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Directionally Euclidean structures of Banach spaces

by

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Abstract. We study Banach spaces with directionally asymptotically controlled ellipsoid-approximations of the unit ball in finite-dimensional sections. Here these ellipsoids are the unique minimum volume ellipsoids, which contain the unit ball of the corresponding finite-dimensional subspace. The directional control here means that we evaluate the ellipsoids by means of a given functional of the dual space. The term 'asymptotical' refers to the fact that we take 'lim sup' over finite-dimensional subspaces.

This leads to isomorphic and isometric characterizations of Hilbert spaces. An application involving Mazur's rotation problem is given. We also discuss the stability of the family of ellipsoids as the dimension and geometry vary. The methods exploit ultrafilter techniques and we also apply them in conjunction with finite Auerbach bases to study the convexity properties of the duality mappings.

1. Introduction. This paper deals with the local theory of Banach spaces. Here that roughly means that we will make deductions about the geometry of Banach spaces by studying the asymptotical behaviour of its finite-dimensional subspaces as the dimension grows.

In a sense, there is a natural way of approximating a norm of a finite-dimensional normed space by a norm induced by an inner product. Namely, by compactness there exists an ellipsoid with the minimal volume (m.v.), i.e. minimal Lebesgue measure, which contains the unit ball. It was an interesting observation made by Auerbach that such a m.v. ellipsoid is unique and thus preserved by linear isometries of the normed space (see [1], [2]). Thus, by now it is a rather standard practice to use the Hilbertian norm induced by the m.v. ellipsoid to approximate the original norm. However, there are several other approaches as well (see [9]).

In Hilbert spaces the minimum volume ellipsoids in finite-dimensional subspaces are precisely the unit balls of the corresponding subspaces and a

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fortiori uniformly bounded. On the other hand, it is not difficult to see that if the Banach space in question is not isomorphic to a Hilbert space, then the diameter of the minimum volume ellipsoids is not uniformly bounded. It turns out that the diameter of the ellipsoids may not be uniformly bounded even if the space is isomorphically Hilbertian. Here we will study spaces such that the ellipsoids are asymptotically bounded in some direction. More precisely, the term 'Directionally Euclidean Structures' in the title refers to spaces X with an $f \in \mathrm{X}^*$, $f \neq 0$, such that that for sufficiently large finite-dimensional subspaces F the corresponding m.v. ellipsoid $\mathcal{E}_F$ has the property that its image under $|f|$ is bounded by a uniform constant. This will be stated accurately shortly. One of the key features here is that the image of m.v. ellipsoids under f is sensitive to the selection of the *particular functional*, as well as to the *geometry* of the unit ball.

To make a short introductory comment about the content of this paper and its connections, this topic has a flavour slightly similar to the study of weak Hilbert spaces and it relies heavily on ultrafilter based analysis. The methodology and the problem setting here are closely related to the papers [5] and [15].

We will characterize Banach spaces isomorphic to Hilbert spaces as those spaces which have directionally Euclidean structure in every direction, up to isomorphism. As an application, we will obtain a result related to Mazur's rotation problem. Interestingly, it turns out that the determination of the ellipsoids is chaotic in a sense with respect to changes in equivalent renormings, even small ones.

1.1. Preliminaries. We refer to [7], [11] and [14] for relevant general background information. The known 'Mazur's rotation problem' appearing in Banach's book is discussed extensively in the survey [3], and for local theory of Banach spaces involving ellipsoids we refer to [13].

Here X and Y stand for real Banach spaces, $\mathbf{B}_{\mathrm{X}}$ is the closed unit ball and $\mathbf{S}_{\mathrm{X}}$ the unit sphere of X. We denote by $\mathrm{Aut}(\mathrm{X})$ the group of isomorphisms $T\colon \mathrm{X} \to \mathrm{X}$. The *rotation group* $\mathcal{G}_{\mathrm{X}}$ is the subgroup of all the isometries in $\mathrm{Aut}(\mathrm{X})$. The identity map $I\colon \mathrm{X} \to \mathrm{X}$ is the neutral element. The group of finite-dimensional perturbations of the identity is given by

$$\mathcal{G}_F = \{T \in \mathcal{G}_{\mathrm{X}} : \mathrm{Rank}(I - T) < \infty\}.$$

We say that X is *convex-transitive* with respect to $\mathcal{G} \subset \mathcal{G}_{\mathrm{X}}$ if $\overline{\mathrm{conv}}(\{Tx : T \in \mathcal{G}\}) = \mathbf{B}_{\mathrm{X}}$ for all $x \in \mathbf{S}_{\mathrm{X}}$. A stronger condition is *almost transitivity* with respect to $\mathcal{G}$, namely that $\overline{\{Tx : T \in \mathcal{G}\}} = \mathbf{S}_{\mathrm{X}}$ for each $x \in \mathbf{S}_{\mathrm{X}}$.

We call a bilinear form $B\colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ with $B(x,x) \geq 0$ for $x \in \mathrm{X}$ *positive semidefinite* and in the symmetric case a *degenerate inner product* informally. If additionally $B(x,x) > 0$ for all $x \neq 0$, then B is an inner product. In such a case we also call B, slightly overemphasising, a non-degenerate inner product. Following e.g. [1], an *ellipsoid* is a set of the form $\{x \in E : (x|x) \leq 1\}$, where $(\cdot|\cdot)$ is an inner product on a finite-dimensional (sub)space E. The set of finite-dimensional subspaces of X will usually be denoted by $\mathcal{F} = \mathcal{F}(\mathrm{X})$.

2. Directionally bounded ellipsoids. For a finite-dimensional subspace $E \subset \mathrm{X}$ and $f \in \mathbf{S}_{\mathrm{X}^*}$ put

$$\eta(f, E) = \max\{|fy| : y \in \mathcal{E}_E\},$$

where $\mathcal{E}_E \subset E$ is the unique minimum volume ellipsoid containing $\mathbf{B}_{\mathrm{X}} \cap E$ (following [1]). We say that a Banach space X has *Directionally Euclidean Structure* (DES) in direction $f \in \mathbf{S}_{\mathrm{X}^*}$ if

$$\inf_F \sup_E \eta(f, E) < \infty, \tag{2.1}$$

where the infimum is taken over finite-dimensional spaces $F \subset \mathrm{X}$ and the supremum is taken over finite-dimensional subspaces $E \subset \mathrm{X}$ with $F \subset E$. If the infimum in (2.1) is $\lambda \in [1, \infty)$, then we say that X has λ-DES in direction f.

Clearly finite-dimensional spaces have DES in every direction, since the unique minimum volume ellipsoid containing the unit ball is bounded ([1]). We do not know whether the property of having DES in all directions is inherited by subspaces. The directionally Euclidean structure is preserved under the isometry group in the sense that if X has DES in direction $f \in \mathbf{S}_{\mathrm{X}^*}$, and $T \in \mathcal{G}_{\mathrm{X}}$, then X also has DES in direction $T^* f$. On the other hand, DES is not preserved under isomorphisms. For instance, for each $x \in \mathbf{S}_{\mathrm{X}}$ one can present X isomorphically as $[x] \oplus_2 \mathrm{Y}$ for suitable $\mathrm{Y} \subset \mathrm{X}$ and it turns out that $[x] \oplus_2 \mathrm{Y}$ has DES in the direction of the axis $[x]$, roughly speaking (see Theorem 2.6).

Let us begin with an observation on the existence of continuous inner products with controlled spread in one direction.

THEOREM 2.1. *Let $f \in \mathbf{S}_{\mathrm{X}^*}$ and consider the following conditions:*

1. X *has DES in direction f.*
2. $\sup_F \inf_E \eta(f, E) < \infty$, *where $F \subset E$ are finite-dimensional.*
3. *There exists a (possibly degenerate) inner product $(\cdot|\cdot)$ on X such that $(x|y) \leq \|x\| \cdot \|y\|$ for $x, y \in \mathrm{X}$ and*

$$(z|z) \geq \left(\frac{|f(z)|}{\sup_F \inf_E \eta(f, E)} \right)^2 \quad \textit{for each } z \in \mathrm{X}.$$

Then (1)⇒(2)⇒(3).

In the proof of this result we will use the following fact, which is easy to see by examining the set $\{z \in X : f(z) \preceq y\}$.

FACT 2.2. *Let $(X, \leq)$ be a directed set, $(Y, \preceq)$ a poset and $f\colon X \to Y$ a map. Suppose that there exists $y \in Y$ with the following property: For each $x \in X$ there exists $z \in X$ such that $x \leq z$ and $f(z) \preceq y$. Then there exists a subset $Z \subset X$ satisfying the following conditions:*

(i) *$(Z, \leq)$ is a directed set.*
(ii) *For each $x \in X$ there is $z \in Z$ with $x \leq z$.*
(iii) *$f(z) \preceq y$ for each $z \in Z$.*

Proof of Theorem 2.1. Instead of proving (1)⇒(2) we will prove a stronger statement, namely that $\sup_F \inf_E \eta(f, E) \leq \inf_F \sup_E \eta(f, E)$. Towards this, fix $\epsilon > 0$. Observe that if F_0 is a finite-dimensional subspace such that $\sup_{E_0} \eta(f, E_0) + \epsilon \leq \inf_F \sup_E \eta(f, E)$, where $F_0 \subset E_0$ are finite-dimensional, then

$$\sup_F \inf_E \eta(f, E) + \epsilon \leq \inf_F \sup_E \eta(f, E)$$

because on the left the infimum can be taken over E such that $F, F_0 \subset E$. Thus the statement holds as ϵ was arbitrary.

To check (2)⇒(3), let $\mathcal{F}$ be the set of all finite-dimensional subspaces of X ordered by inclusion. Consider the map $\alpha\colon \mathcal{F} \to [0, \infty)$ given by $E \mapsto \eta(f, E)$.

According to Fact 2.2 and the fact $\sup_F \inf_E \alpha(E) < \infty$ we find that there is for each $i \in \mathbb{N}$ a directed subset $\mathcal{F}_i \subset \mathcal{F}$ satisfying the statements (i), (ii) of the fact and $\alpha(E) \leq \sup_F \inf_E \alpha(E) + 1/i$ for $E \in \mathcal{F}_i$. Let $\mathcal{M} = \bigcup_i \mathcal{F}_i$.

For each $E \in \mathcal{M}$ let $(\cdot|\cdot)_E$ be the inner product on E corresponding to the minimum volume ellipsoid $\mathcal{E}_E$. For each $E \in \mathcal{M}$ let $P_E\colon \mathrm{X} \to E$ be a (bounded) linear projection. Consider $\mathbb{R}^{\mathcal{M}}$ with the pointwise linear structure. Define a map $B\colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}^{\mathcal{M}}$ by $B(x, y)(E) = (P_E x | P_E y)_E$. Clearly B is a bilinear map.

The family

$$\{\{E \in \mathcal{M} : F \subset E,\ \alpha(E) \leq \inf_E \alpha(E) + i^{-1}\}\}_{(F,i) \in \mathcal{M} \times \mathbb{N}}$$

is a filter base on $\mathcal{M}$. Let $\mathcal{U}$ be an ultrafilter extending the above filter base. Put $(x|y)_{\mathrm{X}} = \lim_{E,\mathcal{U}} B(x, y)(E)$ for $x, y \in \mathrm{X}$. It is easy to see that $(\cdot|\cdot)_{\mathrm{X}}$ is a bilinear mapping.

Let us check that $(x|y)_{\mathrm{X}} \leq \|x\| \cdot \|y\|$ for $x, y \in \mathrm{X}$. Indeed, first observe that the set $\{E \in \mathcal{M} : x, y \in E\}$ contains sets in the filter base and thus this set is in $\mathcal{U}$. For this reason $\lim_{E,\mathcal{U}}(P_E(x) - x) = 0$ and $\lim_{E,\mathcal{U}}(P_E(y) - y) = 0$. Since $\mathbf{B}_E \subset \mathcal{E}_E$ for each E, we see that $(x|x)_E \leq \|x\|^2$ for $x \in \mathrm{X}$ and $E \in \mathcal{M}$

such that $x \in E$. Thus

$$\begin{aligned}\lim_{E,\mathcal{U}}(P_E x|P_E y)_E &\leq \lim_{E,\mathcal{U}} \sqrt{(P_E x|P_E x)_E (P_E y|P_E y)_E} \\ &\leq \lim_{E,\mathcal{U}} \sqrt{\|x\|^2 \cdot \|y\|^2} = \|x\| \cdot \|y\|.\end{aligned}$$

By our selection of the filter base, we have

$$\lim_{D,\mathcal{U}}(\alpha(D) - \sup_F \inf_E \eta(f,E)) = 0.$$

Let $\delta_E(x) = \sup\{a > 0 : ax \in \mathcal{E}_E\}$ for $x \in E$, $x \neq 0$, $E \in \mathcal{M}$. Since $|f(\delta_E(x)x)| \leq \alpha(E)$ for each E such that $x \in E$, we obtain

$$\frac{|f(x)|}{\alpha(E)} \leq \frac{1}{\delta_E(x)},$$

which yields

$$\frac{|f(x)|}{\sup_F \inf_E \alpha(E)} = \lim_{E,\mathcal{U}} \frac{|f(x)|}{\alpha(E)} \leq \lim_{E,\mathcal{U}} \frac{1}{\delta_E(x)} = \sqrt{(x|x)_{\mathrm{X}}}. \quad \blacksquare$$

There is one considerable difference between the lim inf and lim sup quantities above, that is, $\sup_F \inf_E \eta(f,E)$ and $\inf_F \sup_E \eta(f,E)$. Namely, by using the former, weaker, control we may construct inner products which do not vanish in the given direction f. Later we wish to construct inner products with a control simultaneously in several directions, and then the latter, stronger, concept is required. Next, it turns out that DES in every direction is a strong enough property of Banach spaces to characterize Hilbert spaces up to isomorphism.

Theorem 2.3. *Let* X *be a Banach space. The following conditions are equivalent:*

(i) X *is isomorphic to a Hilbert space.*
(ii) X *is isomorphic to a space* Y *having DES in every direction* $f \in \mathbf{S}_{\mathrm{Y}^*}$.

Moreover, X *is isometric to a Hilbert space if and only if it has* 1*-DES in every direction* $f \in \mathbf{S}_{\mathrm{X}^*}$.

Proof. First observe that a Hilbert space clearly has 1-DES in every direction. This covers the implication (i)⇒(ii) and the last 'only if' statement.

Suppose that X is isomorphic to Y having DES in all directions $f \in \mathbf{S}_{\mathrm{Y}^*}$. Let us resume the notation of the proof of Theorem 2.1: we let $\mathcal{F}$ be the set of finite-dimensional subspaces of Y ordered by inclusion. Let $\mathcal{U}$ here be an ultrafilter on $\mathcal{F}$ containing the filter base $\{\{E \in \mathcal{F} : F \subset E\}\}_{F \in \mathcal{F}}$. We still denote $(x|y)_{\mathrm{Y}} = \lim_{E,\mathcal{U}}(P_E x|P_E y)_E$. By construction, $(\cdot|\cdot)_{\mathrm{Y}}$ is a bilinear map with $(x|y)_{\mathrm{Y}} \leq \|x\| \cdot \|y\|$ for $x, y \in \mathrm{X}$.

Now let us study the 'ball' $B = \{y \in \mathrm{Y} : (y|y)_{\mathrm{Y}} \le 1\}$. Pick $b \in B$ with $\lim_{E,\mathcal{U}} (P_E b | P_E b)_E < 1$. We obtain

$$|f(b)| \le \lim_{E,\mathcal{U}} \sup_{c \in \mathcal{E}_E} |f(c)| \le \inf_F \sup_E \eta(f, E). \tag{2.2}$$

If Y has DES in all directions, then the right hand side of (2.2) is finite. Thus we obtain $\sup_{b \in B} |f(b)| < \infty$. Now, since $f \in \mathbf{S}_{\mathrm{Y}^*}$ was arbitrary, the Uniform Boundedness Principle implies that B is norm-bounded. Observe that $\mathbf{B}_Y \subset B$ by the selection of the ellipsoids $\mathcal{E}_E$. This can be rephrased as follows: there exists $1 \le C < \infty$ such that

$$\|y\|^2 \le (y|y)_{\mathrm{Y}} \le C\|y\|^2 \quad \text{for } y \in \mathrm{Y}.$$

To check the last claim of the theorem, we will apply the above argument with $\mathrm{X} = \mathrm{Y}$, where this space has 1-DES in all directions. It follows that $|f(b)| \le 1$ in (2.2). Since this holds for all $b \in B$ and all $f \in \mathbf{S}_{\mathrm{X}^*}$, we get $B \subset \mathbf{B}_{\mathrm{X}}$. Thus $B = \mathbf{B}_{\mathrm{X}}$ and hence $\|x\|^2 = (x|x)_{\mathrm{X}}$ for $x \in \mathrm{X}$. ∎

Remark 2.4. We note that in Theorem 2.3 the condition (ii) could be replaced by an equivalent condition (ii′): X is isomorphic to a space Y for which there exists a norming subspace $\mathrm{Z} \subset \mathrm{Y}^*$ such that Y has DES in every direction $f \in \mathbf{S}_{\mathrm{Z}}$.

2.1. On boundedness of ellipsoids containing the unit ball. In a Banach space a continuous non-degenerate inner product corresponds to a convex body that might be unbounded but does not contain a 1-dimensional linear subspace. There are plenty of such convex bodies, or inner products, according to the following fact, which is probably folklore.

Proposition 2.5. *Let* X *be a Banach space with a ω^*-separable dual space. Then there exists an inner product $(\cdot|\cdot)$ on* X *such that $(x|x) \le \|x\|^2$ and $(x|x) > 0$ for $x \in \mathrm{X}$, $x \neq 0$. This in turn implies that there exists a continuous linear injection from* X *into $c_0(\Gamma)$ for some set Γ.*

Proof. Let $(x_n^*) \subset \mathrm{X}^*$ be a sequence which is ω^*-dense in X^* and does not contain 0. It is easy to see that the mapping $g\colon \mathrm{X} \to \ell^\infty$ given by $x \mapsto (x_n^*(x)/\|x_n^*\|)$ is linear, contractive and injective. The injectivity follows from the fact that (x_n^*) separates each $x \in \mathrm{X}$, $x \neq 0$, as it is ω^*-dense. Next, observe that $f\colon \ell^\infty \to \ell^2$ given by $(z_n) \mapsto (2^{-n} z_n)$ is linear, contractive and injective. Now, the required inner product on X is induced by one in ℓ^2 via the composite mapping $f \circ g$.

For the last claim of the proposition, we first form the completion H of $(\mathrm{X}, \|\cdot\|_2)$ and consider its orthonormal basis $\{e_\gamma\}_{\gamma \in \Gamma}$. The required map $\mathrm{X} \to c_0(\Gamma)$ is then given by $x \mapsto \{(x|e_\gamma)\}_{\gamma \in \Gamma}$. ∎

Spaces that admit a continuous linear injection into $c_0(\Gamma)$ have been studied quite a bit (see e.g. [10]). We do not know exactly what kind of

Banach spaces admit a continuous (non-degenerate) inner product. Note that a continuous inner product will induce a continuous norm on X, but according to the Open Mapping Principle this norm will be *complete* if and only if the original norm and the induced norm are equivalent.

Even though there are often continuous inner products on Banach spaces, the way the inner products are produced here, by applying DES, results in constructions that turn out to be sensitive to small changes of the norm. We also note that continuous inner products are typically not invariant under isometries of the space. However, directionally Euclidean structure allows us to construct continuous inner products that will be invariant under suitable isometries.

The following result suggests that the disposition of the minimum volume ellipsoids becomes in a sense chaotic as the subspaces vary. It follows in particular that the property of having DES in all directions is not preserved under isomorphisms, even for small Banach–Mazur distances.

THEOREM 2.6. *Let* X *be a Banach space and* $F, E \subset \mathrm{X}$ *be subspaces such that* $\mathrm{X} = F \oplus_p E$ *isometrically for some* $1 \leq p \leq \infty$*. If* $1 \leq p \leq 2$ *and* $\dim(F) < \infty$*, then* X *has DES in every direction* $(f, 0) \in \mathbf{S}_{F^* \oplus_{p^*} E^*}$*. On the other hand, if* $2 < p \leq \infty$ *and* $E = \ell^2$*, then* X *does not have DES in any direction* $(f, 0) \in \mathbf{S}_{F^* \oplus_{p^*} \ell^2}$*. However, if* $\mathrm{X} = \ell^2 \oplus_p \ell^2$*,* $2 < p \leq \infty$*, and* $g = (f, 0) \in \mathbf{S}_{\ell^2 \oplus_{p^*} \ell^2}$*, then* $\sup_F \inf_E \eta(g, E) < \infty$*.*

Proof. For both the cases $p \leq 2$ and $p > 2$ we are interested in finite-dimensional subspaces of the type $F \oplus_p E_n$, where $E_n \subset E$ is an n-dimensional subspace. This is so because each finite-dimensional subspace $\mathrm{Y} \subset \mathrm{X}$ is contained in a finite-dimensional subspace of the above type.

Denote by $\mathcal{E} \subset F \oplus_p E_n$ the unique minimal volume ellipsoid containing $\mathbf{B}_{F \oplus_p E_n}$. Let $(\cdot|\cdot)_{\mathcal{E}}$ be the corresponding inner product. According to Auerbach's results this ellipsoid is invariant under linear isometries of $F \oplus_p E_n$ onto itself. In particular, given a linear projection $P_F \colon F \oplus_p E_n \to F$, the ellipsoid $\mathcal{E}$ is invariant under the isometric reflection mapping $\mathbf{I} - 2P_F$. Denote $Q_F = \mathbf{I} - P_F$.

CLAIM 1. *For each* $x \in F \oplus_p E_n$*,*

$$(x|x)_{\mathcal{E}} = (P_F x|P_F x)_{\mathcal{E}} + (Q_F x|Q_F x)_{\mathcal{E}}.$$

Indeed, by using invariance we obtain

$$\begin{aligned}(x|x)_{\mathcal{E}} &= ((\mathbf{I} - 2P_F)x|(\mathbf{I} - 2P_F)x) \\ &= (x|x)_{\mathcal{E}} - 2(x|P_F x)_{\mathcal{E}} - 2(P_F x|x)_{\mathcal{E}} + 4(P_F x|P_F x)_{\mathcal{E}},\end{aligned}$$

which gives $(x|P_F x)_{\mathcal{E}} = (P_F x|P_F x)_{\mathcal{E}}$, and this yields the claim.

Next, recall that each ellipsoid is given by the formula $x_1^2/C_1 + \cdots + x_m^2/C_m \leq 1$ where one fixes a suitable coordinate system. If one normalizes the measure by fixing the volume of $\mathbf{B}_{F\oplus_p E_n}$, then the volume of the ellipsoid does not depend on the selection of the coordinate system. With the above notation the volume of the ellipsoid $\mathcal{E}$ is

$$\mathrm{Vol}(\mathcal{E}) = \beta C_1 \dots C_m,$$

where β is a constant depending on the dimension and the coordinate system.

Let us identify $F \oplus_p E_n$ with $\mathbb{R}^m$. Thus we will regard the volume as the standard m-dimensional Lebesgue measure. According to Claim 1 we may assume that the coordinate system is such that the first $\dim(F)$ coordinates of $\mathbb{R}^m$ support the ellipsoid $P_F(\mathcal{E})$ and the last n coordinates support the ellipsoid $Q_F(\mathcal{E})$. We may assume without loss of generality, by choosing the coordinate system suitably, that $\mathbf{B}_{\ell^2(k+n)} \subset \mathbf{B}_{F\oplus_p E}$. Clearly, the fact that $\mathbf{B}_{\ell^2(k)} \subset P_F(\mathcal{E})$ and $\mathbf{B}_{\ell^2(n)} \subset Q_F(\mathcal{E})$ implies that $C_1, \dots, C_{k+n} \geq 1$.

Fix the minimal volume ellipsoids $\mathcal{E}_1 \subset F$, $\mathcal{E}_2 \subset E_n$ containing $\mathbf{B}_F$ and $\mathbf{B}_{E_n}$, respectively. Then the corresponding constants satisfy $C_1^{(F)} \dots C_k^{(F)} \leq C_1 \dots C_k$ and $C_{k+1}^{(E)} C_{k+2}^{(E)} \dots C_m^{(E)} \leq C_{k+1} C_{k+2} \dots C_m$.

Let us verify the statement involving $p \leq 2$. We obtain

$$\mathbf{B}_{F\oplus_p E_n} \subset \{x \in F \oplus_p E_n : (P_F x|P_F x)_{\mathcal{E}_1} + (Q_F x|Q_F x)_{\mathcal{E}_2} \leq 1\}.$$

We conclude that

$$(x|x)_{\mathcal{E}} = (P_F x|P_F x)_{\mathcal{E}_1} + (Q_F x|Q_F x)_{\mathcal{E}_2} \quad \text{for } x \in F \oplus_p E_n.$$

This means that $P_F(\mathcal{E})$ is a norm bounded set, which does not depend on $n = \dim(E_n)$. Consequently, we have the first part of the statement.

Let us check the latter statement, where $p > 2$ and E is a Hilbert space. The invariance of $\mathcal{E}$ under isometries yields $C_{k+1} = C_{k+2} = \dots = C_m$. Indeed, here we consider isometries of the form $\mathbf{I} \oplus T$, where $\mathbf{I}\colon F \to F$ is the identity map, T is a linear isometry of $E_n = \ell^2(n)$ onto itself, and we apply the fact that the isometry group of a Hilbert space acts transitively on the unit sphere.

Claim 2. *Given $b > 1$ and $2 < p < \infty$, set*

$$\alpha_p(b) = \inf\left\{a > 1 : \left(\frac{x^2}{a} + \frac{y^2}{b}\right)^{1/2} \leq (x^p + y^p)^{1/p} \text{ for } x, y > 0\right\}.$$

Then

$$\frac{b}{2^{2/p}b - 1} \leq \alpha_p(b) \leq \frac{b}{b-1}. \tag{2.3}$$

This is proved by analyzing the test points $(2^{-1/p}, 2^{-1/p})$ and $(1, 1)$. Moreover, it is fairly easy to see that

$$\alpha_p(b) \to \infty \quad \text{as } b \to 1^+ \text{ for } 2 < p \leq \infty. \tag{2.4}$$

Note that according to Claim 2 the constants $C_1, \ldots, C_k$ satisfy the inequality

$$a\frac{C_{k+1}}{2^{2/p}C_{k+1}-1} \le C_i \le b\frac{C_{k+1}}{C_{k+1}-1}, \quad 1 \le i \le k, \tag{2.5}$$

for suitable constants $a, b > 0$ depending only on the disposition of $\mathbf{B}_F$ in $\mathbb{R}^k$, and not on p or the actual value of k or n.

Since $C_{k+1} = C_{k+2} = \cdots = C_m$, it follows by using (2.5) that the expression of the minimal volume,

$$\mathrm{Vol}(\mathcal{E}) = \beta^{(n)} C_1^{(n)} \ldots C_k^{(n)} (C_{k+1}^{(n)})^n,$$

must satisfy $C_{k+1}^{(n)} \to 1^+$ as $n \to \infty$. Thus (2.4) implies that $C_i^{(n)} \to \infty$ as $n \to \infty$ for $1 \le i \le k$. This yields the second claim of the theorem.

For the last claim we may select finite-dimensional subspaces E appearing in $\sup_F \inf_E \eta(g, E)$ to be of the form $E = E_0 \oplus_p E_0$. Then by a simple symmetry argument we obtain $\eta(g, E) = \sup h(\mathcal{E})$, where $\mathcal{E} \subset \ell^p(2)$ is the m.v. ellipsoid containing the unit ball and $h = (1, 0) \in \ell^{p^*}(2)$. ∎

We do not know whether the property of having DES in all (or some) directions passes on to subspaces, ultrapowers, ultra-roots, or to almost isometric copies. Consider Banach spaces X with the property that the minimum volume ellipsoids $\mathcal{E}$ of finite-dimensional subspaces of X are uniformly bounded. We note that this property passes on to the above-mentioned structures related to X. The proof is omitted but we will list some observations which lead to the claims involving ultrapowers.

Observation 1. *The determination of the volume of the minimum volume ellipsoid is continuous with respect to the norm.* This can be formulated more precisely as follows: Suppose that E and F are n-dimensional spaces with Lebesgue measures μ and ν, respectively, normalized so that $\mu(\mathbf{B}_E) = \nu(\mathbf{B}_F) = 1$. If $f\colon E \to F$ is an isomorphism and $\|x\| \le \|f(x)\| \le C\|x\|$ for $x \in E$, then $\mu(A) \le \nu(f(A)) \le C^n \mu(A)$ for each Lebesgue measurable $A \subset E$.

Observation 2. *Each finite-dimensional subspace of* $\mathrm{X}^{\mathcal{U}}$ *is an ultralimit of suitable finite-dimensional subspaces of* X *in the sense of the Banach-Mazur distance.* Let $x_1, \ldots, x_n \in \mathrm{X}^{\mathcal{U}}$ be non-zero, linearly independent vectors. Then for each $1 \le i \le n$ there is a sequence $(z_k^{(i)}) \subset \mathrm{X}$ such that $\lim_{k,\mathcal{U}} \|z_k^{(i)} - x_i\| = 0$. Let $E_k = [z_k^{(1)}, z_k^{(2)}, \ldots, z_k^{(n)}]$ for $k \in \mathbb{N}$. It is not difficult to see that there exists $K \in \mathcal{U}$ such that $T_k\colon E_k \to E$, $T_k(\sum_i a_i z_k^{(i)}) = \sum_i a_i x_i$ defines a linear isomorphism for $k \in K$. Moreover, $\lim_{k,\mathcal{U}} \|T_k\| = 1$ and $\lim_{k,\mathcal{U}} \|T_k^{-1}\| = 1$.

For each k we denote by μ_k the Lebesgue measure on E_k normalized so that $\mu_k(E_k) = 1$. We will denote by μ the minimum volume ellipsoid of E_k containing $\mathbf{B}_{E_k}$ by $\mathcal{E}_k$. The minimum volume ellipsoid of E is denoted by $\mathcal{E}$ and the corresponding normalized measure is denoted by μ.

Observation 3. By using the previous observations we obtain

$$\lim_{k,\mathcal{U}} \mu_k(\mathcal{E}_k) = \mu(\mathcal{E}).$$

Observation 4. By using the previous observations, compactness and the uniqueness of the minimum volume ellipsoid in E, we have

$$\lim_{k,\mathcal{U}} (T_k^{-1}x|T_k^{-1}y)_{\mathcal{E}_k} = (x|y)_{\mathcal{E}} \quad \text{for } x, y \in E.$$

The details are omitted.

3. An application involving Mazur's problem. In the presence of a high degree of symmetry the nice local properties of a Banach space tend to self-improve (see e.g. [8], also [3]). For example, a convex-transitive Banach space with the RNP is already uniformly convex, uniformly smooth and almost transitive.

In [5] it was asked whether a Banach space almost transitive with respect to isometric finite-dimensional perturbations of the identity is isometric to a Hilbert space. The answer is affirmative if the space has DES in some direction.

Theorem 3.1. *Let* X *be a Banach space, which has DES at some direction* $f \in \mathbf{S}_{\mathrm{X}^*}$ *and assume that* X *is convex-transitive with respect to* $\mathcal{G}_F$*. Then* X *is isometrically a Hilbert space.*

Proof. We will construct an inner product $(\cdot|\cdot)_{\mathrm{X}}$ on X such that $|x| = \sqrt{(x|x)_{\mathrm{X}}}$ defines a norm which is continuous with respect to $\|\cdot\|$ and such that $|x| = |Tx|$ for $x \in \mathrm{X}$ and $T \in \mathcal{G}_F$. By using that X is convex-transitive with respect to $\mathcal{G}_F$ it follows that there exists a constant $c > 0$ such that $\|\cdot\| = c|\cdot|$ (see [6]), which yields the claim.

The argument here closely resembles that of [5]. Let Γ be the set of all finite subsets γ of $\mathcal{G}_F$ such that $T \in \gamma \Rightarrow T^{-1} \in \gamma$. Note that Γ can be viewed as a lattice when ordered by inclusion $\subseteq$.

By using a simple argument concerning the Hamel basis of X we find that for each finite-dimensional subspace $A \subset \mathrm{X}$ and each $\gamma \in \Gamma$ there is a finite-dimensional subspace $F \supset A$ and a finite-codimensional subspace E such that $\mathrm{X} = F \oplus E$, $\operatorname{span} \bigcup_{T\in\gamma} (\mathbf{I} - T)(\mathrm{X}) \subset F$, $E \subset \bigcap_{T\in\gamma} \ker(\mathbf{I} - T)$ and $T(F) = F$ for $T \in \gamma \in \Gamma$. Note that $F \neq \{0\}$ for any $\gamma \neq \{\mathbf{I}\}$. Denote by $\mathcal{F}_{A,\gamma}$ the collection of all pairs (F, γ) which satisfy the above conditions.

For each $(F, \gamma) \in \mathcal{F}_{A,\gamma}$ let $\mathcal{E}_F$ be the unique minimum volume ellipsoid in F which contains $\mathbf{B}_{\mathrm{X}} \cap F$. We denote by $(\cdot|\cdot)_F \colon F \times F \to \mathbb{R}$ the inner product induced by the ellipsoid $\mathcal{E}_F$. Observe that the uniqueness of the minimum value ellipsoid implies invariance under isometries and thus $(Tx|Ty)_F \le \sqrt{(Tx|Tx)_F}\sqrt{(Ty|Ty)_F} = \sqrt{(x|x)_F}\sqrt{(y|y)_F} \le \|x\|\cdot\|y\|$ for $x, y \in F$, $T \in \gamma$.

For technical reasons, for each finite-dimensional $F \subset \mathrm{X}$ we denote a linear projection $\mathrm{X} \to F$ by P_F without specifying exactly which projection we mean. Let $\mathcal{M} = \bigcup_{A,\gamma} \mathcal{F}_{A,\gamma}$, where the union is taken over finite-dimensional subspaces $A \subset \mathrm{X}$ and $\gamma \in \Gamma$. We may define a partial order on $\mathcal{M}$ by declaring $(F, \gamma) \le (F', \gamma')$ if $F \subset F'$ and $\gamma \subset \gamma'$.

Define $[\cdot|\cdot]\colon \mathrm{X} \to \mathbb{R}^{\mathcal{M}}$ by letting $[x|y]$, evaluated at (F, γ), be equal to $(P_F x|P_F y)_F$. Observe that the family

$$\{\{(F,\gamma) \in \mathcal{M} : \delta \subset \gamma,\ A \subset F\}\}_{(A,\delta)\in\mathcal{M}},$$

where A ranges over finite-dimensional spaces, defines a filter base on $\mathcal{M}$.

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathcal{M}$ extending this filter base. Define $(\cdot|\cdot)\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}$ by $(x|y) = \lim_{\mathcal{U}}[x|y]$. It is easy to check to that $(\cdot|\cdot)$ is well-defined, bilinear and $(x|y) \le \|x\|\cdot\|y\|$, $(x|x) \ge 0$, $(Tx|Ty) = (x|y)$ for $x, y \in E$, $T \in \mathcal{G}_F$. Indeed, pick $(F, \gamma) \in \mathcal{M}$ such that $x, y \in F$, $T \in \gamma$. Firstly, $[\cdot|\cdot]$ evaluated at $(F', \gamma') \ge (F, \gamma)$ satisfies the conditions described above. Secondly, note that the set of pairs $(F', \gamma') \ge (F, \gamma)$ belongs to $\mathcal{U}$. This means that the ultralimit $(\cdot|\cdot) = \lim_{\mathcal{U}}[\cdot|\cdot]$ satisfies the above-mentioned conditions.

Finally, we will check that $(x|x) > 0$ for $x \in \mathrm{X}$, $x \ne 0$. Suppose that X has λ-DES in direction $f \in \mathbf{S}_{\mathrm{X}^*}$. Then $\inf_F \sup_{F'} \eta(f, F') < \infty$, which means that we may select F such that $\sup_{F'\supset F} \eta(f, F') = \alpha < \infty$. Pick $x \in \mathrm{X}$ such that $f(x) \ge \alpha$. Then $[x|x]$ evaluated at any pair $(F', \gamma') \ge (F, \gamma)$ is at least 1. Similarly, since the set of pairs $(F', \gamma') \ge (F, \gamma)$ belongs to the filter, we find that $(x|x) \ge 1$. This means that $\mathrm{Y} = \{x \in \mathrm{X} : (x|x) = 0\}$ is not the whole space X. Observe that Y is invariant under $\mathcal{G}_F$. It is easy to see that the convex-transitivity with respect to $\mathcal{G}_F$ implies that only a trivial subspace, i.e. $\{0\}$ or X, can be invariant under $\mathcal{G}_F$. Since $\mathrm{Y} \ne \mathrm{X}$, we conclude that $\mathrm{Y} = \{0\}$. ∎

THEOREM 3.2. *Let* X *be a Banach space which is convex-transitive with respect to* $\mathcal{G}_{\mathrm{X}}$ *and has DES in some direction* $f \in \mathbf{S}_{\mathrm{X}^*}$. *Then* X *is isomorphic to a Hilbert space. Moreover, if* X *has* 1-*DES in direction* f, *then* X *is isometric to a Hilbert space.*

Proof. It is known (see [3]) that X is convex-transitive if and only if $\overline{\mathrm{conv}}^{\omega^*}(\{T^*g : T \in \mathcal{G}_{\mathrm{X}}\}) = \mathbf{B}_{\mathrm{X}^*}$ for $g \in \mathbf{S}_{\mathrm{X}^*}$. This means that $\{T^*f : T \in \mathcal{G}_{\mathrm{X}}\}$ is a 1-norming set. Then one can construct, as in the proof of Theorem 2.3, an inner product $(\cdot|\cdot)_{\mathrm{X}}$ on X with $(x|x)_{\mathrm{X}} \le \|x\|^2$ for $x \in \mathrm{X}$. It follows from the assumptions by inspecting the construction of $(\cdot|\cdot)_{\mathrm{X}}$ that $\{x \in \mathrm{X} : (x|x)_{\mathrm{X}} \le 1\} \subset \lambda\mathbf{B}_{\mathrm{X}}$. Thus we have the claim. ∎

Some authors have asked whether an almost transitive Banach space that is isomorphic to a Hilbert space, is in fact isometric to one (see e.g. [4]). This question appears not to have been settled yet.

4. Final remark: near-convexity of the duality mapping. Recall that the *duality mapping* $J\colon \mathrm{X} \to 2^{\mathrm{X}^*}$ is a multivalued mapping defined by

$$J(x) = \{x^* \in \mathrm{X}^* : \|x\|^2 = \|x^*\|^2 = x^*(x)\} \quad \text{for } x \in \mathrm{X}.$$

If X is a Gateaux-smooth space, then J becomes a point-to-point mapping. Recall that for Hilbert spaces the duality map is an isometric isomorphism. We always have $\overline{J(\mathbf{B}_{\mathrm{X}})} = \mathbf{B}_{\mathrm{X}^*}$ according to the Bishop–Phelps theorem, and in the reflexive case $J(\mathbf{B}_{\mathrm{X}}) = \mathbf{B}_{\mathrm{X}^*}$ by James's characterization of reflexivity. However, it can easily happen that the image of a convex set under J is not convex. Next, we will study spaces whose duality mapping does not distort convex sets very far from being convex. It turns out that such spaces are isomorphically Hilbertian.

Theorem 4.1. *Let* X *be a smooth Banach space and let* $J\colon \mathrm{X} \to \mathrm{X}^*$ *be the duality mapping. Suppose that there exists a constant* $0 \le C < 1$ *such that*

$$\Big\|J\Big(\sum x_n\Big) - \sum J(x_n)\Big\| \le C\Big\|\sum x_n\Big\| \quad \textit{for } x_1, \ldots, x_n \in \mathrm{X}.$$

Then X *is isomorphic to a Hilbert space. Moreover, if J is a convex map, then* X *is isometric to a Hilbert space.*

Proof. By using an ultrafilter construction similar to that in the proof of Theorem 2.3 it suffices to check that in any finite-dimensional subspace $F \subset \mathrm{X}$ there exists an inner product $(\cdot|\cdot)\colon F^2 \to \mathbb{R}$ such that

$$(1 - C)\|x\|^2 \le (x|x) \le (1 + C)\|x\|^2 \quad \text{for } x \in F. \tag{4.1}$$

Let $F \subset \mathrm{X}$ be a finite-dimensional subspace. Then there exists an *Auerbach basis* on F, that is, a biorthogonal system $\{(e_i, e_i^*)\}_{i=1}^n \in (\mathbf{S}_{\mathrm{X}} \times \mathbf{S}_{\mathrm{X}^*})^n$. Define a mapping $g\colon F^2 \to \mathbb{R}$ by $g(x, y) = \sum a_i e_i^*(y)$, where $\sum a_i e_i$ is the unique expression of x. Note that g is bilinear. Define $B\colon F^2 \to \mathbb{R}$ by $B(x, y) = (g(x, y) + g(y, x))/2$ for $x, y \in F$. Now B is clearly a symmetric bilinear form.

Fix $x = \sum a_i e_i \in F$. Observe that

$$\begin{aligned} \big|\|x\|^2 - g(x,x)\big| &= |J(x)(x) - g(x,x)| \\ &= \Big\|\Big(J\Big(\sum a_i e_i\Big) - \sum a_i J(e_i)\Big)(x)\Big\| \le C\|x\| \cdot \|x\|. \end{aligned}$$

Thus $\big|\|x\|^2 - B(x,x)\big| \le C\|x\|^2$, so that B is a non-degenerate inner product on F and (4.1) holds. ∎

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