STUDIA MATHEMATICA 224 (3) (2014)

Estimates for oscillatory singular integrals on Hardy spaces

by

HUSSAIN AL-QASSEM (Doha), LESLIE CHENG (Bryn Mawr, PA)
and YIBIAO PAN (Pittsburgh, PA)

Abstract. For any $n \in \mathbb{N}$, we obtain a bound for oscillatory singular integral operators with polynomial phases on the Hardy space $H^1(\mathbb{R}^n)$. Our estimate, expressed in terms of the coefficients of the phase polynomial, establishes the H^1 boundedness of such operators in all dimensions when the degree of the phase polynomial is greater than one. It also subsumes a uniform boundedness result of Hu and Pan (1992) for phase polynomials which do not contain any linear terms. Furthermore, the bound is shown to be valid on weighted Hardy spaces as well if the weights belong to the Muckenhoupt class A_1.

1. Introduction. Let $n \in \mathbb{N}$. Consider the following oscillatory singular integral operator:

$$T_P : f \mapsto \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x-y)} K(x-y) f(y)\, dy, \tag{1}$$

where P is a polynomial in n variables with real coefficients and K is a Calderón–Zygmund kernel (see Definition 2.3). Because the focus of our investigation is on the $H^1 \to H^1$ boundedness, operators studied in this paper are of convolution type.

The operators given in (1) are known to be bounded on L^p spaces $(1 < p < \infty)$ and of weak type $(1,1)$, thanks to the work of Ricci–Stein [5] and Chanillo–Christ [1]. Additionally, the $L^p \to L^p$ and $L^1 \to L^{1,\infty}$ bounds obtained in [5] and [1] are dependent on the degree of the phase polynomial only, and not on its coefficients.

On the other hand, the picture for the corresponding $H^1 \to H^1$ problem has not been as clear. First of all, when P is a polynomial of degree one, the operator T_P is generally not bounded on $H^1(\mathbb{R}^n)$ (see [4], [3]). Yet, the following are known to be true:

2010 *Mathematics Subject Classification*: Primary 42B20; Secondary 42B30.
Key words and phrases: Hardy spaces, oscillatory integrals, singular integrals.

DOI: 10.4064/sm224-3-5

THEOREM 1.1.

(i) *Let K be a Calderón–Zygmund convolution kernel on $\mathbb{R}^n$, and P a polynomial in n variables with real coefficients with $\nabla P(0) = 0$. Then T_P is bounded on $H^1(\mathbb{R}^n)$. Moreover, the following uniform boundedness holds for each $m \in \mathbb{N}$:*

$$\sup\{\|T_P\|_{H^1(\mathbb{R}^n)\to H^1(\mathbb{R}^n)} : \nabla P(0) = 0 \text{ and } \deg(P) \le m\} < \infty. \tag{2}$$

(ii) *If $n = 1$, $K(x) = 1/x$, and P is a real polynomial of a single variable with $\deg(P) \ge 2$, then T_P is bounded on $H^1(\mathbb{R})$.*

Theorem 1.1(i) was proved in [4]. Theorem 1.1(ii), in which the condition $P'(0) = 0$ is not imposed, is a consequence of [3, Theorem 1.2] which deals with the class of rational phase functions of a single variable. By comments preceding Theorem 1.1, it is obvious that the bound on $\|T_P\|_{H^1(\mathbb{R})\to H^1(\mathbb{R})}$ in (ii) cannot be independent of the coefficients of P.

In light of (i) and (ii), one is naturally led to the following question: for $n > 1$, is T_P always bounded on $H^1(\mathbb{R}^n)$ if $\deg(P) \ge 2$?

In the theorem below we provide an estimate which not only answers the above question in the affirmative, but also reveals the difference between the roles played by linear and nonlinear terms of the phase polynomial.

THEOREM 1.2. *Let $n \in \mathbb{N}$, $m \ge 2$ and $P(x) = \sum_{0\le|\alpha|\le m} a_\alpha x^\alpha$ be a polynomial of degree m in $\mathbb{R}^n$ with real coefficients. Let K be a Calderón–Zygmund kernel and T_P be given as in* (1). *Then there exists a positive constant C such that*

$$\|T_P f\|_{H^1(\mathbb{R}^n)} \le C\left(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\le|\alpha|\le m} |a_\alpha|^{1/|\alpha|}}\right)\|f\|_{H^1(\mathbb{R}^n)} \tag{3}$$

for all $f \in H^1(\mathbb{R}^n)$. The constant C may depend on n, m and K, but is independent of the coefficients $\{a_\alpha\}$ of P.

REMARKS. (a) When $\nabla P(0) = 0$, we have $a_\alpha = 0$ for all $|\alpha| = 1$. In this case, (3) recovers the result in [4].

(b) In general, (3) shows that even when $\nabla P(0) \ne 0$, it is possible to control $\|T_P\|_{H^1\to H^1}$ as long as the coefficients of the first order terms in $P(x)$ are not too large relative to those of the higher order terms.

(c) It is not known whether the bound

$$1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\le|\alpha|\le m} |a_\alpha|^{1/|\alpha|}} = \frac{\sum_{1\le|\alpha|\le m} |a_\alpha|^{1/|\alpha|}}{\sum_{2\le|\alpha|\le m} |a_\alpha|^{1/|\alpha|}}$$

in (3) is the best possible. A logarithmic lower bound will be established at the end of the paper.

2. Weighted Hardy spaces. Theorem 1.2 admits an extension to the setting of weighted Hardy spaces with A_1 weights. We will present it after recalling some definitions.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$, and $|B(x,r)|$ denotes the Euclidean volume of $B(x,r)$. For a weight function w, we let

$$w(B(x,r)) = \int_{B(x,r)} w(y)\, dy.$$

Definition 2.1. A locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be in the *Muckenhoupt weight class* $A_1(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that

$$\frac{1}{|B|}\int_B w(y)\, dy \le Cw(x) \tag{4}$$

for all balls B and a.e. $x \in B$. The smallest constant C in (4) is called the A_1 *constant* of w.

Let ϕ be a function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$. For each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we let

$$M_\phi f(x) = \sup_{s>0} |(f * \phi_s)(x)|$$

where $\phi_s(x) = s^{-n}\phi(x/s)$.

Definition 2.2. For a nonnegative, locally integrable function w on $\mathbb{R}^n$, we define the *weighted Hardy space* $H^1_w(\mathbb{R}^n)$ by

$$H^1_w(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|M_\phi f\|_{L^1_w} < \infty\}$$

and we set $\|f\|_{H^1_w(\mathbb{R}^n)} = \|M_\phi f\|_{L^1_w} = \int_{\mathbb{R}^n} M_\phi f(x) w(x)\, dx$.

Definition 2.3. A C^1 function $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called a *Calderón–Zygmund kernel* if the following are true:

(i) There exists a $C > 0$ such that

$$|K(x)| + |x|\,|\nabla K(x)| \le C|x|^{-n} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{5}$$

(ii) For all $b > a > 0$,

$$\int_{a<|x|<b} K(x)\, dx = 0. \tag{6}$$

We are now ready to state the weighted version of Theorem 1.2.

Theorem 2.1. *Let $n \in \mathbb{N}$, $m \ge 2$, $w \in A_1(\mathbb{R}^n)$ and*

$$P(x) = \sum_{0\le|\alpha|\le m} a_\alpha x^\alpha$$

be a polynomial of degree m in $\mathbb{R}^n$ with real coefficients. Let K be a Calderón–Zygmund kernel and T_P be as in (1). *Then there exists a positive constant C such that*

$$\|T_P f\|_{H^1_w(\mathbb{R}^n)} \le C\Big(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\le|\alpha|\le m} |a_\alpha|^{1/|\alpha|}}\Big)\|f\|_{H^1_w(\mathbb{R}^n)} \tag{7}$$

for all $f \in H^1_w(\mathbb{R}^n)$. The constant C may depend on n, m, K and the A_1 constant of w, but is independent of the coefficients $\{a_\alpha\}$ of P.

3. Proof of Theorem 2.1. We shall let C denote a constant whose value may change from line to line. The constant may depend on the dimension n, the degree of the phase polynomial, the A_1 bound of a given weight, but is independent of the coefficients of the phase polynomial.

LEMMA 3.1. *Let $m \ge 2$ and $P(x) = \sum_{0\le|\alpha|\le m} a_\alpha x^\alpha$ be a polynomial of degree m in $\mathbb{R}^n$ with real coefficients. For $j \in \mathbb{N}$, define the operator $S_{P,j}$ by*

$$(S_{P,j} f)(x) = \chi_{[2^j,2^{j+1})}(|x|) \int\limits_{B(0,1)} e^{iP(x-y)} f(y)\, dy. \tag{8}$$

Then, for $2 \le p < \infty$, there exists a $C_p = C(n, m, p) > 0$ such that

$$\|S_{P,j}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le C_p 2^{\frac{j(2n-1)}{2p}} \Big(\sum_{|\alpha|=m} |a_\alpha|\Big)^{-\frac{1}{2p(m-1)}}. \tag{9}$$

Proof. Let β be a multi-index such that $|\beta| = m$ and $|a_\beta| = \max\{|a_\alpha| : |\alpha| = m\}$. By [4, Lemma 4.3], we have

$$\|S_{P,j}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C2^{jn/2}(|a_\beta|2^{j(m-1)})^{-\frac{1}{4(m-1)}}. \tag{10}$$

From

$$|a_\beta| \ge \binom{m+n-1}{n-1}^{-1} \Big(\sum_{|\alpha|=m} |a_\alpha|\Big),$$

we obtain

$$\|S_{P,j}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C2^{j(2n-1)/4}\Big(\sum_{|\alpha|=m} |a_\alpha|\Big)^{-\frac{1}{4(m-1)}}. \tag{11}$$

Now (9) follows by interpolation between (11) and

$$\|S_{P,j}\|_{L^\infty(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \le |B(0, 1)|. \ \blacksquare$$

Let $w \in A_1(\mathbb{R}^n)$. Recall that a measurable function g on $\mathbb{R}^n$ is called an H^1_w *atom* if there exist $\zeta \in \mathbb{R}^n$ and $r > 0$ such that

$$\operatorname{supp}(g) \subseteq B(\zeta, r), \tag{12}$$

$$\|g\|_\infty \le \frac{1}{w(B(\zeta, r))}, \tag{13}$$

$$\int\limits_{\mathbb{R}^n} g(y)\, dy = 0. \tag{14}$$

LEMMA 3.2. *Let P, K be as in Theorem 2.1, $w \in A_1(\mathbb{R}^n)$, and $g(\cdot)$ be a function which satisfies (12)–(13). Then there exist $C, \theta > 0$ such that*

$$\int_{(B(\zeta,h))^c} |Tg(x)|w(x)\,dx \le C\Big(1 + \Big(\sum_{|\alpha|=m} |a_\alpha| r h^{m-1}\Big)^{-\theta}\Big) \tag{15}$$

for all $h \ge 2r$. The constants C and θ may depend on n, m, K and the A_1 constant of w, but are independent of $\{a_\alpha\}$, ζ, r and h.

Proof. By a result in [2], there exists a $\delta \in (0,1)$ such that $w^{1+\delta} \in A_1(\mathbb{R}^n)$. Both δ and the A_1 constant of $w^{1+\delta}$ depend on the A_1 constant of w only. Let

$$\theta = \frac{\delta}{2(1+\delta)(m-1)}.$$

Let $h \ge 2r$ and write

$$\int_{(B(\zeta,h))^c} |Tg(x)|w(x)\,dx = I_1 + I_2,$$

where

$$I_1 = \int_{(B(\zeta,h))^c} \Big| \int_{B(\zeta,r)} e^{iP(x-y)} g(y)\,dy \Big| |K(x-\zeta)| w(x)\,dx,$$

$$I_2 = \int_{(B(\zeta,h))^c} \Big| \int_{B(\zeta,r)} e^{iP(x-y)} (K(x-y) - K(x-\zeta)) g(y)\,dy \Big| w(x)\,dx.$$

When $x \in (B(\zeta,h))^c$ and $y \in B(\zeta,r)$, by (5) we have

$$|K(x-y) - K(x-\zeta)| \le \frac{C|y-\zeta|}{|x-\zeta|^{n+1}}.$$

Thus,

$$\begin{aligned} I_2 &\le \frac{Cr|B(\zeta,r)|}{w(B(\zeta,r))} \int_{|x-\zeta|\ge 2r} \frac{w(x)\,dx}{|x-\zeta|^{n+1}} \le \frac{Cr|B(\zeta,r)|}{w(B(\zeta,r))} \sum_{j=2}^{\infty} \int_{B(\zeta,2^{j+1}r)} \frac{w(x)\,dx}{(2^j r)^{n+1}} \\ &\le \frac{Cr|B(\zeta,r)|}{w(B(\zeta,r))} \sum_{j=2}^{\infty} \frac{|B(\zeta,2^{j+1}r)| w(B(\zeta,r))}{(2^j r)^{n+1} |B(\zeta,r)|} \le C \sum_{j=2}^{\infty} 2^{n-j} \le C. \end{aligned}$$

In order to treat the term I_1, we let $Q(x) = P(rx)$ and $\tilde{g}(y) = g(\zeta + ry)$. Let $l = [\log_2(h/r)]$. By Hölder's inequality and Lemma 3.1, we have

$$\begin{aligned} I_1 \le C \sum_{j=l}^{\infty} &\Big(\int_{2^j r \le |x-\zeta| \le 2^{j+1} r} \Big| \int_{B(\zeta,r)} e^{iP(x-y)} g(y)\,dy \Big|^{\frac{1+\delta}{\delta}} dx \Big)^{\frac{\delta}{1+\delta}} \\ &\times \Big(\int_{2^j r \le |x-\zeta| \le 2^{j+1} r} \frac{(w(x))^{1+\delta} dx}{|x-\zeta|^{n(1+\delta)}} \Big)^{\frac{1}{1+\delta}} \end{aligned}$$

$$\leq Cr^{\frac{n(1+2\delta)}{1+\delta}} \sum_{j=l}^{\infty} (\|S_{Q,j}\tilde{g}\|_{L^{(1+\delta)/\delta}(\mathbb{R}^n)}) \bigg(\frac{w(B(\zeta,r))|B(\zeta,2^{j+1}r)|^{\frac{1}{1+\delta}}}{(2^j r)^n |B(\zeta,r)|} \bigg)$$

$$\leq Cr^{\frac{n(1+2\delta)}{1+\delta}} \Big(\sum_{|\alpha|=m} |a_\alpha| r^m \Big)^{-\theta}$$

$$\times \sum_{j=l}^{\infty} \frac{2^{\frac{j\delta(2n-1)}{2(1+\delta)}}}{w(B(\zeta,r))} \bigg(\frac{w(B(\zeta,r))(2^{j+1}r)^{\frac{n}{1+\delta}}}{(2^j r)^n r^n} \bigg)$$

$$\leq Cr^{-m\theta} \Big(\sum_{|\alpha|=m} |a_\alpha| \Big)^{-\theta} \sum_{j=l}^{\infty} 2^{-\frac{j\delta}{2(1+\delta)}}$$

$$\leq Cr^{-m\theta} \Big(\sum_{|\alpha|=m} |a_\alpha| \Big)^{-\theta} (h/r)^{-(m-1)\theta} = C \Big(\sum_{|\alpha|=m} |a_\alpha| r h^{m-1} \Big)^{-\theta}.$$

This completes the proof of Lemma 3.2. ∎

Lemma 3.3. *Let P, K be as in Theorem 2.1, $w \in A_1(\mathbb{R}^n)$, and $g(\cdot)$ be an H_w^1 atom which satisfies (12)–(14). Then there exists $C > 0$ such that*

$$\int_{\mathbb{R}^n} |Tg(x)| w(x)\, dx \leq C \bigg(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2 \leq |\alpha| \leq m} |a_\alpha|^{1/|\alpha|}} \bigg). \tag{16}$$

The constant C may depend on n, m, K and the A_1 constant of w, but is independent of $\{a_\alpha\}$, ζ, r and h.

Proof. We shall prove (16) by induction on $m = \deg(P)$.

When $m = 2$, there are two cases:

$$\text{(i) } 0 < r \leq \Big(2 \sum_{|\alpha|=2} |a_\alpha| \Big)^{-1/2}; \qquad \text{(ii) } r > \Big(2 \sum_{|\alpha|=2} |a_\alpha| \Big)^{-1/2}.$$

Case (i). By the uniform L_w^2 boundedness of T_P, we have

$$\int_{B(\zeta,2r)} |Tg(x)| w(x)\, dx \leq \|Tg\|_{L_w^2(\mathbb{R}^n)} (w(B(\zeta,2r)))^{1/2} \tag{17}$$

$$\leq C \bigg(\frac{w(B(\zeta,2r))}{w(B(\zeta,r))} \bigg)^{1/2} \leq 2^{n/2} C.$$

It should be pointed out that the condition $\deg(P) = 2$ was not used in establishing (17). All one needs is that $\deg(P)$ is bounded.

Let $h = (\sum_{|\alpha|=2} |a_\alpha| r)^{-1}$ and $l = [\log_2(h/r)]$. Then $h \geq 2r$. By Lemma 3.2, we have

$$\int_{(B(\zeta,h))^c} |Tg(x)| w(x)\, dx \leq C. \tag{18}$$

On the other hand, we have

$$(19)\quad \int\limits_{2r\le|x-\zeta|<h}\int\limits_{|y-\zeta|<r}\Big|\sum_{|\alpha|=1}a_\alpha(x-y)^\alpha-\sum_{|\alpha|=1}a_\alpha(x-\zeta)^\alpha\Big||g(y)|\,dy\,\frac{w(x)\,dx}{|x-\zeta|^n}$$
$$\le\Big(\sum_{|\alpha|=1}|a_\alpha|r\Big)\Big(\int\limits_{2r\le|x-\zeta|\le h}\frac{w(x)\,dx}{|x-\zeta|^n}\Big)\frac{|B(\zeta,r)|}{w(B(\zeta,r))}$$
$$\le\Big(\sum_{|\alpha|=1}|a_\alpha|r\Big)\sum_{j=1}^{l}\frac{w(B(\zeta,2^{j+1}r))|B(\zeta,r)|}{(2^jr)^nw(B(\zeta,r))}$$
$$\le Cr\ln\Big(\frac{h}{r}\Big)\sum_{|\alpha|=1}|a_\alpha|.$$

Similarly,

$$(20)\quad \int\limits_{2r\le|x-\zeta|<h}\int\limits_{|y-\zeta|<r}\Big|\sum_{|\alpha|=2}a_\alpha(x-y)^\alpha-\sum_{|\alpha|=2}a_\alpha(x-\zeta)^\alpha\Big||g(y)|\,dy\,\frac{w(x)\,dx}{|x-\zeta|^n}$$
$$\le\Big(\sum_{|\alpha|=2}|a_\alpha|r\Big)\Big(\int\limits_{2r\le|x-\zeta|<h}\frac{w(x)\,dx}{|x-\zeta|^{n-1}}\Big)\frac{|B(\zeta,r)|}{w(B(\zeta,r))}$$
$$\le\Big(\sum_{|\alpha|=2}|a_\alpha|r\Big)\sum_{j=1}^{l}\frac{w(B(\zeta,2^{j+1}r))|B(\zeta,r)|}{(2^jr)^{n-1}w(B(\zeta,r))}\le C\sum_{|\alpha|=2}|a_\alpha|rh.$$

Observe that

$$\int\limits_{B(\zeta,h)\setminus B(\zeta,2r)}|Tg(x)|w(x)\,dx\le\tilde I_1+\tilde I_2,$$

where

$$\tilde I_1=\int\limits_{2r\le|x-\zeta|<h}\Big|\int\limits_{|y-\zeta|\le r}e^{iP(x-y)}g(y)\,dy\Big||K(x-\zeta)|w(x)\,dx,$$
$$\tilde I_2=\int\limits_{2r\le|x-\zeta|<h}\Big|\int\limits_{|y-\zeta|\le r}e^{iP(x-y)}(K(x-y)-K(x-\zeta))g(y)\,dy\Big|w(x)\,dx.$$

By the treatment used for I_2 in the proof of Lemma 3.2, we obtain

$$\tilde I_2\le C.$$

By (14),

$$\tilde I_1=\int\limits_{2r\le|x-\zeta|<h}\Big|\int\limits_{|y-\zeta|\le r}(e^{iP(x-y)}-e^{iP(x-\zeta)})g(y)\,dy\Big||K(x-\zeta)|w(x)\,dx$$
$$\le\int\limits_{2r\le|x-\zeta|<h}\int\limits_{|y-\zeta|\le r}\Big|\sum_{1\le|\alpha|\le2}a_\alpha(x-y)^\alpha-\sum_{1\le|\alpha|\le2}a_\alpha(x-\zeta)^\alpha\Big||g(y)|\,dy\,\frac{w(x)dx}{|x-\zeta|^n}$$

$$\leq \int\limits_{2r\leq|x-\zeta|<h}\int\limits_{|y-\zeta|<r}\Big|\sum_{|\alpha|=1}a_\alpha(x-y)^\alpha-\sum_{|\alpha|=1}a_\alpha(x-\zeta)^\alpha\Big||g(y)|\,dy\,\frac{w(x)\,dx}{|x-\zeta|^n}$$
$$+\int\limits_{2r\leq|x-\zeta|<h}\int\limits_{|y-\zeta|<r}\Big|\sum_{|\alpha|=2}a_\alpha(x-y)^\alpha-\sum_{|\alpha|=2}a_\alpha(x-\zeta)^\alpha\Big||g(y)|\,dy\,\frac{w(x)\,dx}{|x-\zeta|^n}.$$

Now, by (19) and (20), we have

$$\begin{aligned}(21)\qquad \int\limits_{B(\zeta,h)\setminus B(\zeta,2r)}|Tg(x)|w(x)\,dx &\leq \tilde{I}_1+\tilde{I}_2\\ &\leq C+Cr\ln\Big(\frac{h}{r}\Big)\sum_{|\alpha|=1}|a_\alpha|+C\sum_{|\alpha|=2}|a_\alpha|rh\\ &\leq C+C\frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{|\alpha|=2}|a_\alpha|^{1/|\alpha|}}\Big(\sum_{|\alpha|=2}|a_\alpha|\Big)^{1/2}r\ln\Big(\frac{h}{r}\Big)\\ &\leq C+C\frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{|\alpha|=2}|a_\alpha|^{1/|\alpha|}}\Big[\Big(\frac{r}{h}\Big)^{1/2}\ln\Big(\frac{h}{r}\Big)\Big]\\ &\leq C\Big(1+\frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{|\alpha|=2}|a_\alpha|^{1/|\alpha|}}\Big).\end{aligned}$$

By combining (17), (18) and (21), we see that (16) holds in this case.

Case (ii). By Lemma 3.2 and $r^2>(2\sum_{|\alpha|=2}|a_\alpha|)^{-1}$, we get

$$(22)\qquad \int\limits_{(B(\zeta,2r))^c}|Tg(x)|w(x)\,dx\leq C\Big(1+\Big(\sum_{|\alpha|=2}2|a_\alpha|r^2\Big)^{-\theta}\Big)\leq C.$$

By (17) and (22), we see that (16) also holds in this case. This concludes the verification of (16) for $\deg(P)=2$.

Suppose that $m\geq 3$ and that (16) holds for all polynomials P which satisfy $2\leq\deg(P)\leq m-1$.

Now we shall prove that (16) holds for any polynomial $P(x)=\sum_{|\alpha|\leq m}a_\alpha x^\alpha$ with $\sum_{|\alpha|=m}|a_\alpha|\neq 0$. Let $d=\max\{2r,(r\sum_{|\alpha|=m}|a_\alpha|)^{-1/(m-1)}\}$. Then

$$\int\limits_{\mathbb{R}^n}|T_Pg(x)|w(x)\,dx\leq J_1+J_2+J_3,$$

where

$$J_1=\int\limits_{B(\zeta,2r)}|T_Pg(x)|w(x)\,dx,\qquad J_2=\int\limits_{B(\zeta,d)\setminus B(\zeta,2r)}|T_Pg(x)|w(x)\,dx,$$
$$J_3=\int\limits_{(B(\zeta,d))^c}|T_Pg(x)|w(x)\,dx.$$

By (17) and Lemma 3.2, we have

$$J_1 + J_3 \leq C + C\Big(\sum_{|\alpha|=m} |a_\alpha| r d^{m-1}\Big)^{-\theta} \leq C. \tag{23}$$

Thus, our remaining task is to establish

$$J_2 \leq C\bigg(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\leq|\alpha|\leq m} |a_\alpha|^{1/|\alpha|}}\bigg). \tag{24}$$

Since $J_2 = 0$ when $d = 2r$, from this point on we may assume that

$$d = \Big(r \sum_{|\alpha|=m} |a_\alpha|\Big)^{-1/(m-1)} > 2r. \tag{25}$$

Let $s = r(\sum_{|\alpha|=m} |a_\alpha|)^{1/m}$. It is easy to see that $s = (r/d)^{(m-1)/m} < 1$.

To prove (24), we shall consider the following two cases separately:

(a) $\displaystyle\sum_{2\leq|\alpha|\leq m-1} |a_\alpha|^{1/|\alpha|} \geq \sum_{|\alpha|=m} |a_\alpha|^{1/|\alpha|}$;

(b) $\displaystyle\sum_{2\leq|\alpha|\leq m-1} |a_\alpha|^{1/|\alpha|} < \sum_{|\alpha|=m} |a_\alpha|^{1/|\alpha|}$.

Case (a). In this case, let $\Phi(x) = \sum_{|\alpha|\leq m-1} a_\alpha x^\alpha$. Then $2 \leq \deg(\Phi) \leq m-1$. Thus

$$\begin{aligned}\int_{\mathbb{R}^n} |T_\Phi g(x)| w(x)\,dx &\leq C\bigg(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\leq|\alpha|\leq m-1} |a_\alpha|^{1/|\alpha|}}\bigg) \\ &\leq 2C\bigg(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\leq|\alpha|\leq m} |a_\alpha|^{1/|\alpha|}}\bigg).\end{aligned} \tag{26}$$

Therefore, by (26),

$$\begin{aligned}J_2 &\leq \int_{B(\zeta,d)\setminus B(\zeta,2r)} |T_\Phi g(x)| w(x)\,dx \\ &\quad + \int_{B(\zeta,d)\setminus B(\zeta,2r)} |T_P g(x) - e^{i\sum_{|\alpha|=m} a_\alpha (x-\zeta)^\alpha} T_\Phi g(x)| w(x)\,dx \\ &\leq C\bigg(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\leq|\alpha|\leq m} |a_\alpha|^{1/|\alpha|}}\bigg) \\ &\quad + C\sum_{|\alpha|=m} |a_\alpha| \Big(\int_{2r\leq|x-\zeta|<d} r|x-\zeta|^{m-1-n} w(x)\,dx\Big) \frac{|B(\zeta,r)|}{w(B(\zeta,r))}\end{aligned}$$

$$\le C\Big(1 + \frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{2\le|\alpha|\le m}|a_\alpha|^{1/|\alpha|}} + \sum_{|\alpha|=m}|a_\alpha| r d^{m-1}\Big)$$
$$\le C\Big(1 + \frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{2\le|\alpha|\le m}|a_\alpha|^{1/|\alpha|}}\Big).$$

Thus, (24) holds in this case.

Case (b). In this case we have, for all $2 \le |\alpha| \le m$,

$$|a_\alpha|^{1/|\alpha|} \le \sum_{|\beta|=m}|a_\beta|^{1/m} \le C\Big(\sum_{|\beta|=m}|a_\beta|\Big)^{1/m}.$$

By applying (25) we obtain

$$|a_\alpha| r d^{|\alpha|-1} \le C(r/d)^{1-|\alpha|/m} \le C \tag{27}$$

for all $2 \le |\alpha| \le m$. Let

$$\Psi(x,y,\zeta) = \sum_{2\le|\alpha|\le m} a_\alpha(x-\zeta)^\alpha + \sum_{|\alpha|\le 1} a_\alpha(x-y)^\alpha.$$

By (14), (19), (27), and an argument similar to the proof of (20), we have

$$J_2 \le C + \int_{2r\le|x-\zeta|<d}\Big|\int_{|y-\zeta|<r} e^{P(x-y)}g(y)\,dy\Big|\frac{w(x)\,dx}{|x-\zeta|^n}$$
$$\le C + \int_{2r\le|x-\zeta|<d}\Big|\int_{|y-\zeta|<r}(e^{P(x-y)} - e^{i\Psi(x,y,\zeta)})g(y)\,dy\Big|\frac{w(x)\,dx}{|x-\zeta|^n}$$
$$+ \int_{2r\le|x-\zeta|<d}\Big|\int_{|y-\zeta|<r}(e^{i\sum_{|\alpha|\le1}a_\alpha(x-y)^\alpha} - e^{i\sum_{|\alpha|\le1}a_\alpha(x-\zeta)^\alpha})g(y)\,dy\Big|\frac{w(x)\,dx}{|x-\zeta|^n}$$
$$\le C + \int_{2r\le|x-\zeta|<d}\int_{|y-\zeta|<r}\Big|\sum_{2\le|\alpha|\le m}a_\alpha(x-y)^\alpha - \sum_{2\le|\alpha|\le m}a_\alpha(x-\zeta)^\alpha\Big||g(y)|\,dy\,\frac{w(x)\,dx}{|x-\zeta|^n}$$
$$+ \int_{2r\le|x-\zeta|<d}\int_{|y-\zeta|<r}\Big|\sum_{|\alpha|=1}a_\alpha(x-y)^\alpha - \sum_{|\alpha|=1}a_\alpha(x-\zeta)^\alpha\Big||g(y)|\,dy\,\frac{w(x)\,dx}{|x-\zeta|^n}$$
$$\le C + C\sum_{2\le|\alpha|\le m}|a_\alpha| r d^{|\alpha|-1} + Cr\ln\Big(\frac{d}{r}\Big)\sum_{|\alpha|=1}|a_\alpha|$$
$$\le C + \frac{C\sum_{|\alpha|=1}|a_\alpha|}{(\sum_{|\alpha|=m}|a_\alpha|)^{1/m}}\Big[s\ln\Big(\frac{1}{s}\Big)\Big] \le C\Big(1 + \frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{2\le|\alpha|\le m}|a_\alpha|^{1/|\alpha|}}\Big).$$

The proof of Lemma 3.3 is now complete. ∎

Proof of Theorem 2.1. Let $f \in H_w^1(\mathbb{R}^n)$. Then by the atomic decomposition for $H_w^1(\mathbb{R}^n)$ (see [6]), there exist H_w^1 atoms $\{g_j\}_{j=1}^\infty$ and complex numbers $\{c_j\}_{j=1}^{\infty}$ such that

$$f = \sum_{j=1}^{\infty} c_j g_j \quad \text{and} \quad \sum_{j=1}^{\infty} |c_j| \leq C\|f\|_{H_w^1(\mathbb{R}^n)}.$$

By Lemma 3.3, we have

$$\begin{aligned}
\|T_P f\|_{L_w^1(\mathbb{R}^n)} &\leq \sum_{j=1}^{\infty} |c_j| \, \|T_P g_j\|_{L_w^1(\mathbb{R}^n)} \\
&\leq C\left(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\leq|\alpha|\leq m} |a_\alpha|^{1/|\alpha|}}\right) \sum_{j=1}^{\infty} |c_j| \\
&\leq C\left(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\leq|\alpha|\leq m} |a_\alpha|^{1/|\alpha|}}\right) \|f\|_{H_w^1(\mathbb{R}^n)}.
\end{aligned}$$

For $1 \leq j \leq n$, let R_j denote the jth Riesz transform on $\mathbb{R}^n$. By results in [6] and [7], each R_j is bounded on $H_w^1(\mathbb{R}^n)$ and

$$\|f\|_{H_w^1(\mathbb{R}^n)} \approx \|f\|_{L_w^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L_w^1(\mathbb{R}^n)}.$$

Thus,

$$\begin{aligned}
\|T_P f\|_{H_w^1(\mathbb{R}^n)} &\leq C\Big(\|T_P f\|_{L_w^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j T_P f\|_{L_w^1(\mathbb{R}^n)}\Big) \\
&= C\Big(\|T_P f\|_{L_w^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|T_P R_j f\|_{L_w^1(\mathbb{R}^n)}\Big) \\
&\leq C\left(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\leq|\alpha|\leq m} |a_\alpha|^{1/|\alpha|}}\right)\Big(\|f\|_{H_w^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{H_w^1(\mathbb{R}^n)}\Big) \\
&\leq C\left(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\leq|\alpha|\leq m} |a_\alpha|^{1/|\alpha|}}\right) \|f\|_{H_w^1(\mathbb{R}^n)}. \ \blacksquare
\end{aligned}$$

4. An example. By considering a class of polynomials on $\mathbb{R}^1$, below we shall show that any proposed substitute for the bound

$$\frac{\sum_{1\leq|\alpha|=m} |a_\alpha|^{1/|\alpha|}}{\sum_{2\leq|\alpha|\leq m} |a_\alpha|^{1/|\alpha|}}$$

in Theorem 2.1 (or Theorem 1.2) cannot be smaller than

$$C \log\left(1 + \frac{\sum_{1\leq|\alpha|=m} |a_\alpha|^{1/|\alpha|}}{\sum_{2\leq|\alpha|\leq m} |a_\alpha|^{1/|\alpha|}}\right).$$

Let $K(x) = 1/x$. For $\lambda \geq 2$, let $P_\lambda(x) = x + x^2/\lambda^2$. Then we have $a_1 = 1$, $a_2 = 1/\lambda^2$ and $|a_1|/|a_2|^{1/2} = \lambda$.

Let $g(\cdot)$ be an H^1 atom such that $\mathrm{supp}(g) \subseteq [-1,1]$, $\|g\|_\infty \leq 1$, and $\hat{g}(1) \neq 0$. Then

$$\begin{aligned}
&\|T_{P_\lambda} g\|_{H^1} \geq \|T_{P_\lambda} g\|_{L^1} \geq \int_2^\lambda |T_{P_\lambda} g(x)|\,dx \\
&= \int_2^\lambda \bigg| \int_{-1}^1 \bigg[e^{iP_\lambda(x-y)} \bigg(\frac{1}{x-y} - \frac{1}{x} \bigg) + \frac{e^{iP_\lambda(x-y)} - e^{i(x-y)}}{x} + \frac{e^{i(x-y)}}{x} \bigg] g(y)\,dy \bigg| dx \\
&\geq \int_2^\lambda \bigg| e^{ix} \int_{-1}^1 e^{-iy} g(y)\,dy \bigg| \frac{dx}{x} - \int_2^\lambda \int_{-1}^1 \bigg| \frac{1}{x-y} - \frac{1}{x} \bigg| |g(y)|\,dy\,dx \\
&\quad - \int_2^\lambda \int_{-1}^1 \frac{(x-y)^2}{\lambda^2 x} |g(y)|\,dy\,dx \\
&\geq (\ln\lambda - \ln 2)|\hat{g}(1)| - \int_2^\lambda \frac{2\,dx}{x^2} - \int_2^\lambda \frac{4x\,dx}{\lambda^2} \geq C(\ln\lambda)\|g\|_{H^1},
\end{aligned}$$

for λ sufficiently large.

Acknowledgements. The authors thank the referee for his/her helpful comments.

References

[1] S. Chanillo and M. Christ, *Weak* $(1,1)$ *bounds for oscillatory singular integrals*, Duke Math. J. 55 (1987), 141–155.

[2] R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250.

[3] M. Folch-Gabayet and J. Wright, *Weak* $(1,1)$ *bounds for oscillatory singular integrals with rational phases*, Studia Math. 210 (2012), 57–76.

[4] Y. Hu and Y. Pan, *Boundedness of oscillatory singular integrals on Hardy spaces*, Ark. Mat. 30 (1992), 311–320.

[5] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals*, J. Funct. Anal. 73 (1987), 179–194.

[6] J. O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer, 1989.

[7] R. Wheeden, *A boundary value characterization of weighted* H^1, Enseign. Math. 24 (1976), 121–134.

Hussain Al-Qassem
Department of Mathematics and Physics
Qatar University
Doha, Qatar
E-mail: husseink@qu.edu.qa

Leslie Cheng
Department of Mathematics
Bryn Mawr College
Bryn Mawr, PA 19010, U.S.A.
E-mail: lcheng@brynmawr.edu

Yibiao Pan
Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260, U.S.A.
E-mail: yibiao@pitt.edu

Received July 18, 2014
Revised version November 14, 2014 (8023)