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Suitable domains to define fractional integrals of Weyl via fractional powers of operators

by

Celso Martínez (València), Antonia Redondo (Albacete) and Miguel Sanz (València)

Abstract. We present a new method to study the classical fractional integrals of Weyl. This new approach basically consists in considering these operators in the largest space where they make sense. In particular, we construct a theory of fractional integrals of Weyl by studying these operators in an appropriate Fréchet space. This is a function space which contains the $L^p(\mathbb{R})$-spaces, and it appears in a natural way if we wish to identify these fractional operators with fractional powers of a suitable non-negative operator. This identification allows us to give a unified view of the theory and provides some elegant proofs of some well-known results on the fractional integrals of Weyl.

1. Introduction and results in $L^1_{\mathrm{loc}}(\mathbb{R})$. Many of the fractional operators that appear in Fractional Calculus can be described as fractional powers of some non-negative operator considered on a suitable function space. Frequently this space is not a Banach space. However, if it is a sequentially complete locally convex space, the theory of fractional powers can provide global results applicable to the part of the operator in certain classical spaces such as the Lebesgue spaces L^p. This approach to Fractional Calculus has already been considered by the authors (see [11]–[13]).

The theory of fractional powers in Banach spaces is well-known and it was developed mainly by A. V. Balakrishnan [1] and H. Komatsu [5] for non-negative operators (see also [9] and [3]). An extensive bibliography of this theory can be found in [9] and [16]. The concept of a non-negative operator in a Fréchet space and the corresponding theory of fractional powers were developed in [10] and [8] (see also [9]). Previously, W. Lamb [6] constructed another more restrictive extension.

Fractional integrals and derivatives of Weyl have been widely studied (see [4] and [14] and the books [2] and [15]), but usually they have not been

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considered as operators defined in suitable function spaces, and so their properties have only been proved for certain kinds of functions.

In this paper we develop a method to study the classical Weyl fractional integrals. The basic idea is to consider these operators in the largest space where they make sense and where these operators can be described as fractional powers of a certain non-negative operator.

From now on, given a function $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ which is not necessarily integrable, the expression $\int_a^\infty g(t)\,dt$ will be understood as an improper integral. $\mathbb{C}_+$ will stand for the set of complex numbers with strictly positive real part. Given a linear operator A, $D(A)$ and $R(A)$ will denote, respectively, the domain and the range of A.

Given a complex number $\alpha \in \mathbb{C}_+$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, if there exist points $a > x_0$ such that the integral

$$\int_a^\infty (t-x_0)^{\alpha-1} f(t)\,dt$$

is convergent, then we deduce without difficulty (see [11, Lemmas 3.1, 3.2 and 3.3]) that for almost every $x \in \mathbb{R}$, the integral

$$\int_x^\infty (t-x)^{\alpha-1} f(t)\,dt \tag{1}$$

also converges, and the function

$$K_w^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\,dt, \qquad \text{a.e. } x \in \mathbb{R},$$

belongs to the space $L^1_{\mathrm{loc}}(\mathbb{R})$. The function $K_w^\alpha f$ is called the *classical fractional integral of Weyl* of order α of the function f. So, if we do not make any restrictions, the natural space where we can consider the linear operator $K_w^\alpha : f \mapsto K_w^\alpha f$ is $L^1_{\mathrm{loc}}(\mathbb{R})$, and its natural domain is

$$D(K_w^\alpha) := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}) : \int_x^\infty (t-x)^{\alpha-1} f(t)\,dt \text{ exists for a.e. } x \in \mathbb{R} \Big\}.$$

This was already indicated in [11, Remark 3.5], where we also noted that, from the merely algebraic point of view, the operator K_w^1 (which we will call K) cannot be a non-negative operator in $L^1_{\mathrm{loc}}(\mathbb{R})$, since $1+K$ is not surjective. We cannot work with the closure operator either, since K is not closable, and thus fractional powers of K do not make sense in $L^1_{\mathrm{loc}}(\mathbb{R})$, and obviously, we cannot establish relationships between them and fractional integrals of Weyl either.

In this paper we overcome these difficulties by constructing a suitable function space which entirely contains the domain and the range of K. This space will be a Fréchet space and K will be a non-negative operator in this space. The aforementioned space contains all the Lebesgue spaces, and in algebraic sense, it is maximal with respect to the property of non-negativity of K.

In [11, Lemma 3.2 and Theorem 3.6] we proved the following results:

LEMMA 1.1. *Let $\alpha \in \mathbb{C}_+$ and $a, x, x_0 \in \mathbb{R}$ with $x, x_0 \in \,]-\infty, a[$. If $f \in D(K_w^\alpha)$, then*

$$\Big|\int_a^\infty (t-x)^{\alpha-1} f(t)\,dt\Big| \le \left(A + B\left(\frac{a-x}{a-x_0}\right)^{\operatorname{Re}\alpha-1}\right) \sup_{t\ge a}\Big|\int_t^\infty (s-x_0)^{\alpha-1} f(s)\,ds\Big|$$

when $\operatorname{Re}\alpha \neq 1$, *and*

$$\Big|\int_a^\infty (t-x)^{\alpha-1} f(t)\,dt\Big| \le \left(1 + C\Big|\log\left(\frac{a-x}{a-x_0}\right)\Big|\right) \sup_{t\ge a}\Big|\int_t^\infty (s-x_0)^{\alpha-1} f(s)\,ds\Big|$$

when $\operatorname{Re}\alpha = 1$, *where A, B, C are positive constants which only depend on the complex number α.*

LEMMA 1.2. *Given $\alpha, \beta \in \mathbb{C}_+$ such that $\operatorname{Re}\alpha > \operatorname{Re}\beta$, we have $D(K_w^\alpha) \subset D(K_w^\beta)$, and for $a, x \in \mathbb{R}$ with $x \in \,]-\infty, a[$ the following estimate holds:*

$$\Big|\int_a^\infty (t-x)^{\beta-1} f(t)\,dt\Big| \le E(a-x)^{\operatorname{Re}(\beta-\alpha)} \sup_{t\ge a}\Big|\int_t^\infty (s-x)^{\alpha-1} f(s)\,ds\Big|$$

where E is a constant which only depends on α and β.

By means of these two results we proved (see [11, Theorem 3.7]) that for $\alpha, \beta \in \mathbb{C}_+$ with $\operatorname{Re}\beta < 1$, the operator $K_w^\beta K_w^\alpha$ is an extension of $K_w^{\alpha+\beta}$. In what follows, with a more careful use of these same results, we will prove that the restriction on the size of the real part of β can be removed.

THEOREM 1.3. *If $\alpha, \beta \in \mathbb{C}_+$, then the operator $K_w^\beta K_w^\alpha$ is an extension of $K_w^{\alpha+\beta}$.*

Proof. Given $f \in D(K_w^{\alpha+\beta})$, let x in $\mathbb{R}$ be a point where the function $K_w^{\alpha+\beta} f$ is defined, and let $a > x$. At the beginning of the proof of [11, Theorem 3.7] we proved that

$$\int_x^a (t-x)^{\beta-1}(K_w^\alpha f)(t)\,dt = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_x^a (t-x)^{\alpha+\beta-1} f(t)\,dt + \frac{1}{\Gamma(\alpha)}\int_x^a (t-x)^{\beta-1}\Big(\int_a^\infty (s-t)^{\alpha-1} f(s)\,ds\Big)\,dt$$

for any $\alpha, \beta \in \mathbb{C}_+$. So, in order to prove our statement, it is sufficient to show that the last iterated integral converges to zero as $a \to \infty$. For a fixed $x_0 < a$, we denote

$$M(a) := \sup_{t\ge a}\Big|\int_t^\infty (s-x_0)^{\alpha+\beta-1} f(s)\,ds\Big| \tag{2}$$

and we write $\sigma = \operatorname{Re}\alpha$ and $\tau = \operatorname{Re}\beta$. By applying successively Lemmas 1.1 and 1.2 we find that there exist positive constants P, Q, P_1, Q_1 such that

$$\Big|\int_a^\infty (s-t)^{\alpha-1} f(s)\,ds\Big| \le \Big(P + Q\Big(\frac{a-t}{a-x_0}\Big)^{\sigma-1}\Big)(a-x_0)^{-\tau} M(a)$$

if $\sigma \ne 1$, and

$$\Big|\int_a^\infty (s-t)^{\alpha-1} f(s)\,ds\Big| \le \Big(P_1 + Q_1 \log\Big(\frac{a-x_0}{a-t}\Big)\Big)(a-x_0)^{-\tau} M(a)$$

if $\sigma = 1$. Since $\lim_{a\to\infty} M(a) = 0$, it is sufficient to prove that the expressions

$$M_1(a) := (a-x_0)^{-\tau}\int_x^a (t-x)^{\tau-1}\,dt,$$

$$M_2(a) := (a-x_0)^{-\tau-\sigma+1}\int_x^a (t-x)^{\tau-1}(a-t)^{\sigma-1}\,dt,$$

$$M_3(a) := (a-x_0)^{-\tau}\int_x^a (t-x)^{\tau-1}\log\Big(\frac{a-x_0}{a-t}\Big)\,dt$$

are bounded in a. But this is evident, since it is easy to see that

$$M_1(a) = \frac{1}{\tau}\Big(\frac{a-x}{a-x_0}\Big)^{\tau},$$

$$M_2(a) = \Big(\frac{a-x}{a-x_0}\Big)^{\sigma+\tau-1}\int_0^1 v^{\tau-1}(1-v)^{\sigma-1}\,dv,$$

$$M_3(a) = \Big(\frac{a-x}{a-x_0}\Big)^{\tau}\Big[\log\Big(\frac{a-x_0}{a-x}\Big) - \int_0^1 v^{\tau-1}\log(1-v)\,dv\Big],$$

which completes the proof. ∎

Remark 1.4. In general, the operator $K_w^{\alpha+\beta}$ is not an extension of the composition $K_w^\alpha K_w^\beta$. Set $f(x) := e^{-2\pi ix}$. If $0 < \operatorname{Re}\alpha < 1$ we have

$$(3)\quad (K_w^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1} f(t)\,dt = \frac{1}{\Gamma(\alpha)} e^{-2\pi ix}\int_0^\infty e^{-2\pi is} s^{\alpha-1}\,ds.$$

In view of the identities

$$s^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)}\int_0^\infty e^{-vs} v^{-\alpha}\,dv,$$

$$\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^\infty \left(\frac{1}{v+2\pi i}\right) v^{-\alpha}\,dv = (2\pi i)^{-\alpha},$$

the last integral in (3) is equal to

$$\frac{1}{\Gamma(1-\alpha)}\int_0^\infty e^{-2\pi is}\Big(\int_0^\infty e^{-vs} v^{-\alpha}\,dv\Big)\,ds$$
$$= \frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{1}{v+2\pi i} v^{-\alpha}\,dv = \Gamma(\alpha)(2\pi i)^{-\alpha},$$

hence

$$(K_w^\alpha f)(x) = (2\pi i)^{-\alpha} f(x).$$

Thus f belongs to the domain of $D(K_w^{1/2}K_w^{1/2})$. However, it is evident that $f \notin D(K)$.

The following partial result of additivity will be useful later on.

Theorem 1.5. *Given* $\alpha,\beta,\gamma \in \mathbb{C}_+$ *such that* $\operatorname{Re}(\alpha+\beta) = \operatorname{Re}\gamma$, *if* $f \in D(K_w^\gamma)$ *and* $K_w^\alpha f \in D(K_w^\beta)$, *then*

$$f \in D(K_w^{\alpha+\beta}) \quad \textit{and} \quad K_w^{\alpha+\beta} f = K_w^\beta K_w^\alpha f.$$

Proof. By applying similar arguments to those in the proof of Theorem 1.3, we first note that it is sufficient to show that, given a point $x \in \mathbb{R}$ such that $(K_w^\beta K_w^\alpha f)(x)$ is defined, we have

$$\lim_{a\to\infty}\int_x^a (t-x)^{\beta-1}\Big(\int_a^\infty (s-t)^{\alpha-1} f(s)\,ds\Big)dt = 0.$$

From this, we reason in the same way as in Theorem 1.3, but taking

$$M(a) := \sup_{t\ge a}\Big|\int_t^\infty (s-x_0)^{\gamma-1} f(s)\,ds\Big|$$

instead of (2). ∎

In order to establish the next two results, given $\lambda > 0$, we introduce the set

$$D(H_\lambda) := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}) : \int_x^\infty e^{-\lambda t} f(t)\, dt \text{ is convergent} \Big\}$$

and we consider the operator $H_\lambda : D(H_\lambda) \to L^1_{\mathrm{loc}}(\mathbb{R})$ defined by

$$(H_\lambda f)(x) := e^{\lambda x} \int_x^\infty e^{-\lambda t} f(t)\, dt \quad \text{for all } f \in D(H_\lambda) \text{ and } x \in \mathbb{R}.$$

Proposition 1.6. *If* $\alpha \in \mathbb{C}_+$, *then* $D(K_w^\alpha) \subset D(H_\lambda)$, *and*

$$\lim_{x\to\infty} x^{\alpha-1}(H_\lambda f)(x) = 0 \quad \textit{for all } f \in D(K_w^\alpha). \tag{4}$$

In addition, if $\operatorname{Re}\alpha > 1$, *then*

$$\lim_{x\to\infty} x^{\alpha-1}(Kf)(x) = 0 \quad \textit{for all } f \in D(K_w^\alpha). \tag{5}$$

Proof. Let $f \in D(K_w^\alpha)$ and $x \in \mathbb{R}$. When $c < x < a$, we have

$$\int_x^a e^{-\lambda t} f(t)\, dt = (x-c)^{1-\alpha} e^{-\lambda x} \int_x^a (t-c)^{\alpha-1} f(t)\, dt + \int_x^a \frac{((1-\alpha) - \lambda(t-c))}{e^{\lambda t}(t-c)^\alpha} \Big(\int_t^a (s-c)^{\alpha-1} f(s)\, ds \Big)\, dt,$$

which easily implies that $f \in D(H_\lambda)$. Moreover

$$|x^{\alpha-1}(H_\lambda f)(x)| \le \Big| x^{\alpha-1}(x-c)^{1-\alpha} \int_x^\infty (t-c)^{\alpha-1} f(t)\, dt \Big| + S_x(\alpha) \Big| x^{\alpha-1} e^{\lambda x} \int_x^\infty e^{-\lambda t}((1-\alpha)(t-c)^{-\alpha} - \lambda(t-c)^{1-\alpha})\, dt \Big|$$

where

$$S_x(\alpha) := \sup_{t \ge x} \Big| \int_t^\infty (s-c)^{\alpha-1} f(s)\, ds \Big|.$$

By taking into account that

$$\lim_{x\to\infty} x^{\alpha-1}(x-c)^{1-\alpha} = 1 \quad \text{and} \quad \lim_{x\to\infty} S_x(\alpha) = 0$$

and using L'Hôpital's rule, we obtain

$$\limsup_{x\to\infty} \Big| x^{\alpha-1} e^{\lambda x} \int_x^\infty e^{-\lambda t}((1-\alpha)(t-c)^{-\alpha} - \lambda(t-c)^{1-\alpha})\, dt \Big| \le 1$$

and therefore $\lim_{x\to\infty} x^{\alpha-1}(H_\lambda f)(x) = 0$.

If $\operatorname{Re}\alpha > 1$, from Lemma 1.2 we deduce that $f \in D(K)$ and in the same way the relation (5) is proved. ∎

In the case $0 < \operatorname{Re}\alpha < 1$ we have $D(K) \subset D(K_w^\alpha)$. However, as we will see in the next proposition, we cannot guarantee a similar inclusion when $\operatorname{Re}\alpha > 1$. As usual, we will denote

$$D(K^\infty) := \bigcap_{n\in\mathbb{N}} D(K^n), \qquad R(K^\infty) := \bigcap_{n\in\mathbb{N}} R(K^n).$$

PROPOSITION 1.7. *For all $\alpha \in \mathbb{C}_+$ with* $\operatorname{Re}\alpha > 1$,

$$D(K^\infty) \cap R(K^\infty) \not\subseteq D(K_w^\alpha).$$

Proof. First, let us see that the function

$$f_\beta(x) := \begin{cases} e^{2\pi i x} x^{-\beta}, & x > 1, \\ 0, & x \le 1, \end{cases}$$

belongs to $D(K^r)$ for all $\beta \in \mathbb{C}_+$ and for all positive integers r. This is obvious for $r = 1$, since by integration by parts, it is easy to prove that $f_\beta \in D(K)$. If we reiterate this process p times, we obtain an expression

$$K f_\beta = c_0 f_\beta + c_1 f_{\beta+1} + \cdots + c_{p-1} f_{\beta+p-1} + c_p K f_{\beta+p}, \tag{6}$$

valid for $x > 1$, where $c_0, \ldots, c_p$ are complex constants.

Now, we are going to prove that $f_\beta \in D(K^r)$. In fact, if we assume the result up to exponent $r-1$, we can apply this induction hypothesis to the functions $f_\beta, \ldots, f_{\beta+p-1}$ and we only need to prove that $K f_{\beta+p} \in D(K^{r-1})$, which is obvious when $p \ge r$, since

$$|K f_{\beta+p}(x)| \le \gamma x^{-\operatorname{Re}\beta - p + 1} \quad \text{for } x > 1,$$

where γ is a positive constant. Consequently, $f_\beta \in D(K^\infty)$.

Letting $p = 1$, directly from (6) it follows that $f_\beta \in R(K)$, and by induction we can easily prove that $f_\beta \in R(K^r)$ for all positive integers r. In particular, given $\alpha \in \mathbb{C}_+$ with $\operatorname{Re}\alpha > 1$, the function $f_{\alpha-1}$ belongs to $D(K^\infty) \cap R(K^\infty)$. However, it does not belong to $D(K_w^\alpha)$. In fact, from (6), taking $\beta = \alpha - 1$ and $p = 2$, by means of L'Hôpital's rule we easily find that

$$x^{\alpha-1}(K f_{\alpha-1})(x) = -\frac{1}{2\pi i} e^{2\pi i x} + O(x^{-1}),$$

and thus $f_{\alpha-1}$ does not satisfy (5). ■

THEOREM 1.8. *If $\alpha \in \mathbb{C}_+$ and $f \in D(K_w^\alpha)$, then*

$$K_w^\alpha f \in D(H_\lambda) \iff H_\lambda f \in D(K_w^\alpha),$$

and in that case $H_\lambda K_w^\alpha f = K_w^\alpha H_\lambda f$.

Proof. From Proposition 1.6, if $f \in D(K_w^\alpha)$ then $f \in D(H_\lambda)$. Given $x > 0$ such that the function $K_w^\alpha f$ is defined at x, for all $a > x$, applying the Tonelli theorem, we obtain the identities

$$\int_x^a (t-x)^{\alpha-1} e^{\lambda t}\Big(\int_t^a e^{-\lambda s} f(s)\,ds\Big)\,dt = \int_x^a e^{-\lambda s} f(s)\Big(\int_x^s (t-x)^{\alpha-1} e^{\lambda t}\,dt\Big)\,ds \tag{7}$$

and

$$\int_x^a e^{-\lambda t}\Big(\int_t^a (s-t)^{\alpha-1} f(s)\,ds\Big)\,dt = \int_x^a f(s)\Big(\int_t^a (s-t)^{\alpha-1} e^{-\lambda t}\,dt\Big)\,ds.$$

Moreover

$$e^{-\lambda s}\int_x^s (t-x)^{\alpha-1} e^{\lambda t}\,dt = e^{\lambda(x-s)}\int_0^{s-x} \tau^{\alpha-1} e^{\lambda\tau}\,d\tau = e^{\lambda x}\int_x^s (s-t)^{\alpha-1} e^{-\lambda t}\,dt.$$

So, the left member of (7) is equal to

$$e^{\lambda x}\int_x^a e^{-\lambda t}\Big(\int_t^a (s-t)^{\alpha-1} f(s)\,ds\Big)\,dt.$$

Now, it only remains to prove that the expressions

$$\begin{gathered}\int_x^a (t-x)^{\alpha-1} e^{\lambda t}\,dt \int_a^\infty e^{-\lambda s} f(s)\,ds \quad \text{and} \\ e^{\lambda x}\int_x^a e^{-\lambda t}\Big(\int_a^\infty (s-t)^{\alpha-1} f(s)\,ds\Big)\,dt\end{gathered} \tag{8}$$

converge to zero as a goes to infinity.

In order to prove the convergence of the first expression, it is sufficient to write it in this way:

$$\frac{\int_0^{a-x} \xi^{\alpha-1} e^{\lambda\xi}\,d\xi}{a^{\alpha-1} e^{\lambda(a-x)}} \cdot a^{\alpha-1} e^{\lambda a}\int_a^\infty e^{-\lambda s} f(s)\,ds$$

and to apply L'Hôpital's rule on the left and Proposition 1.6 on the right.

On the other hand, given c such that $c < x$, we have

$$\begin{aligned}\int_a^\infty (s-t)^{\alpha-1} f(s)\,ds &= \Big(\frac{a-c}{a-t}\Big)^{1-\alpha}\int_a^\infty (s-c)^{\alpha-1} f(s)\,ds \\ &\quad - (1-\alpha)\int_a^\infty \Big(\frac{s-c}{s-t}\Big)^{-\alpha}\frac{t-c}{(s-t)^2}\Big(\int_s^\infty (\xi-c)^{\alpha-1} f(\xi)\,d\xi\Big)\,ds,\end{aligned}$$

which implies the estimate

$$\Big|\int_a^\infty (s-t)^{\alpha-1} f(s)\,ds\Big| \le k(a,c)\Big[(p(\alpha)+1)\Big(\frac{a-c}{a-t}\Big)^{1-\operatorname{Re}\alpha} - p(\alpha)\Big]$$

where

$$k(a,c) := \sup_{s\ge a}\Big|\int\limits_s^\infty (\xi-c)^{\alpha-1} f(\xi)\,d\xi\Big| \quad \text{and} \quad p(\alpha) := \frac{|1-\alpha|}{\operatorname{Re}\alpha}.$$

Then, using L'Hôpital's rule, we find

$$\lim_{a\to\infty}(a-c)^{1-\sigma}\int\limits_a^x e^{-\lambda t}(a-t)^{\sigma-1}\,dt = \lim_{a\to\infty}\frac{\int_0^{a-x}\tau^{\sigma-1}e^{\lambda\tau}\,d\tau}{(a-c)^{\sigma-1}e^{\lambda a}} = e^{-\lambda x}$$

with $\sigma = \operatorname{Re}\alpha$. Since $f \in D(K_w^\alpha)$, we have $\lim_{a\to\infty} k(a,c) = 0$. Consequently, the second expression in (8) also converges to zero as a goes to infinity. ∎

2. Relation between the Weyl operators and the fractional powers of K. In this section, we are going to construct a Fréchet space contained in $L^1_{\mathrm{loc}}(\mathbb{R})$ and containing the domain and the range of K and where this operator will be non-negative. This space will not be a topological subspace of $L^1_{\mathrm{loc}}(\mathbb{R})$. These conditions uniquely characterize the aforementioned space. We will study the relationships between the fractional powers of K and Weyl's operators.

PROPOSITION 2.1. *Let $\lambda > 0$. In the space $L^1_{\mathrm{loc}}(\mathbb{R})$ the operator $1+\lambda K$ is injective and its range is*

$$\begin{aligned} X &:= \{f \in D(H_\lambda) : \lim_{x\to\infty}(H_\lambda f)(x) = 0\}\\ &= \Big\{f \in L^1_{\mathrm{loc}}(\mathbb{R}) : \int\limits_x^\infty e^{-\lambda t} f(t)\,dt \text{ exists and } \lim_{x\to\infty}\int\limits_x^\infty e^{\lambda(x-t)} f(t)\,dt = 0\Big\},\end{aligned}$$

which is independent of $\lambda > 0$. Furthermore,

$$D(K)\cup R(K) \subsetneq X.$$

Proof. Given $f \in X$, the function

$$h(x) := e^{\lambda x}\int\limits_x^\infty e^{-\lambda t} f(t)\,dt \tag{9}$$

is absolutely continuous and it is very easy to prove that

$$g = -h' \in D(K) \quad \text{and} \quad (1+\lambda K)g = f.$$

Conversely, given $f \in R(1+\lambda K)$ and $g \in D(K)$ satisfying $(1+\lambda K)g = f$, the absolutely continuous function $h = Kg$ satisfies the differential equation

$$-h'(x) + \lambda h(x) = f(x) \quad (\text{a.e. } x \in \mathbb{R})$$

and in addition $\lim_{x\to\infty} h(x) = 0$. An elementary calculation yields the unique solution of this Cauchy problem, and it establishes that h can be expressed by means of (9), hence $f \in X$. The uniqueness of h implies that of g, so the operator $1+\lambda K$ is injective.

Let us see that the range of the operator $1+\lambda K$ is independent of $\lambda > 0$. In fact, given $\lambda, \mu > 0$ and $f \in R(1+\lambda K)$, if we consider c, d such that $c \le d$, by integration by parts we get

$$\int_c^d e^{-\mu t} f(t)\,dt = e^{c(\lambda-\mu)} \int_c^d e^{-\lambda t} f(t)\,dt + (\lambda-\mu) \int_c^d e^{t(\lambda-\mu)} \Big(\int_t^d e^{-\lambda s} f(s)\,ds \Big)\,dt,$$

which leads to

$$\Big| \int_c^d e^{-\mu t} f(t)\,dt \Big| \le e^{-\mu c} \Big| e^{\lambda c} \int_c^d e^{-\lambda t} f(t)\,dt \Big| + \frac{|\lambda-\mu|(e^{-\mu c} - e^{-\mu d})}{\mu} \sup_{t\in[c,d]} \Big| e^{\lambda t} \int_t^d e^{-\lambda s} f(s)\,ds \Big|.$$

This estimate implies the convergence of the integral $\int_x^\infty e^{-\mu t} f(t)\,dt$ for all $x \in \mathbb{R}$. Likewise, we deduce that

$$\Big| e^{\mu c} \int_c^\infty e^{-\mu t} f(t)\,dt \Big| \le (1+\mu^{-1}|\lambda-\mu|) \sup_{t\ge c} \Big| e^{\lambda t} \int_t^\infty e^{-\lambda s} f(s)\,ds \Big|, \tag{10}$$

which implies that the first member converges to zero as c goes to infinity.

Therefore, we can conclude that $f \in R(1+\mu K)$, and so $R(1+\lambda K)$ is contained in $R(1+\mu K)$. This proves that $R(1+\lambda K)$ is in fact independent of $\lambda > 0$.

In order to show that $D(K) \subset X$, we reason in a similar way. Note that the inclusion $R(K) \subset X$ is a direct consequence of the fact that all the functions in $R(K)$ have zero limit as $x \to \infty$.

On the other hand, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1, & x \in [n, n+n^{-1}[,\ n = 1, 2, \ldots, \\ 0, & x \in \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} [n, n+n^{-1}[, \end{cases}$$

does not belong to $D(K)$ since

$$\int_n^\infty f(t)\,dt = \sum_{m\ge n} 1/m = \infty;$$

also $f \notin R(K)$, since $f(x)$ does not converge to zero as x goes to ∞. Nevertheless, $f \in X$ since for $x \in [n, n+1[$ we have

$$e^x \int_x^\infty e^{-t} f(t)\,dt \le e^{x-n} e^n \int_n^\infty e^{-t} f(t)\,dt \le \frac{e}{n} \sum_{m \ge 0} e^{-m}$$

and the right side tends to zero as $n \to \infty$. Thus, $D(K) \cup R(K) \subsetneq X$.

Remark 2.2. It is evident that $X \subsetneq L^1_{\mathrm{loc}}(\mathbb{R})$, since the function $f(x) := e^x$ does not belong to X.

Proposition 2.3. *If $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$, then the following relations hold:*

(i) $\bigcap_{0<\operatorname{Re}\alpha<1} (D(K_w^\alpha) \cap R(K_w^\alpha)) \setminus X \neq \emptyset$.
(ii) $\bigcup_{\operatorname{Re}\alpha=1} D(K_w^\alpha) \subset X$.
(iii) $\bigcup_{\operatorname{Re}\alpha>n} D(K_w^\alpha) \subset D(K_w^n) \subset D(K^n) \subset X$.
(iv) $\bigcup_{\operatorname{Re}\alpha>n} R(K_w^\alpha) \subset R(K_w^n) \subset R(K^n) \subset X$.

Proof. In Remark 1.4, we have considered $f(x) := e^{-2\pi i x}$, which obviously does not belong to X. In addition, $K_w^\alpha f = (2\pi i)^{-\alpha} f$ whenever $0 < \operatorname{Re}\alpha < 1$. Consequently, $f \in D(K_w^\alpha) \cap R(K_w^\alpha)$, which proves (i).

If $\operatorname{Re}\alpha = 1$ and $f \in D(K_w^\alpha)$, then taking into account (4), from Proposition 1.6 we get

$$\lim_{x\to\infty} (H_1 f)(x) = 0,$$

which means that $f \in X$ and proves (ii). Finally, the inclusions in (iii) and (iv) are straightforward consequences of Theorem 1.3. ∎

Remark 2.4. For $\alpha \in \mathbb{C}_+$, $\lambda > 0$ and $f \in D([K_w^\alpha]_X)$ (the part of K_w^α in X, see Remark 2.10), we have $K_w^\alpha f \in X \subset D(H_\lambda)$. Theorem 1.8 implies that $H_\lambda f = K(1+\lambda K)^{-1} f \in D(K_w^\alpha)$ and

$$K_w^\alpha K(1+\lambda K)^{-1} f = K(1+\lambda K)^{-1} K_w^\alpha f,$$

therefore $(1+\lambda K)^{-1} f \in D(K_w^\alpha)$ and

$$K_w^\alpha (1+\lambda K)^{-1} f = (1+\lambda K)^{-1} K_w^\alpha f.$$

By repeating this argument, we find that $(1+\lambda K)^{-n} f \in D(K_w^\alpha)$ for all $n \in \mathbb{N}$ and

$$K_w^\alpha (1+\lambda K)^{-n} f = (1+\lambda K)^{-n} K_w^\alpha f.$$

But Theorem 1.8 does not ensure that the conditions

$$f \in X \cap D(K_w^\alpha) \quad \text{and} \quad (1+\lambda K)^{-1} f \in D(K_w^\alpha)$$

imply that $K_w^\alpha f \in X$. We can only assert that

$$K_w^\alpha K(1+\lambda K)^{-1} f = H_\lambda K_w^\alpha f.$$

That is,

$$K_w^\alpha (1+\lambda K)^{-1} f = (1-\lambda H_\lambda) K_w^\alpha f.$$

THEOREM 2.5. *The set X, endowed with the seminorms*

$$\|f\|_{\infty,a} := \sup_{x\ge a} |[K(K+1)^{-1}f](x)| = \sup_{x\ge a} \Big|e^x \int_x^\infty e^{-t} f(t)\,dt\Big| \quad \text{for } a\in\mathbb{R},$$

$$\|f\|_{1,[a,b]} := \int_a^b |f(t)|\,dt \quad \text{for } b>a,$$

is a Fréchet space.

Proof. Let $\{g_n\}_{n=1}^\infty$ be a Cauchy sequence in X. By the definition of the topology of X, and the fact that the space $L^1_{\mathrm{loc}}(\mathbb{R})$ is complete, this sequence is convergent in $L^1_{\mathrm{loc}}(\mathbb{R})$. In order to prove that X is complete, it is enough to see that its limit f belongs to X.

Given $\varepsilon>0$ and $a\in\mathbb{R}$, there exists n_0 such that

$$\|g_m - g_{n_0}\|_{\infty,b} \le \varepsilon \quad \text{for } m\ge n_0,\ b\ge a.$$

On the other hand, taking $b\ge a$ such that

$$\|g_{n_0}\|_{\infty,b} \le \varepsilon,$$

we get

$$\|g_m\|_{\infty,c} \le \varepsilon + \|g_{n_0}\|_{\infty,c} \le 2\varepsilon \quad \text{for all } m\ge n_0 \text{ and } c\ge b.$$

Let d,c be such that $d\ge c\ge b$, and $m_0\ge n_0$ such that

$$\|g_{m_0} - f\|_{1,[c,d]} \le \varepsilon.$$

Then

$$\begin{aligned}\Big|e^c\int_c^d e^{-t}f(t)\,dt - e^c\int_c^d e^{-t}g_{m_0}(t)\,dt\Big| &\le e^c\int_c^d e^{-t}|f(t)-g_{m_0}(t)|\,dt \le \int_c^d |f(t)-g_{m_0}(t)|\,dt\\ &= \|g_{m_0}-f\|_{1,[c,d]} \le \varepsilon,\end{aligned}$$

hence

$$\begin{aligned}\Big|\int_c^d e^{-t}f(t)\,dt\Big| &\le \Big|\int_c^\infty e^{-t}g_{m_0}(t)\,dt\Big| + \Big|\int_d^\infty e^{-t}g_{m_0}(t)\,dt\Big| + e^{-c}\varepsilon\\ &= e^{-c}\|g_{m_o}\|_{\infty,c} + e^{-d}\|g_{m_o}\|_{\infty,d} + e^{-c}\varepsilon \le 3\varepsilon e^{-c} + 2\varepsilon e^{-d}.\end{aligned} \tag{11}$$

Consequently, for fixed $x\in\mathbb{R}$, the integral $\int_x^\infty e^{-t}f(t)\,dt$ always converges.

Now, multiplying (11) by e^c and taking limits as d goes to infinity, we obtain

$$\Big|e^c \int\limits_c^\infty e^{-t} f(t)\,dt\Big| \le 3\varepsilon \quad \text{for all } c \ge b.$$

Thus $f \in X$. ∎

Lemma 2.6. *The set $C_0^1(\mathbb{R})$ of functions of class C^1 with compact support is dense in the Fréchet space X.*

Proof. Given $g \in X$, the sets

$$\{f \in X : \|g-f\|_{1,[a,b]} \le \varepsilon \text{ and } \|g-f\|_{\infty,b} \le \varepsilon\} \quad (\varepsilon > 0,\ a < b)$$

are a neighborhood base of g in the topology of X. Now, given $\varepsilon > 0$ and $a < b$, choose $c \ge b$ such that $\|g\|_{\infty,c} \le \varepsilon$. As $C_0^\infty(]a,c[)$ is dense in $L^1([a,c])$, there exists $f \in C_0^\infty(]a,c[)$ such that

$$\|g-f\|_{1,[a,c]} \le \varepsilon,$$

so we get

$$\|g-f\|_{1,[a,b]} \le \|g-f\|_{1,[a,c]} \le \varepsilon$$

and

$$\|g-f\|_{\infty,b} \le \|g-f\|_{\infty,a} \le \|g-f\|_{1,[a,c]} + \|g\|_{\infty,c} \le 2\varepsilon,$$

which implies that $C_0^1(\mathbb{R})$ is dense in X. ∎

Theorem 2.7. *The operator $K : D(K) \to X$ is a densely defined non-negative operator in the space X and its range is dense.*

Proof. The density of $D(K)$ and $R(K)$ is a direct consequence of the preceding lemma. In fact, it is evident that $C_0^1(\mathbb{R}) \subset D(K)$, and for $f \in C_0^1(\mathbb{R})$, we have

$$f(x) = -\int\limits_x^\infty f'(s)\,ds \quad \text{for all } x \in \mathbb{R},$$

which means that $f \in R(K)$, so $C_0^1(\mathbb{R}) \subset R(K)$.

Now let us prove that K is non-negative. Given $a \in \mathbb{R}$, for all $\lambda > 0$ we get

$$\begin{aligned}
\|\lambda K(1+\lambda K)^{-1} f\|_{\infty,a} &= \sup_{x\ge a} |[K(K+1)^{-1}\lambda K(1+\lambda K)^{-1} f](x)| \\
&= \sup_{x\ge a} |[\lambda K(1+\lambda K)^{-1} K(K+1)^{-1} f](x)| \\
&= \sup_{x\ge a} \Big|\lambda e^{\lambda x} \int\limits_x^\infty e^{-\lambda t} [K(K+1)^{-1} f](t)\,dt\Big| \\
&\le \sup_{x\ge a} |[K(K+1)^{-1} f](x)| \lambda e^{\lambda x} \int\limits_x^\infty e^{-\lambda t}\,dt = \|f\|_{\infty,a}.
\end{aligned}$$

Assuming that $b > a$, in order to estimate

$$\|\lambda K(1+\lambda K)^{-1}f\|_{1,[a,b]}$$

we write

$$[\lambda K(1+\lambda K)^{-1}f](x) = g(x) + h(x),$$

with

$$g(x) := \lambda e^{\lambda x}\int_x^b e^{-\lambda t} f(t)\,dt, \quad h(x) := \lambda e^{\lambda x}\int_b^\infty e^{-\lambda t} f(t)\,dt \qquad (x \in [a,b]).$$

Note that, taking into account (10), we obtain

$$|h(x)| \le (\lambda + |\lambda - 1|)e^{-\lambda(b-x)}\|f\|_{\infty,b}$$

where $\lambda + |\lambda - 1| = \max\{1, 2\lambda - 1\}$. This implies

$$\int_a^b |h(x)|\,dx \le \begin{cases} (b-a)\|f\|_{\infty,b}, & 0 < \lambda < 1, \\ 2(1-e^{-\lambda(b-a)})\|f\|_{\infty,b}, & \lambda \ge 1, \end{cases}$$

hence

$$\int_a^b |h(x)|\,dx \le \max\{2, b-a\}\|f\|_{\infty,b}.$$

On the other hand

$$\begin{aligned}\int_a^b |g(x)|\,dx &\le \int_a^b \lambda e^{\lambda x}\Big(\int_x^b e^{-\lambda t}|f(t)|\,dt\Big)\,dx \\ &= \int_a^b e^{-\lambda t}|f(t)|\Big(\int_a^t \lambda e^{\lambda x}\,dx\Big)\,dt \\ &= \int_a^b (1-e^{\lambda(a-t)})|f(t)|\,dt \le \|f\|_{1,[a,b]}.\end{aligned}$$

Thus, it is shown that

$$\begin{aligned}\|\lambda K(1+\lambda K)^{-1}f\|_{1,[a,b]} &\le \int_a^b |g(x)|\,dx + \int_a^b |h(x)|\,dx \\ &\le \|f\|_{1,[a,b]} + \max\{2, b-a\}\|f\|_{\infty,b},\end{aligned}$$

which proves that the operator K is non-negative. ■

REMARK 2.8. The inverse of K is the distributional derivative operator on the domain of absolutely continuous functions which vanish at infinity.

REMARK 2.9. We will say that a topological space of locally integrable functions has the *property of convergence almost everywhere* if the convergence of a sequence in this space implies the pointwise convergence of some subsequence except on a set of measure zero. As is well known, this happens in Lebesgue spaces and in $L^1_{\mathrm{loc}}(\mathbb{R})$, with the usual topologies. This property also holds for the topology of X since it is finer than the one induced by $L^1_{\mathrm{loc}}(\mathbb{R})$.

REMARK 2.10. Given a linear operator $A : D(A) \subset X \to X$ and a linear subspace Y of X, we denote by A_Y the operator with domain

$$D(A_Y) := \{\phi \in D(A) \cap Y : A\phi \in Y\}$$

and defined as $A_Y\phi := A\phi$ for $\phi \in D(A_Y)$. The operator A_Y is called the *part of A in Y*.

From Proposition 1.7 we find that $[K^\alpha_w]_X \neq K^\alpha$ for all $\alpha \in \mathbb{C}_+$ with $\operatorname{Re}\alpha > 1$, since we know that $D(K^\infty) \subset D(K^\alpha)$, and $D(K^\alpha)\backslash D(K^\alpha_w) \neq \emptyset$. However, as we will see later, there exists a certain relation between both operators.

PROPOSITION 2.11. *If $0 < \operatorname{Re}\alpha < 1$ and $f \in D(K)$, then $K^\alpha_w f = K^\alpha f$.*

Proof. We know that $D(K)$ is contained in $D(K^\alpha)$. From Theorem 1.3, it is also contained in the domain of K^α_w. Let $f \in D(K)$ and

$$\Omega := \{x > 0 : (K^\alpha_w f)(x) \text{ exists}\}.$$

The set $\mathbb{R}\setminus\Omega$ has measure zero. Given $x \in \Omega$, we are going to prove that the function $\lambda \mapsto \lambda^{-\alpha}[K(1+\lambda K)^{-1}f](x)$ is integrable in $]0,\infty[$. Given $b > x$, writing $[K(1+\lambda K)^{-1}f](x) = g(\lambda) + h(\lambda)$, where

$$g(\lambda) := e^{\lambda x}\int_x^b e^{-\lambda t} f(t)\,dt, \qquad h(\lambda) := e^{\lambda x}\int_b^\infty e^{-\lambda t} f(t)\,dt,$$

and applying the Tonelli theorem, we see that $\lambda^{-\alpha}g(\lambda)$ is integrable in $]0,\infty[$, and

$$\begin{aligned}\int_0^\infty \lambda^{-\alpha} g(\lambda)\,d\lambda &= \int_x^b\Big(\int_0^\infty \lambda^{-\alpha}e^{-\lambda(t-x)}\,d\lambda\Big) f(t)\,dt\\ &= \Gamma(1-\alpha)\int_x^b (t-x)^{\alpha-1} f(t)\,dt.\end{aligned}$$

On the other hand, by using integration by parts, we can prove without difficulty the estimate

$$|h(\lambda)| \le 2e^{-\lambda(b-x)}\sup_{t\ge b}\Big|\int_t^\infty f(s)\,ds\Big|,$$

which implies that the function $\lambda^{-\alpha}h(\lambda)$ is integrable and

$$\Big|\int_0^\infty \lambda^{-\alpha}h(\lambda)\,d\lambda\Big| \le 2\Gamma(1-\operatorname{Re}\alpha)(b-x)^{\operatorname{Re}\alpha-1}\sup_{t\ge b}\Big|\int_t^\infty f(s)\,ds\Big|.$$

The right side tends to 0 as $b \to \infty$. So, the function $\lambda^{-\alpha}[K(1+\lambda K)^{-1}f](x)$ is integrable with respect to λ on the interval $]0,\infty[$, and we have

$$\begin{aligned}\int_0^\infty \lambda^{-\alpha}[K(1+\lambda K)^{-1}f](x)\,d\lambda &= \Gamma(1-\alpha)\lim_{b\to\infty}\int_x^b (t-x)^{\alpha-1}f(t)\,dt \\ &= \Gamma(1-\alpha)\Gamma(\alpha)(K_w^\alpha f)(x).\end{aligned}$$

Finally, since the topology of X has the property of convergence almost everywhere, it is easy to prove that there exists a set $\Omega_0 \subset \Omega$ such that $\Omega \setminus \Omega_0$ has measure zero and for all $x \in \Omega_0$ we have

$$\Big(\int_0^\infty \lambda^{-\alpha}[K(1+\lambda K)^{-1}f]\,d\lambda\Big)(x) = \int_0^\infty \lambda^{-\alpha}[K(1+\lambda K)^{-1}f](x)\,d\lambda.$$

Therefore, $(K_w^\alpha f)(x) = (K^\alpha f)(x)$ for all $x \in \Omega_0$, and the proposition is proved. ∎

Theorem 2.12. *For all $\alpha \in \mathbb{C}_+$ the fractional power K^α is an extension of $[K_w^\alpha]_X$, and so the latter operator is closable. If $0 < \operatorname{Re}\alpha < 1$, both operators are equal. If $\operatorname{Re}\alpha = 1$, the power operator K^α coincides with the closure of the operator $[K_w^\alpha]_X$.*

Proof. To see that K^α extends $[K_w^\alpha]_X$, let n be an integer such that $n > \operatorname{Re}\alpha$. Given $f \in D([K_w^\alpha]_X)$ we will prove by induction on n that $(1+K)^{-1}f \in D(K^\alpha)$ and

$$(1+K)^{-1}K_w^\alpha f = K^\alpha(1+K)^{-1}f.$$

For $n = 1$ this is a direct consequence of Proposition 2.11 and Remark 2.4. Now, assuming the assertion holds up to $n-1$, consider α such that $0 < \operatorname{Re}\alpha < n$. Then taking into account the induction hypothesis, jointly with Theorem 1.3 and the additivity of fractional powers, we obtain

$$\begin{aligned}(1+K)^{-1}K_w^\alpha f &= K_w^{\alpha-\alpha/n}K_w^{\alpha/n}(1+K)^{-1}f \\ &= K_w^{\alpha-\alpha/n}K^{\alpha/n}(1+K)^{-1}f \\ &= K^{\alpha-\alpha/n}K^{\alpha/n}(1+K)^{-1}f = K^\alpha(1+K)^{-1}f.\end{aligned}$$

By the characterization of fractional powers (see [7, Def. 2.1 and Corollary 3.5] and [9, Th. 5.2.1]), the relation obtained shows that $f \in D(K^\alpha)$ and that $K_w^\alpha f$ and $K^\alpha f$ coincide.

Suppose that $0 < \operatorname{Re}\alpha < 1$ and $f \in D(K^\alpha)$. As

$$(K+1)^{-1}K^\alpha f \in D(K),$$

from Theorem 1.3 it follows that the function $\phi = K_w^{1-\alpha}(K+1)^{-1}K^\alpha f$ belongs to $D([K_w^\alpha]_X)$. On the other hand, applying Proposition 2.11 and the additivity of fractional powers, we obtain

$$\begin{aligned}\phi &= K_w^{1-\alpha}(K+1)^{-1}K^\alpha f = K^{1-\alpha}(K+1)^{-1}K^\alpha f\\ &= K^{1-\alpha}K^\alpha(K+1)^{-1}f = K(K+1)^{-1}f.\end{aligned}$$

Then, as $K(K+1)^{-1}f = f-(K+1)^{-1}f$ and $(K+1)^{-1}f \in D(K) \subset D([K_w^\alpha]_X)$, we obtain $f \in D([K_w^\alpha]_X)$.

Finally, in the case $\operatorname{Re}\alpha = 1$, given $f \in D(K^\alpha)$, for all $\lambda > 0$ we have

$$K^\alpha(1+\lambda K)^{-1}f = K^{\alpha/2}K^{\alpha/2}(1+\lambda K)^{-1}f.$$

Then, from Proposition 2.11, it follows that

$$K^{\alpha/2}(1+\lambda K)^{-1}f = K_w^{\alpha/2}(1+\lambda K)^{-1}f$$

and, by the arguments in the case of exponents less than one,

$$K^{\alpha/2}K_w^{\alpha/2}(1+\lambda K)^{-1}f = K_w^{\alpha/2}K_w^{\alpha/2}(1+\lambda K)^{-1}f.$$

In addition, from Theorem 1.5 we obtain

$$(1+\lambda K)^{-1}f \in D(K_w^\alpha), \quad K_w^\alpha(1+\lambda K)^{-1}f = (1+\lambda K)^{-1}K^\alpha f. \tag{12}$$

Letting $\lambda \to 0$ and taking into account that $D(K)$ is dense in X and that K^α is a closed operator, we obtain $f \in D(\overline{[K_w^\alpha]_X})$. ∎

REMARK 2.13. When $\operatorname{Re}\alpha \geq 1$, we do not know if $[K_w^\alpha]_X$ is a closed operator. Nor do we know if $\overline{[K_w^\alpha]_X} = K^\alpha$ when $\operatorname{Re}\alpha > 1$. In the case $\operatorname{Re}\alpha > 1$, if we try to prove this last property, we have no relation of type (12). In this case, the use of an integer power of the resolvent operator does not improve the situation. In fact, if for all $f \in D(K^\alpha)$ there exists an integer $m > 0$ such that $(1+\lambda K)^{-m}f \in D(K_w^\alpha)$, then by considering the function $(1+\lambda K)^m f_{\alpha-1} \in D(K^\infty) \subset D(K^\alpha)$, from Proposition 1.7 we would get a contradiction, since as we saw in the proof of that proposition, $f_{\alpha-1} \notin D(K_w^\alpha)$.

3. Fractional integral of Weyl in $L^p(\mathbb{R})$ and other subspaces of X. In this section, we will use the following result about fractional powers in sequentially complete locally convex spaces (see [9, Th. 12.1.6]).

THEOREM 3.1. *Let E be a sequentially complete locally convex space, and $A : D(A) \subset E \to E$ a non-negative linear operator. Let $F \subset E$ be a vector subspace of the same type, not necessarily with the same topology, such that the operator A_F (the part of A in F) is non-negative. Let $\alpha \in \mathbb{C}_+$. If there exists an integer $n >$ $\operatorname{Re}\alpha$ such that $A^\alpha\phi = (A_F)^\alpha\phi$ for all $\phi \in D[(A_F)^n]$, then*

$$(A^\alpha)_F = (A_F)^\alpha.$$

From this theorem jointly with Theorem 2.12, we deduce the following result about the part of the operator K_w^α in certain subsets of X, such as the $L^p(\mathbb{R})$ spaces.

As usual, given $a \in \mathbb{R}$, the characteristic function of the interval $[a, \infty[$ is denoted by χ_a.

Theorem 3.2. *Let $F \subset X$ be a sequentially complete locally convex vector space whose topology has the property of pointwise convergence almost everywhere, such that the operator K_F is non-negative. Then*

$$\begin{aligned} [K_w^\alpha]_F &= (K_F)^\alpha \quad &&\textit{for } 0 < \operatorname{Re}\alpha < 1, \\ \overline{[K_w^\alpha]_F} &= (K_F)^\alpha \quad &&\textit{for } \operatorname{Re}\alpha = 1, \\ (K_F)^\alpha &\supset [K_w^\alpha]_F \quad &&\textit{for } \operatorname{Re}\alpha > 1. \end{aligned}$$

In particular, this is valid for $F = L^p(\mathbb{R})$ with $1 \le p < \infty$, and for the space $C_{+\infty}(\mathbb{R})$ of continuous functions which vanish at $+\infty$, endowed with the seminorms $\{\|\chi_a f\|_\infty : a \in \mathbb{R}\}$.

Proof. Let us denote by τ_X and τ_F the topologies of the spaces X and F. First, we are going to show that $(K_F)^\alpha = [K^\alpha]_F$ for $\alpha \in \mathbb{C}_+$. For this, taking into account Theorem 3.1, it is sufficient to prove that $(K_F)^\alpha f = K^\alpha f$ for $f \in D[(K_F)^n]$ with $n > \operatorname{Re}\alpha$. That is, we will see that the value of $\int_0^\infty \lambda^{n-\alpha} K^n (1+\lambda K)^{-n} f\, d\lambda$ does not depend on the topology, τ_X or τ_F, considered.

This fact is a direct consequence of the property of pointwise convergence almost everywhere, since the function $\lambda^{n-\alpha} K^n (1+\lambda K)^{-n} f$ is integrable in $]0,\infty[$ with respect to both topologies, the operators K and K_F are non-negative, and $f \in D[(K_F)^n]$. From the identity $(K_F)^\alpha = [K^\alpha]_F$ and Theorem 2.12 we deduce that $[K_w^\alpha]_F = (K_F)^\alpha$ when $0 < \operatorname{Re}\alpha < 1$, and the second operator is an extension of the first when $\operatorname{Re}\alpha \ge 1$.

In the case $\operatorname{Re}\alpha = 1$, in order to prove the identity $\overline{[K_w^\alpha]_F}$ and $(K_F)^\alpha$ it is enough to observe that the relation (12) implies that, given $f \in D((K_F)^\alpha)$, we have

$$(1+\lambda K_F)^{-1} f \in D(K_w^\alpha)$$

and

$$K_w^\alpha (1+\lambda K_F)^{-1} f = (1+\lambda K_F)^{-1} (K_F)^\alpha f.$$

From this, we proceed as in Theorem 2.12, now taking limits in F as λ goes to infinity.

Next, we are going to see that the result can be applied to the concrete spaces mentioned above. In fact, if $1 \le p < \infty$, for $f \in L^p(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$\Big| \int_x^\infty e^{-(t-x)} f(t)\, dt \Big| \le \begin{cases} q^{-1/q} \|\chi_x f\|_p & \text{if } p > 1 \text{ (where } 1/p + 1/q = 1), \\ \|\chi_x f\|_1 & \text{if } p = 1, \end{cases}$$

which proves that $L^p(\mathbb{R}) \subset X$. Likewise, Young's inequality implies that, for all $\lambda > 0$, $K(1+\lambda K)^{-1} f \in L^p(\mathbb{R})$ and

$$\|\lambda K(1+\lambda K)^{-1} f\|_p \leq \|f\|_p.$$

For the space $C_{+\infty}(\mathbb{R})$, we can reason in the same way. ∎

REMARK 3.3. In [12, Th. 3.13], we proved that for $F = L^p(\mathbb{R})$ with $1 < p < \infty$ and $\alpha \in \mathbb{C}_+$,

$$\overline{[K_w^\alpha]_F} = (K_F)^\alpha.$$

But in this case, this result follows from the fact that the integral operator of Weyl is the adjoint operator of the integral operator of Riemann–Liouville (see [9, Th. 2.1.4]), that the fractional powers commute with the adjoint ([9, Corollary 5.2.4]), and the fractional integral of Riemann–Liouville coincides with the corresponding power of the integral operator of Riemann–Liouville [12, Th. 2.6]).

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Celso Martínez, Miguel Sanz
Departament de Matemàtica Aplicada
Universitat de València
46100 Burjassot, València, Spain
E-mail: martinel@uv.es
sanzma@uv.es

Antonia Redondo
Departamento de Matemáticas
I.E.S. Bachiller Sabuco
02002 Albacete, Spain
E-mail: aredondo@sabuco.com

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