove the conclusions on a small time interval. We divide the proof into several steps.

(1) Define

$$p(t,x;s,y) = p_{a,b}(t,x;s,y) + \sum_{n=1}^{\infty}\varTheta_n(t,x;s,y).$$

Since $c \in \mathbb{K}_d^1$, we have

$$\lim_{\varepsilon\downarrow 0}\ell_1^c(\varepsilon) = 0.$$

Hence, for any given $\varepsilon \in (0,1)$, one can choose $T_\varepsilon \in (0,1)$ small enough such that for all $0 \le t < s \le 1$ with $s-t \le T_\varepsilon$,

$$\ell_1^c(s-t) \le \varepsilon/\varLambda,$$

where $\varLambda$ is the constant from Lemma 5.1. Thus,

$$\begin{aligned} |p(t,x;s,y) - p_{a,b}(t,x;s,y)| &\le \sum_{n=1}^{\infty}|\varTheta_n(t,x;s,y)| \\ &\le \frac{\varLambda\ell_1^c(s-t)}{1-\varLambda\ell_1^c(s-t)}\varrho_1^0(t,x;s,y) \\ &\le \frac{\varepsilon}{1-\varepsilon}\varrho_1^0(t,x;s,y), \end{aligned}$$

which together with (3.41) and (4.7) gives (1.6) and (1.7) for all $0 \le t < s \le 1$ with $s - t \le T_\varepsilon$, provided ε small enough. Moreover, noticing that

$$\sum_{n=0}^{m} \Theta_n(t,x;s,y) = p_{a,b}(t,x;s,y) + \int_t^s \int_{\mathbb{R}^d} p_{a,b}(t,x;r,z)c(r,z) \sum_{n=0}^{m-1} \Theta_n(r,z;s,y)\,dz\,dr,$$

by taking limits, we obtain (5.1). Moreover, estimates (1.8) and (1.9) follow from (5.4), (3.42) and (5.5), (3.43).

(2) Define

$$P_{t,s}f(x) := \int_{\mathbb{R}^d} p(t,x;s,y)f(y)\,dy,$$
$$P^{a,b}_{t,s}f(x) := \int_{\mathbb{R}^d} p_{a,b}(t,x;s,y)f(y)\,dy.$$

To prove (1.2), it suffices to show that for any $f \in C_0^\infty(\mathbb{R}^d)$,

$$P_{t,s}f(x) = P_{t,r}P_{r,s}f(x), \quad t < r < s. \tag{5.7}$$

By (5.1) and (3.54), we have

$$\begin{aligned} P_{t,s}f(x) &= P^{a,b}_{t,s}f(x) + \int_t^s P^{a,b}_{t,r'}\big(c(r',\cdot)P_{r',s}f\big)(x)\,dr' \\ &= P^{a,b}_{t,r}P^{a,b}_{r,s}f(x) + \int_r^s P^{a,b}_{t,r}P^{a,b}_{r,r'}\big(c(r',\cdot)P_{r',s}f\big)(x)\,dr' \\ &\quad + \int_t^r P^{a,b}_{t,r'}\big(c(r',\cdot)P_{r',s}f\big)(x)\,dr' \\ &= P^{a,b}_{t,r}P_{r,s}f(x) + \int_t^r P^{a,b}_{t,r'}\big(c(r',\cdot)P_{r',s}f\big)(x)\,dr'. \end{aligned}$$

On the other hand,

$$P_{t,r}P_{r,s}f(x) = P^{a,b}_{t,r}P_{r,s}f(x) + \int_t^r P^{a,b}_{t,r'}\big(c(r',\cdot)P_{r',r}P_{r,s}f\big)(x)\,dr'.$$

Fix $t < r$ and set

$$u_t(x) := P_{t,r}P_{r,s}f(x) - P_{t,s}f(x).$$

Then

$$u_t(x) = \int_t^r \int_{\mathbb{R}^d} p_{a,b}(t,x;r',y)c(r',y)u_{r'}(y)\,dy\,dr'.$$

By (3.41), we have

$$\|u_t\|_\infty \le \sup_{r'\in[t,r]} \|u_{r'}\|_\infty \int_t^r \int_{\mathbb{R}^d} \varrho_1^0(t,x;r',y)|c(r',y)|\,dy\,dr' = \ell_1^c(r-t) \sup_{r'\in[t,r]} \|u_{r'}\|_\infty,$$

which implies that

$$\sup_{r'\in[t,r]} \|u_{r'}\|_\infty \le \sup_{\varepsilon\in(0,r-t]} \ell_1^c(\varepsilon) \sup_{r'\in[t,r]} \|u_{r'}\|_\infty.$$

In particular, if $r-t$ is small enough (say less than ε_0), then

$$\sup_{r'\in[t,r]} \|u_{r'}\|_\infty = 0.$$

Thus, we obtain (5.7) for $r-t<\varepsilon_0$. For general t, we use the same argument repeatedly.

(3) We now prove (1.3). By (5.1) and (3.44), we only need to prove that for any $f \in C_b(\mathbb{R}^d)$,

$$\lim_{t\to s} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \int_t^s \int_{\mathbb{R}^d} p_{a,b}(t,x;r,z) c(r,z) p(r,z;s,y) f(y)\,dz\,dr\,dy \right| = 0.$$

This follows by noticing that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \int_t^s \int_{\mathbb{R}^d} p_{a,b}(t,x;r,z) c(r,z) p(r,z;s,y) f(y)\,dz\,dr\,dy \right| \\
&\quad \preceq \int_{\mathbb{R}^d} \int_t^s \int_{\mathbb{R}^d} \varrho_1^0(t,x;r,z)|c(r,z)|\varrho_1^0(r,z;s,y)|f(y)|\,dz\,dr\,dy \\
&\quad \preceq \int_{\mathbb{R}^d} \Big(\int_t^s \int_{\mathbb{R}^d} \big(\varrho_1^0(t,x;r,z) + \varrho_1^0(r,z;s,y)\big) c(r,z)\,dz\,dr \Big) \varrho_1^0(t,x;s,y)\,dy \\
&\quad \preceq \ell_1^c(|s-t|) \int_{\mathbb{R}^d} \varrho_1^0(t,x;s,y)\,dy \overset{(2.2)}{\le} C\ell_1^c(|s-t|) \to 0, \qquad t\to s.
\end{aligned}$$

(4) We now prove (1.4). Let $f, g \in C_c^2(\mathbb{R}^d)$. By (5.1), we make the following decomposition:

$$\frac{P_{t,s}f(x) - f(x)}{s-t} - \mathscr{L}_{s,x} f(x) = \frac{1}{s-t} \int_t^s \big(P_{t,r}^{a,b}(c(r,\cdot)P_{r,s}f)(x) - c(r,x)P_{r,s}f(x)\big)\,dr$$

$$
\begin{aligned}
&+ \frac{1}{s-t}\int_t^s \big(c(r,x)-c(s,x)\big)P_{r,s}f(x)\,dr \\
&+ \frac{1}{s-t}\int_t^s c(s,x)\big(P_{r,s}f(x)-f(x)\big)\,dr \\
&+ \left(\frac{P^{a,b}_{t,s}f(x)-f(x)}{s-t} - \mathscr{L}^{a,b}_{s,x}f(x)\right) \\
&=: I_1(t,s,x)+I_2(t,s,x)+I_3(t,s,x)+I_4(t,s,x).
\end{aligned}
$$

For $I_1(t,s,x)$, if we write

$$
(P^{a,b}_{t,r})^* g(y) := \int_{\mathbb{R}^d} p_{a,b}(t,x;r,y)g(x)\,dx,
$$

then

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^d} g(x)I_1(t,s,x)\,dx\right| \\
&\quad \le \left|\frac{1}{s-t}\int_t^s\int_{\mathbb{R}^d}\big((P^{a,b}_{t,r})^*g(x)-(P^{a,b}_{t,r})^*1(x)\cdot g(x)\big)c(r,x)P_{r,s}f(x)\,dx\,dr\right| \\
&\qquad + \left|\frac{1}{s-t}\int_t^s\int_{\mathbb{R}^d}\big((P^{a,b}_{t,r})^*1-1\big)(x)g(x)c(r,x)P_{r,s}f(x)\,dx\,dr\right| \\
&\quad =: J_1(t,s)+J_2(t,s).
\end{aligned}
$$

For $J_1(t,s)$, noticing that

$$
\begin{aligned}
|(P^{a,b}_{t,r})^*g(y)-(P^{a,b}_{t,r})^*1(y)\cdot g(y)| &= \left|\int_{\mathbb{R}^d} p_{a,b}(t,x;r,y)(g(x)-g(y))\,dx\right| \\
&\overset{(3.41)}{\le} C\|g\|_{\mathbb{H}^1}\int_{\mathbb{R}^d}\varrho^0_1(t,x;r,y)(|x-y|\wedge 1)\,dx \\
&\overset{(2.2)}{\le} C\|g\|_{\mathbb{H}^1}|r-t|,
\end{aligned}
$$

by the definition of $P_{r,s}f$ and (1.6), we have

$$
\begin{aligned}
J_1(t,s) &\le C\|g\|_{\mathbb{H}^1}\int_t^s\int_{\mathbb{R}^d}|c(r,x)|\cdot|P_{r,s}f(x)|\,dx\,dr \\
&\le C\|g\|_{\mathbb{H}^1}\int_t^s\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|c(r,x)|\varrho^0_1(r,x;s,y)|f(y)|\,dy\,dx\,dr \\
&\le C\|g\|_{\mathbb{H}^1}\ell^c_1(s-t)\int_{\mathbb{R}^d}|f(y)|\,dy \to 0, \qquad t\uparrow s.
\end{aligned}
$$

For $J_2(t,s)$, since $c \in C([0,\infty); L^1_{\rm loc}(\mathbb{R}^d))$, by (3.45) and the dominated convergence theorem, we have

$$\lim_{t\uparrow s} J_2(t,s) = 0.$$

For the same reason,

$$\lim_{t\uparrow s} \int_{\mathbb{R}^d} g(x)(I_2(t,s,x) + I_3(t,s,x))\, dx = 0.$$

Moreover, by (3.46), for almost all $s > 0$,

$$\lim_{t\uparrow s} \int_{\mathbb{R}^d} g(x) I_4(t,s,x)\, dx = 0.$$

Combining the above limits, we obtain (1.4). The limit (1.5) is similar. ∎

Acknowledgements. The authors would like to thank Professors Zhen-Qing Chen, Renming Song and Feng-Yu Wang for useful conversations. The referee's helpful suggestions are also acknowledged. This work is supported by NSF of China (Nos. 11271294, 11325105), Program for New Century Excellent Talents in University (NCET-10-0654) and the Fundamental Research Funds for the Central Universities (No. 2014201020208).

References

[1] M. Aizenman and B. Simon, *Brownian motion and Harnack inequaity for Schrödinger operators*, Comm. Pure Appl. Math. 35 (1982), 209–273.

[2] B. Barrios, E. Colorado, A. de Pablo and U. Sánchez, *On some critical problems for the fractional Laplacian operator*, J. Differential Equations 252 (2012), 6133–6162.

[3] K. Bogdan and T. Jakubowski, *Estimates of heat kernel of fractional Laplacian perturbed by gradient operators*, Comm. Math. Phys. 271 (2007), 179–198.

[4] K. Bogdan and T. Jakubowski, *Estimates of Green function for the fractional Laplacian perturbed by gradient*, Potential Anal. 36 (2012), 455–481.

[5] L. Caffarelli and A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasigeostrophic equation*, Ann. of Math. 171 (2010), 1903–1930.

[6] Z. Q. Chen, P. Kim and R. Song, *Dirichlet heat kernel estimates for fractional Laplacian with gradient perturbation*, Ann. Probab. 40 2012, 2483–2538.

[7] Z. Q. Chen, P. Kim and R. Song, *Stability of Dirichlet heat kernel estimates for non-local operators under Feynman–Kac perturbation*, arXiv:1112.3401v1 (2011).

[8] Z. Q. Chen and T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stoch. Process. Appl. 108 (2003), 27–62.

[9] Z. Q. Chen and T. Kumagai, *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probab. Theory Related Fields 140 (2008), 277–317.

[10] Z. Q. Chen and R. Song, *Estimates on Green functions and Poisson kernels for symmetric stable processes*, Math. Ann. 312 (1998), 465–501.

[11] Z. Q. Chen and J. Wang, *Perturbation by non-local operators*, arXiv:1312.7594 (2013).

[12] Z. Q. Chen and X. Zhang, *Heat kernels and analyticity of non-symmetric jump diffusion semigroups*, arXiv:1306.5015 (2013).

[13] S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization and Convergence*, Wiley, New York, 1986.

[14] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1975.

[15] T. Jakubowski, *Fractional Laplacian with singular drift*, Studia Math. 207 (2011), 257–273.

[16] T. Jakubowski and K. Szczypkowski, *Estimates of gradient perturbation series*, J. Math. Anal. Appl. 389 (2012), 452–460.

[17] T. Jakubowski and K. Szczypkowski, *Time-dependent gradient perturbations of fractional Laplacian*, J. Evolution Equations 10 (2010), 319–339.

[18] A. Kiselev, F. Nazarov and R. Shterenberg, *Blow up and regularity for fractal Burgers equation*, Dynam. Partial Differential Equations 5 (2008), 211–240.

[19] A. Kiselev, F. Nazarov and A. Volberg, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation*, Invent. Math. 167 (2007), 445–453.

[20] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, Amer. Math. Soc., 1968.

[21] Y. Maekawa and H. Miura, *Upper bounds for fundamental solutions to non-local diffusion equations with divergence free drift*, J. Funct. Anal. 264 (2013), 2245–2268.

[22] E. Priola, *Pathwise uniqueness for singular SDEs driven by stable processes*, Osaka J. Math. 49 (2012), 421–447.

[23] L. Silvestre, *On the differentiability of the solution to the Hamilton–Jacobi equation with critical fractional diffusion*, Adv. Math. 226 (2011), 2020–2039.

[24] L. Silvestre, *Hölder estimates for advection fractional-diffusion equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 11 (2012), 843–855.

[25] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[26] F. Y. Wang and X. Zhang, *Heat kernel for fractional diffusion operators with perturbations*, Forum Math., to appear; arXiv:1204.4956 (2014).

[27] K. Yosida, *Functional Analysis*, Springer, Berlin, 1995.

[28] Q. Zhang, *A Harnack inequality for the equation* $\nabla(a\nabla u) + b\nabla u = 0$, *when* $|b| \in K_{n+1}$, Manuscripta Math. 89 (1995), 61–77.

[29] X. Zhang, L^p*-maximal regularity of nonlocal parabolic equation and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire 30 (2013), 573–614.

[30] X. Zhang, *Well-posedness of fully nonlinear and nonlocal critical parabolic equations*, J. Evolution Equations 13 (2013), 135–162.

Longjie Xie, Xicheng Zhang
School of Mathematics and Statistics
Wuhan University
Wuhan, Hubei 430072, P.R. China
E-mail: xlj.98@whu.edu.cn
XichengZhang@gmail.com

Received January 30, 2014
Revised version December 8, 2014 (7907)