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Resolvent conditions and powers of operators

by

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Abstract. We discuss the relation between the growth of the resolvent near the unit circle and bounds for the powers of the operator. Resolvent conditions like those of Ritt and Kreiss are combined with growth conditions measuring the resolvent as a meromorphic function.

0. Introduction. In this paper we discuss powers of bounded operators whose spectrum lies in the closed unit disc. The general theme is to relate growth conditions on the resolvent near the unit disc to bounds for the powers and their differences.

Much of my present interest in these questions originated from a question of J. Zemánek who asked whether there are quasinilpotent operators Q such that $A = 1 + Q$ would satisfy the *Ritt resolvent condition*, i.e.

$$\|(\lambda - A)^{-1}\| \leq \frac{C}{|\lambda - 1|} \quad \text{for } |\lambda| > 1.$$

This condition has now a characterization (see Theorem 10 below) but we still do not know the answer to the original question ([1]). Other properties which can be characterized or estimated quantitatively include:

• The resolvent is uniformly bounded outside the unit disc (see Theorem 1).

• Operators which are power bounded even after a small overrelaxation (see Theorem 8).

• Power boundedness for operators which have meromorphic resolvent in a neighborhood of the unit circle (see Theorem 12).

We should also mention that Strikwerda and Wade [SW] have shown

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([1]) *Editorial note*: See Yu. Lyubich, *The single-point spectrum operators satisfying Ritt's resolvent condition*, this issue, 135–142.

that the *Kreiss resolvent condition*

$$\|(\lambda - A)^{-1}\| \le \frac{C}{|\lambda| - 1} \quad \text{for } |\lambda| > 1$$

is equivalent to boundedness of the second Cesàro sums of the powers of $e^{i\theta}A$.

In addition to powers as such we also look at their differences, in particular the behavior of $A^n(A-1)$. A result of Katznelson and Tzafriri [K] characterizes those power bounded operators for which the differences tend to zero. We give some examples which e.g. show that their conclusion does not hold if power boundedness is replaced by the Kreiss resolvent condition. Also, we look at the condition

$$\|e^{zA}\| \le Ce^{|z|}$$

for complex z which is weaker than power boundedness but stronger than the Kreiss condition (see Theorem 7). Here again it is natural to ask whether these conditions imply much stronger conclusions in the case $A = 1 + Q$. We are only able to show that the bounds can be replaced by the corresponding "little oh" versions if the spectrum touches the unit disc in a set of zero measure. We give a list of results in Section 1, and the proofs in Section 2.

Much of this research was done while the author was visiting the Mittag-Leffler Institute during the winter 1997–1998.

1. Results. We study the growth and decay of powers of operators in Banach spaces. So, let X be a complex Banach space, and A a bounded operator in X such that the spectrum is in the closed unit disc:

$$\sigma(A) \subset \overline{\mathbb{D}}. \tag{1.1}$$

We shall partly specialize to the case where $\sigma(A) = \{1\}$ and then we write $A = 1 + Q$ with $\sigma(Q) = \{0\}$. The resolvent is denoted by $(\lambda - A)^{-1}$ and is analytic outside the unit disc. If we put

$$M(r) := \sup_{|\lambda| \ge r} \|(\lambda - A)^{-1}\| \tag{1.2}$$

then (1.1) can be written equivalently as

$$M(r) < \infty \quad \text{for } r > 1. \tag{1.3}$$

Since

$$A^n = \frac{1}{2\pi i} \int_{|\lambda| = r} \lambda^n (\lambda - A)^{-1}\, d\lambda \quad \text{for } n \ge 0, \tag{1.4}$$

this immediately gives

$$\|A^n\| \le M(r) r^{n+1} \quad \text{for } r > 1, \tag{1.5}$$

and in particular

$$\|A^n\| \le e^{o(n)} \quad \text{as } n \to \infty. \tag{1.6}$$

EXAMPLE 1. Operators of the form $A = 1 + Q$ with Q quasinilpotent can have fast growing powers. In fact, let ω be a positive real, choose $\alpha_j = (1/j)^{1/\omega}$ and let Q be the weighted backward shift $Qe_1 = 0$ while $Qe_{j+1} = \alpha_j e_j$. Then Q is quasinilpotent and its resolvent grows like $e^{\tau_1(\omega)/r^\omega}$, that is, with order ω and with a positive type $\tau_1(\omega)$. The growth of the resolvent and the growth of the Taylor coefficients are related and one shows likewise that $\|(1+Q)^n\|$ grows like $e^{\tau_2(\omega)n^{\omega/(\omega+1)}}$.

We are mainly interested in the situation where $M(r) \to \infty$ as r tends to 1. However, for the sake of completeness, we also formulate a quantitative result for the case where $M(1) < \infty$. Notice that $M(1) < \infty$ is equivalent to $\|A^n\| \to 0$.

THEOREM 1. *If* $M(1) < \infty$, *then*

$$M(1) \le \sum_{n=0}^{\infty} \|A^n\| \le 1 + 4[M(1) - 1]M(1). \tag{1.7}$$

REMARK 1. This result was given in [Ne5] with the constant 6 in place of 4. For example, taking a nilpotent two-dimensional matrix with a small norm we see that the constant must be larger than 2. For large values of $M(1)$ the behavior is necessarily quadratic in $M(1)$ (see [Ne5]).

From now on we assume $M(1) = \infty$. In Example 1 we saw that the growth of powers can be fast even in the case were the spectrum touches the unit circle only at one point. We first pose a restriction of different nature. We assume that the resolvent is of *bounded characteristics* outside the unit disc. To that end put

$$m(r) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ \|(re^{i\phi} - A)^{-1}\| \, d\phi. \tag{1.8}$$

This tool would suffice if we only looked at resolvents which are analytic outside the unit disc and have an essential singularity somewhere on the unit circle. In general, however, one also has to count the number of poles. Assume that the resolvent is meromorphic for $|\lambda| > R$ and let b be a pole, meaning that there is a smallest positive integer μ, the multiplicity of b, such that $(\lambda - b)^\mu(\lambda - A)^{-1}$ is analytic in a neighborhood of b. Number the poles so that their absolute values are nonincreasing, repeating each pole according to its multiplicity, as long as they stay outside the disc of radius R. Then we set

$$N(r) := \sum_j \log^+ \frac{|b_j|}{r} \qquad \text{for } r > R. \tag{1.9}$$

Finally we define

$$T(r) := m(r) + N(r). \tag{1.10}$$

Now one says that the resolvent is of *bounded characteristics* outside the disc of radius R if

$$T(R) := \limsup_{r \to R+} T(r) < \infty. \tag{1.11}$$

One knows that T is nondecreasing and convex in the variable $\log(1/r)$ (see e.g. [Ne4]). Notice that, since we assume that the spectrum is in the unit disc, $T(r) = m(r)$ for $r \geq 1$.

THEOREM 2. *Suppose* (1.1) *holds and* $T(1) < \infty$. *Then*

$$\|A^n\| \leq e^{T(1)} e^{\sqrt{8T(1)(n+1)}} \qquad \textit{for all } n \geq 0. \tag{1.12}$$

REMARK 2. The best constants are not known. The theorem is sharp in the sense that for the operators in Example 1 we have $T(1) < \infty$ for $\omega < 1$ while $T(1) = \infty$ for $\omega > 1$.

EXAMPLE 2. Let V denote integration in $L_2[0,1]$,

$$Vf(t) = \int_0^t f(s)\, ds,$$

and put $A = 1 + V$. Then A^{-1} is a contraction and it follows from the Phragmén–Lindelöf principle that for $\varepsilon > 0$ small enough there does not exist a constant C_ε so that

$$\|A^n\| \leq C_\varepsilon e^{\varepsilon \sqrt{n}} \qquad \text{for all } n$$

(see [A]). On the other hand, one easily sees that

$$\|A^n\| \leq e^{2\sqrt{n}}.$$

Further, $\sigma(1+V) = \{1\}$, $m(1) < \infty$ but V is not in the trace class as the singular values decay like $1/j$.

The next result gives a sufficient condition for $1+K$ to satisfy $T(1) < \infty$. Notice that the result does not depend on the location of the spectrum.

THEOREM 3. *Suppose K is a compact operator in a Hilbert space such that the singular values $\sigma_j(K)$ satisfy*

$$C_*(K) := \sum_{j=1}^{\infty} \frac{\sigma_j(K)}{2} \log^+ \frac{2}{\sigma_j(K)} < \infty. \tag{1.13}$$

Then for $A = 1 + K$ we have

$$T(1) \le C_*(K) + \|K\|_1,$$

where $\|K\|_1 = \sum_{j=1}^{\infty} \sigma_j(K)$.

We now pose a natural growth condition for the resolvent near the unit circle. If A satisfies $\|A^n\| \le C$ for $n \ge 0$ then the expansion

$$(\lambda - A)^{-1} = \sum_{n=0}^{\infty} A^n \lambda^{-1-n} \quad \text{for } |\lambda| > 1$$

implies the *Kreiss resolvent condition*

$$\|(\lambda - A)^{-1}\| \le \frac{C}{|\lambda| - 1} \quad \text{for } |\lambda| > 1,$$

which we write here as

$$M(r) \le \frac{C}{r-1} \quad \text{for } r > 1. \tag{1.14}$$

When (1.14) holds, substituting $r := 1 + 1/n$ into (1.5) gives

$$\|A^n\| \le Ce(n+1), \tag{1.15}$$

which is optimal in the following sense: no constant smaller than Ce is possible and the dependence on n can be linear. In fact, the former can be seen by looking at truncated shift operators multiplied by a very large constant [Le]. Recent sharp forms of this are proved in [Sp]. The dependence on n is central to our theme and we give an example by Shields [Sh] in detail.

EXAMPLE 3 (see [Sh]). Let X denote the space of analytic functions in the open unit disc such that f' has boundary values in the Hardy space H^1, equipped with the norm

$$\|f\| := |f|_\infty + |f'|_1 = \sup_{|z| \le 1} |f(z)| + \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(e^{i\varphi})| \, d\varphi.$$

If M_z denotes the multiplication operator with the independent variable z then $\|M_z^n\| = n + 1$ while the Kreiss condition holds (e.g. $C = 3/2$ will do). Also (see the proof of Example 4),

$$\left(1 + \frac{1}{2}\sqrt{t}\right) e^t \le \|e^{tM_z}\| \le (1 + 2\sqrt{t}) e^t \quad \text{for } t > 0.$$

Also here the growth is as fast as it can be.

THEOREM 4. *If the Kreiss condition* (1.14) *holds, then*

$$\|e^{zA}\| \le C_1 \sqrt{1 + |z|}\, e^{|z|} \qquad \textit{for all } z \tag{1.16}$$

with $C_1 = 2C$. If (1.16) *holds, then*

$$\|A^n\| \le C_1 \sqrt{2\pi}\, (n+1) \quad \textit{for all } n \ge 0. \tag{1.17}$$

In the previous example the spectrum of M_z equals the closed unit disc. If we assume that the spectrum touches the unit circle only in a set of measure zero then a sharpening of (1.15) can be given. Let us denote the usual measure on the unit circle by meas, normalized so that $\operatorname{meas}(\partial\mathbb{D}) = 1$. Notice that if $T(1) < \infty$ then always $\operatorname{meas}(\sigma(A) \cap \partial\mathbb{D}) = 0$.

THEOREM 5. *Assume that the Kreiss condition* (1.14) *holds and that*

$$\operatorname{meas}(\sigma(A) \cap \partial\mathbb{D}) = 0.$$

Then

$$\|A^n\| = o(n) \qquad \text{as } n \to \infty. \tag{1.18}$$

This result has a converse.

THEOREM 6. *There exists a Banach space with the following property. Let E be a closed subset of the unit circle. Then there exists an operator A satisfying the Kreiss condition and*

$$\sigma(A) \cap \partial\mathbb{D} = E,$$

and such that

$$\|A^n\| \geq 1 + \frac{n}{2} \operatorname{meas}(E). \tag{1.19}$$

EXAMPLE 4. We set $B := \frac{1}{2}(1 + M_z)$ where M_z is the multiplication operator in the space X of Example 3. Then the Kreiss condition (1.14) holds and

$$\sigma(B) = \{\lambda \mid |\lambda - 1/2| \leq 1/2\}$$

so that in particular $\sigma(B) \cap \partial\mathbb{D} = \{1\}$, $T(1) < \infty$ and:

- there exists C_1 such that for $t > 0$ we have

$$\left(1 + \frac{1}{C_1}\sqrt{t}\right)e^t \leq \|e^{tB}\| \leq (1 + C_1\sqrt{t})e^t,$$

- there exists C_2 such that

$$\|(\lambda - B)^{-1}\| \leq \frac{C_2}{|\lambda - 1|^2} \qquad \text{for } 1 < |\lambda| < 2,$$

- there exists C_3 such that

$$\frac{1}{C_3}\sqrt{n+1} \leq \|B^n\| \leq C_3\sqrt{n+1} \qquad \text{for } n \geq 0,$$

- there exists C_4 such that

$$\frac{1}{C_4} \leq \|B^n(B - 1)\| \leq C_4 \qquad \text{for } n \geq 0.$$

REMARK 3. We recall ([Ne5], Proposition 1.1, or Theorem 8 below) that the *iterated resolvent condition*

$$\|(\lambda - A)^{-n}\| \le \frac{C}{(|\lambda| - 1)^n} \quad \text{for } n \ge 0,\ |\lambda| > 1 \tag{1.20}$$

is equivalent to the condition

$$\|e^{zA}\| \le Ce^{|z|} \quad \text{for all } z. \tag{1.21}$$

So, in particular, the operator B in Example 4 does not satisfy the iterated resolvent condition.

Notice further that, by the well known theorem by Katznelson and Tzafriri [K], power boundedness and

$$\sigma(A) \cap \partial\mathbb{D} \subset \{1\} \tag{1.22}$$

together imply

$$A^n(A - 1) \to 0. \tag{1.23}$$

Thus we see that power boundedness cannot be replaced in this result by the weaker Kreiss resolvent condition. Whether power boundedness can be replaced by the iterated resolvent condition is open.

THEOREM 7. *If*

$$\|A^n\| \le C \quad \textit{for } n \ge 0 \tag{1.24}$$

then both (1.20) *and* (1.21) *hold. These are equivalent and both imply* (1.14) *and*

$$\|A^n\| \le C\sqrt{2\pi(n+1)}. \tag{1.25}$$

If the iterated resolvent condition (1.20) (*and* (1.21)) *holds and additionally*

$$\operatorname{meas}(\sigma(A) \cap \partial\mathbb{D}) = 0$$

then

$$\|A^n\| = o(\sqrt{n}) \quad \textit{as } n \to \infty. \tag{1.26}$$

EXAMPLE 5. We shall now look at the effect of under- (and over-) relaxation. To that end we start with a simple example. Consider e.g. the continuous 2π-periodic functions with maximum norm and let the operator be multiplication by $e^{i\theta}$ composed with underrelaxation:

$$(A_\omega f)(\theta) := (\omega e^{i\theta} + 1 - \omega) f(\theta)$$

where $0 < \omega < 1$. Then

$$\|e^{te^{i\varphi}A_\omega}\| \le e^{t(1-2(1-\omega)\varphi^2/\pi^2)} \quad \text{for } |\varphi| \le \pi. \tag{1.27}$$

On the other hand, there are constants $c_i(\omega) > 0$ such that

$$\frac{c_1(\omega)}{\sqrt{n+1}} \le \|A_\omega^n(A_\omega - 1)\| \le \frac{c_2(\omega)}{\sqrt{n+1}} \quad \text{for all } n \ge 1. \tag{1.28}$$

Theorem 4.5.3 of [Ne1] says that if contractions are underrelaxed then the differences decay (at least) like $1/\sqrt{n}$. The next theorem says that the same also happens for operators which need not be power bounded as long as they satisfy the iterated resolvent condition. Example 4 shows in particular that there are operators which satisfy the Kreiss condition but do not satisfy the iterated resolvent condition even after underrelaxation and the differences do not decay.

Our next result characterizes the operators which are power bounded even after a small overrelaxation.

THEOREM 8. *Let* $0 < c < 1/\pi^2$ *and* $1 \le C < \infty$ *be fixed. Then the following are equivalent*:

$$\|(re^{i\varphi} - A)^{-n}\| \le \frac{C}{(r + c\varphi^2 - 1)^n} \quad \text{for } n \ge 0,\ r > 1,\ |\varphi| \le \pi, \tag{1.29}$$

$$\|e^{te^{i\varphi}A}\| \le Ce^{t(1-c\varphi^2)} \quad \text{for all } t > 0,\ |\varphi| \le \pi. \tag{1.30}$$

If either of them holds then for $0 \le \varepsilon \le c/(1-2c)$ *and* $A_{1+\varepsilon} := (1+\varepsilon)A - \varepsilon$ *we have*

$$\|A^n_{1+\varepsilon}\| \le C_1 \quad \text{for } n \ge 0, \tag{1.31}$$

where $C_1 = C\sqrt{2/c}$. *Conversely*, *if* (1.31) *holds then* (1.29) *and* (1.30) *hold with* $C = C_1$ *and* $c = \varepsilon/((1+\varepsilon)\pi^2)$. *Further*, *if* (1.29) *holds then*

$$\|A^n(A-1)\| \le \frac{C_2}{\sqrt{n+1}} \tag{1.32}$$

where $C_2 = C/(c\sqrt{\pi})$.

COROLLARY 1. *If* A *satisfies the iterated resolvent condition* (1.20), *then for* $0 \le \omega < 1$ *the underrelaxed operators* $A_\omega := \omega A + 1 - \omega$ *are power bounded*, *and the following estimates hold*:

$$\|A^n_\omega\| \le \frac{C\pi}{\sqrt{1-\omega}} \quad \text{for } n \ge 0 \tag{1.33}$$

and

$$\|e^{te^{i\varphi}A_\omega}\| \le Ce^{t(1-2(1-\omega)\varphi^2/\pi^2)} \quad \text{for all } t > 0,\ |\varphi| \le \pi. \tag{1.34}$$

In fact, since (1.20) and (1.21) are equivalent we have $\|e^{zA}\| \le Ce^{|z|}$. Thus

$$\|e^{te^{i\varphi}A_\omega}\| \le Ce^{t\omega}e^{(1-\omega)t\cos\varphi} \le Ce^{t(1-2(1-\omega)\varphi^2/\pi^2)}.$$

The estimate for powers of A_ω now follows from the previous theorem.

The next result can be compared with Example 4 by setting $\alpha = 1$.

Theorem 9. *Assume the Kreiss condition* (1.14) *holds and that there exist* $\alpha \geq 0$ *and* C_1 *such that*

$$\|(\lambda - A)^{-1}\| \leq \frac{C_1}{|\lambda - 1|^{1+\alpha}} \quad \text{for } 1 < |\lambda| < 2. \tag{1.35}$$

Then for every integer $k \geq 0$ *there exists* M_k *such that*

$$\|A^n(A-1)^k\| \leq \frac{M_k}{(n+1)^{\beta_k}} \quad \text{for all } n \geq 0, \tag{1.36}$$

where $\beta_k = (k - \alpha)/(1 + \alpha)$.

Here the case $\alpha = 0$ corresponds to the *Ritt condition*

$$\|(\lambda - 1)(\lambda - A)^{-1}\| \leq C \quad \text{for } |\lambda| > 1. \tag{1.37}$$

This condition is formally weaker than the sectorial condition studied in [Ne1] and [Ne5]. However, the proof of the previous theorem implies that the condition easily extends to a sectorial set. This has also been shown in [Ly] and [Na]. Finally, we mention that [B] contains an integration argument where the Ritt condition directly implies power boundedness. Ritt originally showed that the condition implies at most $o(n)$ growth, which was improved to $O(\log(n))$ in [T]. Theorem 2.1 of [Ne5] gives four different characterizations for (1.36) to hold with $k = 0$ and 1. We formulate here yet another one (already published in [Na]) as follows.

Theorem 10. *The following are equivalent*:

(i) *There exist* M_0 *and* M_1 *such that for all* $n \geq 0$,

$$\|A^n\| \leq M_0 \tag{1.38}$$

and

$$\|A^n(A-1)\| \leq \frac{M_1}{n+1}. \tag{1.39}$$

(ii) *There exists* C *such that the Ritt condition* (1.37) *holds.*

Then we can still strengthen (1.37) by assuming that the corresponding iterated version holds (a question asked also by D. Tsedenbayar). This then already implies that $A = 1$. In fact, already a somewhat weaker assumption implies this.

Theorem 11. *Assume that the Ritt condition* (1.37) *holds and that for some* $\lambda \leq -1$ *we have*

$$\left\| \left(1 - \frac{1}{\lambda - 1}(A - 1)\right)^{-n} \right\| \leq C_1 \quad \text{for all } n \geq 0. \tag{1.40}$$

Then $A = 1$.

We recall that Spijker has proved that if A is a matrix in a d-dimensional space then the Kreiss condition implies

$$\|A^n\| \le Ced. \tag{1.41}$$

In a d-dimensional space the resolvent is meromorphic in the whole plane. In the general case we can assume that the resolvent is meromorphic outside a disc which is strictly smaller than the unit disc. Then power boundedness can be estimated in terms of $T(r)$ and the constant C in the Kreiss condition.

THEOREM 12. *For every $0 < \theta < 1$ there exists a constant $C_1(\theta)$ such that if $T(\theta) < \infty$ then the Kreiss condition* (1.14) *implies*

$$\|A^n\| \le C_1(\theta)T(\theta)C \quad \textit{for } n \ge 0. \tag{1.42}$$

This is just a reformulation of results in [Ne2]. Here is a corollary for operators in Hilbert spaces.

THEOREM 13. *For every $0 \le \eta < 1$ there exists a constant $c(\eta)$ such that if A is an operator in a separable Hilbert space satisfying the Kreiss condition* (1.14) *and decomposable as $A = B + K$ where $\|B\| \le \eta$ and $\|K\|_1 < \infty$ then*

$$\|A^n\| \le c(\eta)[\|K\|_1 + 1]C \quad \textit{for } n \ge 0. \tag{1.43}$$

An earlier version of Theorem 12 was given in [Ne5], for a different growth function to measure the size of the resolvent as a meromorphic function. The natural growth function here is, however, $T(r)$, and we point out that also other results of Section 4 of [Ne5] have counterparts in this terminology. For example, if we only assume (1.39), then a resolvent condition follows, which implies (1.39) back provided $T(\theta) < \infty$ with some $\theta < 1$.

We end with a remark on small values of $T(1)$. For the identity operator we have $T(1) = \gamma$, where

$$\gamma := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |e^{i\theta} - 1|\, d\theta = 0.323\ldots$$

If $T(1) < \gamma$, then the spectral radius of the operator is necessarily smaller than 1 and we can bound the size of $M(1)$ in terms of $T(1)$.

THEOREM 14. *If $T(1) < \gamma$, and if ξ is the solution of*

$$\xi = \frac{1}{3(\gamma - T(1))}(1 + \log(1 + \xi))$$

then

$$M(1) \le \xi.$$

EXAMPLE 6. We can also look at the continuity from "above". Operators of the form $1 + \varepsilon Q$, where $\varepsilon > 0$ and Q is as in Example 1 with $\omega < 1$, are examples of operators satisfying $\gamma < T(1) < \infty$ and having fast growing powers. Writing

$$(e^{i\theta} - 1 - \varepsilon Q)^{-1} = \frac{1}{e^{i\theta} - 1}\left(1 - \frac{\varepsilon}{e^{i\theta} - 1} Q\right)^{-1}$$

we see that for fixed Q and small ε we have $T(1) < \gamma + C\varepsilon$.

2. Proofs

Proof of Theorem 1. For short, let $M := M(1)$. The idea of the proof is simple. Knowing the value of M allows us to use estimate (1.5),

$$\|A^n\| \leq M(r) r^{n+1},$$

with $r < 1$ such that $(1 - r)M < 1$ (as then the integration path in (1.4) still surrounds the spectrum). In fact, from the identity

$$(\lambda - A)^{-1} = (\lambda_0 - A)^{-1}(1 - (\lambda - \lambda_0)(\lambda_0 - A)^{-1})^{-1}$$

we obtain

$$M(r) \leq \frac{M}{1 - (1 - r)M}.$$

We choose $r = 1 - 1/(2M - 1)$, which gives

$$M(r) \leq \frac{(2M - 1)M}{M - 1}$$

and so

$$\sum_{n=0}^{\infty} \|A^n\| \leq 1 + r \sum_{n=1}^{\infty} M(r) r^n = 1 + r^2 \frac{M(r)}{1 - r} \leq 1 + 4(M - 1)M.$$

Proof of Theorem 2. Since the resolvent is analytic outside the unit circle and bounded at infinity we can bound the maximum norm outside a circle by the logarithmic average along a slightly smaller circle. In fact, since the integrand in m is subharmonic, using the Poisson kernel one gets

$$\log^+ M(r) \leq \frac{\theta + 1}{\theta - 1} m(r/\theta) \quad \text{for } 1 < \theta < r$$

(a formulation for operator-valued functions is given in [Ne4]), which by assumption in the limit $\theta \to r$ gives

$$\log^+ M(r) \leq \frac{r + 1}{r - 1} m(1).$$

We utilize this by substituting $r := 1 + \sqrt{2m(1)/(n + 1)}$ into (1.5) to get

$$\|A^n\| \leq M(r) r^{n+1} \leq r^{n+1} e^{\frac{r+1}{r-1} m(1)} \leq e^{m(1)}\, e^{\sqrt{8m(1)(n+1)}}.$$

Proof of Theorem 3. Here we assume that $A = 1 + K$ where K is a compact operator in a Hilbert space such that

$$\|K\|_1 = \sum \sigma_j < \infty \tag{2.1}$$

and

$$C_*(K) := \sum (\sigma_j/2) \log^+(2/\sigma_j) < \infty. \tag{1.13}$$

Here $\sigma_j = \sigma_j(K)$ are the singular values of the operator (ordered nonincreasingly). (Finite-dimensional cases are included with trivial modifications.) Notice that from the compactness of K it follows that $\sigma_j \to 0$ and so (1.13) implies (2.1).

In [Ne4] two characteristic functions, T_∞ and T_1, for operator-valued meromorphic functions were discussed. The functions were assumed to be meromorphic in a disc $|z| < R$ and normalized to be the identity at origin.

Here we want to estimate $T(1)$ of the resolvent $(\lambda - 1 - K)^{-1}$. In order to be able to refer to the results directly we change the variable $\lambda = 1/z$ and write

$$(\lambda - 1 - K)^{-1} = \frac{z}{1-z}\left(1 - \frac{z}{1-z}K\right)^{-1}.$$

At $z = 1$ we have $T(1) = T_\infty(1)$. Clearly, $\bigl(1 - \frac{z}{1-z}K\bigr)^{-1}$ is meromorphic in the open disc. Thus we have

$$T(1) \le \sup_{t<1} T_\infty\left(t, \left(1 - \frac{z}{1-z}K\right)^{-1}\right).$$

On the other hand, since K is in the trace class, the following estimate holds:

$$T_\infty\left(t, \left(1 - \frac{z}{1-z}K\right)^{-1}\right) \le T_1\left(t, 1 - \frac{z}{1-z}K\right)$$

(see [Ne4]). Since $1 - \frac{z}{1-z}K$ is analytic in the open unit disc, T_1 is here simply

$$T_1\left(t, 1 - \frac{z}{1-z}K\right) = \sum_{j=1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \sigma_j \, d\varphi$$

where now $\sigma_j = \sigma_j\bigl(1 - \frac{te^{i\varphi}}{1-te^{i\varphi}}K\bigr)$ are the singular values.

The estimate

$$\sigma_j\left(1 - \frac{z}{1-z}K\right) \le 1 + \frac{|z|}{|1-z|}\sigma_j(K)$$

follows from the approximation property, from the fact that the singular values are obtained as distances to finite rank operators. Thus, we have to estimate integrals of the form

$$F(\sigma) := \frac{1}{2\pi}\int \log\left(1 + \frac{\sigma}{|1-e^{i\varphi}|}\right) d\varphi$$

because

$$T_1\left(1, 1 - \frac{z}{1-z}K\right) \le \sum_j F(\sigma_j(K)).$$

Estimate here $|1-e^{i\varphi}| \ge (2/\pi)\varphi$ and substitute $\varphi = \pi\sigma/(2u)$. Then estimate $\log(1+u) \le u$ for $u \le 1$, while $\log(1+u) \le \sqrt{u}$ for $u > 1$. This gives

$$F(\sigma) \le \frac{\sigma}{2}\log^+\frac{2}{\sigma} + \sigma,$$

and the bound $T(1) \le C_*(K) + \|K\|_1$ follows.

Proof of Theorem 4. To prove (1.16) we may put $z = t$ as the Kreiss condition holds for A if and only if it holds for $e^{i\theta}A$. Write

$$e^{tA} = \frac{r}{2\pi}\int e^{tre^{i\varphi}}(re^{i\varphi} - A)^{-1}e^{i\varphi}\, d\varphi \tag{2.2}$$

to obtain

$$\|e^{tA}\| \le \frac{Cr}{r-1}\cdot\frac{1}{2\pi}\int |e^{tr\cos\varphi}|\, d\varphi. \tag{2.3}$$

Substituting here $r = 1 + 1/t$ yields the desired bound.

Suppose now that (1.16) holds. We use the following representation for powers of A:

$$A^n = \frac{n!}{2\pi i}\int z^{-1-n}e^{zA}\, dz \tag{2.4}$$

where the integration is around the origin. Choose $|z| = n$. Then we obtain

$$\|A^n\| \le 2C\sqrt{n+1}\, e^n\frac{n!}{n^n},$$

which implies (1.17).

Proof of Theorem 5. Choose $\varepsilon > 0$. We have to show that

$$\limsup \|A^n\|/n \le \varepsilon. \tag{2.5}$$

The set $E := \sigma(A) \cap \partial\mathbb{D}$ is compact and of measure zero. Choose an open cover $\{U_j\}$ of E on the circle such that

$$\sum_j \operatorname{meas}(U_j) < \frac{\varepsilon}{Ce}$$

where C is the constant in the Kreiss condition, and each U_j is an arc along the circle. By the compactness of E the cover can be assumed to be finite, say $j = 1, \ldots, N$, and nonoverlapping. Consider now φ such that $e^{i\varphi} \notin \bigcup_{j=1}^N U_j$

and $re^{i\varphi} \in \sigma(A)$. From the compactness of $\sigma(A)$ it follows that there exists a $\delta > 0$ such that $r \le 1 - \delta$ for all such φ. We use the Cauchy integral

$$A^n = \frac{1}{2\pi i} \int_\Gamma \lambda^n (\lambda - A)^{-1} \, d\lambda \tag{1.4}$$

where Γ consists of a finite number of pieces as follows:

- when $e^{i\varphi} \in U_j$ we choose $\lambda := \left(1 + \frac{1}{n+1}\right)e^{i\varphi}$,
- when $e^{i\varphi} \notin \overline{U}_j$ we choose $\lambda := (1-\delta)e^{i\varphi}$,
- in-between when $e^{i\varphi} \in \partial U_j$ we keep φ fixed and take $\lambda := te^{i\varphi}$.

The first choice corresponds (by the Kreiss condition) to

$$Ce(n+1) \sum_j \operatorname{meas}(U_j) \le \varepsilon(n+1),$$

the second terms all decay with speed $(1-\delta)^n$ while the third terms decay with speed $1/(n+1)$.

This completes the proof.

Proof of Theorem 6. The proof is based on Example 4. Let $E \subset \partial\mathbb{D}$ be a closed set of positive measure. We define a Banach space X_E as follows. For every $e^{i\theta} \in E$ let f_θ be an analytic function in the unit disc with boundary values in H^1. Denote by f the set $\{f_\theta\}_{e^{i\theta} \in E}$. We set

$$\|f\| := \sup_\varphi \sup_\theta |f_\theta(e^{i\varphi})| + \frac{1}{2\pi} \int_{-\pi}^{\pi} \sup_\theta |f'_\theta(e^{i\varphi})| \, d\varphi \tag{2.6}$$

and define Y_E as the closure of those f with finite norm.

Next, our operator A will be a diagonal multiplication operator as follows. The component f_θ is multiplied by $(e^{i\theta} + z)/2$ where z indicates the independent variable. Thus

$$(Af)_\theta(z) = \frac{e^{i\theta} + z}{2} f_\theta(z)$$

and in particular

$$(A^n f)_\theta(e^{i\theta}) = e^{in\theta} f_\theta(e^{i\theta}), \qquad (A^n f)'_\theta(e^{i\theta}) = \frac{n}{2} e^{i(n-1)\theta} f_\theta(e^{i\theta}) + e^{in\theta} f'_\theta(e^{i\theta}).$$

Applying this to the constant vector f with $f_\theta(z) = 1$ we obtain

$$\|A^n f\| \ge 1 + \frac{n}{2} \operatorname{meas}(E).$$

We still have to check that the Kreiss condition holds and that $\sigma(A) \cap \partial\mathbb{D} = E$.

Let $|\lambda| = r > 1$. Since

$$((\lambda - A)^{-1} f)_\theta(z) = \left(\lambda - \frac{e^{i\theta} + z}{2}\right)^{-1} f_\theta(z)$$

we have

$$\sup_{\varphi,\theta} |((\lambda - A)^{-1} f)_\theta(e^{i\varphi})| \le \frac{1}{r-1} \sup_{\varphi,\theta} |f_\theta(e^{i\varphi})|.$$

Similarly,

$$\left(\left(\lambda - \frac{e^{i\theta}+z}{2}\right)^{-1} f_\theta\right)'(z) = \frac{1}{2}\left(\lambda - \frac{e^{i\theta}+z}{2}\right)^{-2} f_\theta(z) + \left(\lambda - \frac{e^{i\theta}+z}{2}\right)^{-1} f_\theta'(z)$$

gives

$$\begin{aligned}
&\frac{1}{2\pi}\int \sup_\theta \left| \left(\left(\lambda - \frac{e^{i\theta}+e^{i\varphi}}{2}\right)^{-1} f_\theta\right)'(e^{i\varphi})\right| d\varphi \\
&\quad \le \frac{1}{2}\cdot\frac{1}{2\pi}\int \sup_\theta \left|\left(\lambda - \frac{e^{i\theta}+e^{i\varphi}}{2}\right)^{-2}\right| d\varphi \sup_{\varphi,\theta} |f_\theta(e^{i\varphi})| + \frac{1}{2\pi}\int \sup_\theta |f_\theta(e^{i\varphi})|\, d\varphi.
\end{aligned}$$

Together these imply

$$\|(\lambda - A)^{-1} f\| \le \frac{1}{r-1}\|f\| + c(r)\sup_{\varphi,\theta}|f_\theta(e^{i\varphi})|$$

where

$$c(r) = \frac{1}{2}\cdot\frac{1}{2\pi}\int \sup_\theta \left|\left(\lambda - \frac{e^{i\theta}+e^{i\varphi}}{2}\right)^{-2}\right| d\varphi \le \frac{1}{2}\cdot\frac{1}{r-1}.$$

Let us now consider the spectrum of A. It is actually the union of discs of the form $|\lambda - e^{i\theta}/2| \le 1/2$ but all we need here is that its intersection with the unit circle equals E. For simplicity, assume that $1 \in E$, other values are similar. Consider the operator M_z in X, as in Example 3. Its spectrum is the closed unit disc. In Example 4 we consider $B = \frac{1}{2}(1 + M_z)$. By the spectral mapping theorem its spectrum is the disc $|\lambda - 1/2| \le 1/2$ and in particular 1 is in the spectrum. But if we restrict A to the one-dimensional subspace corresponding to $\theta = 0$ we see that it operates like B and 1 is also in the spectrum of the restriction of A. Finally, as the resolvent is analytic outside the disc, 1 is a boundary point, and the boundary points of the spectrum of a restricted operator always belong to the spectrum of the full operator.

Take now $e^{i\psi} \notin E$. We have

$$\left| e^{i\psi} - \frac{e^{i\theta}+e^{i\varphi}}{2}\right| \ge \beta|\theta - \psi|^2,$$

for some $\beta > 0$ not depending on φ. This implies

$$\|(e^{i\psi} - A)^{-1}\| \le \frac{C/\beta}{\operatorname{dist}(e^{i\psi}, E)^4}$$

and in particular $e^{i\psi} \notin E$ is a regular value for A.

Above we have introduced for every closed E a Banach space Y_E. Defining $Y := Y_{\partial\mathbb{D}}$ and treating Y_E as a subspace of Y completes the proof.

Details for Example 4. Let M_z and X be as in Example 3 and set $B = \frac{1}{2}(1 + M_z)$. If $b(z) = \frac{1}{2}(1+z)$, then in X we have

$$\|B^n\| = 1 + \sqrt{\frac{n}{2\pi}}\,(1 + o(1)).$$

In fact,

$$\|B^n\| = 1 + |(b^n)'|_1$$

as $B^n 1(z) = b^n(z)$ and

$$\|B^n f\| \le |b^n f|_\infty + |(b^n)' f|_1 + |b^n f'|_1 \le \|f\| + |(b^n)'|_1 |f|_\infty.$$

But $(b^n)' = (n/2)b^{n-1}$ and $|b(e^{i\varphi})| = |\cos(\varphi/2)|$ so that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} |(b^n)'|\,d\varphi &= \frac{n}{2\pi}\int_0^{\pi}\left(\cos\frac{\varphi}{2}\right)^{n-1} d\varphi \\ &= \frac{\sqrt{n-1} + 1/\sqrt{n-1}}{2\pi}\int_0^{\pi\sqrt{n-1}}\left(\cos\frac{t}{2\sqrt{n-1}}\right)^{n-1} dt,\end{aligned}$$

from which the claim follows as the integral tends to $\int_0^\infty e^{-t^2/8}\,dt = \sqrt{2\pi}$.

Likewise, to obtain

$$1/C \le \|B^n(B-1)\| \le C$$

the key term to estimate is $nb^{n-1}(b-1)$.

Consider now the iterated resolvent condition. We first show that the multiplication operator M_z does not satisfy it. This is in fact clear already from the linear growth of the powers as the iterated condition allows at most $\sqrt{n}$ growth. However, the estimate for the exponential function e^{tM_z} has independent interest and then the corresponding result for B follows.

It is clear from the previous discussion that to estimate the operator norm of e^{tM_z} we need to compute the norm of e^{tz} as a function in X.

Thus,

$$\|e^{tM_z}\| = |e^{tz}|_\infty + |te^{tz}|_1,$$

which easily gives the following bound for $t > 0$:

$$\left(1 + \frac{1}{2}\sqrt{t}\right)e^t \le \|e^{tM_z}\| \le (1 + 2\sqrt{t})e^t.$$

From this we obtain the corresponding result for B as $e^{tB} = e^{(t/2)M_z}e^{t/2}$.

Proof of Theorem 7. This result is contained in Proposition 1.1 of [Ne5], except the case of the peripheral spectrum being of measure zero.

We use the following representation for powers of A:

$$A^n = \frac{n!}{2\pi i}\int z^{-1-n} e^{zA}\,dz \tag{2.4}$$

where the integration is around the origin. We assume that

$$\|e^{zA}\| \le Ce^{|z|} \quad \text{for all } z. \tag{1.21}$$

Our aim is to show that together with $\operatorname{meas}(\sigma(A)\cap\partial\mathbb{D}) = 0$ this implies

$$\|A^n\| = o(\sqrt{n}). \tag{1.26}$$

Fix an $\varepsilon > 0$. As in the proof of Theorem 4 we may assume that disjoint open arcs U_j are given on the unit circle such that $\operatorname{meas}(\bigcup_{j=1}^N U_j) \le \varepsilon$ and

$$\sigma(A)\cap\partial\mathbb{D} \subset \bigcup_{j=1}^N U_j \subset \partial\mathbb{D}. \tag{2.5}$$

The integration path consists of several parts. First, let Γ_A denote the following arcs: $z = ne^{i\varphi}$ where $e^{i\varphi} \in \bigcup_{j=1}^N U_j$. Then these contribute to A^n, by (2.4) and (1.21),

$$\left\| \frac{n!}{2\pi i} \int_{\Gamma_A} z^{-1-n} e^{zA}\, dz \right\| \le C(e/n)^n n!\varepsilon \le C\varepsilon\sqrt{2\pi(n+1)}. \tag{2.6}$$

Let now θ stand for an angle such that $e^{-i\theta} \in K := \partial\mathbb{D} - \bigcup_{j=1}^N U_j$. Clearly K is a compact set. We need to estimate $e^{ne^{i\theta}A}$. To that end we fix any such θ_0 and write

$$e^{ne^{i\theta_0}A} = \frac{1}{2\pi i} \int_{\gamma_0} e^{ne^{i\theta_0}\lambda}(\lambda - A)^{-1}\, d\lambda. \tag{2.7}$$

Here γ_0 is a contour around the spectrum such that $\Re\{e^{i\theta_0}\lambda\} \le 1 - \varepsilon_0$ for some positive ε_0 and for all $\lambda \in \gamma_0$. This is possible by the spectral mapping theorem. We obtain

$$\|e^{ne^{i\theta_0}A}\| \le C_0 e^{(1-\varepsilon_0)n} \tag{2.8}$$

where

$$C_0 := \sup_{\lambda\in\gamma_0} \|(\lambda - A)^{-1}\| l(\gamma_0), \qquad l(\gamma_0) := \frac{1}{2\pi} \int_{\gamma_0} |d\lambda|.$$

We claim that there is a $\delta > 0$ such that

$$\|e^{ne^{i\theta}A}\| \le 2C_0 e^{(1-\varepsilon_0)n} \qquad \text{for } |\theta - \theta_0| < \delta. \tag{2.9}$$

In fact, we can integrate along γ consisting of points λ such that $e^{i(\theta-\theta_0)}\lambda \in \gamma_0$. For δ small enough, γ is still a contour around the spectrum and by construction $\Re\{e^{i\theta}\lambda\} \le 1 - \varepsilon_0$. Further, we may also assume that δ is small enough that

$$\sup_{\lambda\in\gamma} \|(\lambda - A)^{-1}\| \le 2 \sup_{\lambda\in\gamma_0} \|(\lambda - A)^{-1}\|.$$

Now, by (2.9) we have an open cover for K and we can choose from a finite subcover a largest C and smallest ε such that

$$\|e^{ne^{i\theta}A}\| \le 2Ce^{(1-\varepsilon)n} \quad \text{for all } n \ge 0 \text{ and all } e^{-i\theta} \in K.$$

Returning to the integral (2.4) we can estimate the contribution of $z/n \in K$. We obtain exponential decay:

$$\left\| \frac{n!}{2\pi i} \int_{nK} z^{-1-n} e^{zA} dz \right\| \le 2Ce^{(1-\varepsilon)n} \frac{n!}{n^n} = O(\sqrt{n}\, e^{-\varepsilon n}).$$

Proof of Theorem 8. Assume (1.29) holds. Then from the exponential formula

$$e^{te^{i\varphi}A} = \lim \left(1 - \frac{te^{i\varphi}}{n} A\right)^{-n}$$

we obtain (1.30). The converse direction follows by writing

$$(r - e^{-i\varphi}A)^{-1} = \int_0^\infty e^{-tr} e^{te^{-i\varphi}A}\, dt.$$

Differentiating this $n-1$ times and using (1.30) implies (1.29).

Suppose now that A_ε is power bounded with constant C_1. Then by Theorem 7 we have

$$\|e^{zA_\varepsilon}\| \le C_1 e^{|z|},$$

which can be written as

$$\|e^{(1+\varepsilon)te^{i\varphi}A} e^{-\varepsilon te^{i\varphi}}\| \le C_1 e^t.$$

This implies

$$\|e^{te^{i\varphi}A}\| \le C_1 e^{t(1-\frac{\varepsilon}{1+\varepsilon}\frac{\varphi^2}{\pi^2})} \quad \text{for } |\varphi| \le \pi.$$

Next we show that if A satisfies (1.30) then A_ε satisfies the same inequality with constants C and $c/2$ if $0 \le \varepsilon \le c/(1-2c)$. In fact, we have

$$\|e^{te^{i\varphi}A_\varepsilon}\| = \|e^{(1+\varepsilon)te^{i\varphi}A} e^{-\varepsilon te^{i\varphi}}\| \le Ce^{(1+\varepsilon)t(1-c\varphi^2)-\varepsilon t(1-\varphi^2/2)} \le Ce^{t(1-c\varphi^2/2)}.$$

It suffices to show that if this holds then A is power bounded as then by the previous result the same holds for A_ε. We use the formula

$$A^n = \frac{n!}{2\pi i} \int z^{-1-n} e^{zA}\, dz$$

and take $|z| = n$. Thus

$$\|A^n\| \le Cn! \left(\frac{e}{n}\right)^n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-cn\varphi^2}\, d\varphi \le \frac{C}{\sqrt{c}} \sqrt{2\pi(n+1)}\, E_0/\sqrt{n}$$

where

$$E_k = \frac{1}{\pi} \int_0^\infty t^k e^{-t^2}\, dt.$$

Finally, the identity

$$A^n(A-1) = \frac{n!}{2\pi i}\int\left(\frac{n+1}{z}-1\right)z^{-1-n}e^{zA}dz$$

gives, upon choosing $|z| = n+1$, the other inequality in the same manner:

$$\|A^n(A-1)\| \le \frac{C}{c}E_1\frac{\sqrt{2\pi(n+2)}}{n+1}.$$

Proof of Theorem 9. Let us start with a lemma.

LEMMA. *If* $\sigma(A) \subset \overline{\mathbb{D}}$ *and*

$$\|(\lambda-A)^{-1}\| \le \frac{C}{|\lambda-1|^{1+\alpha}} \qquad \text{for } |\lambda| = 1$$

then there exists a curve $\gamma = \gamma(\varphi)$ *and* $c > 0$ *such that*

$$\begin{aligned} \gamma(\varphi) &= e^{i\varphi - c\varphi^{1+\alpha}} && \text{for } |\varphi| \le \varphi_0, \\ |\gamma(\varphi)| &\le 1-\delta && \text{for } \varphi_0 \le |\varphi| \le \pi, \end{aligned}$$

$\gamma(-\pi) = \gamma(\pi)$ *and for* $\lambda \in \operatorname{Ext}(\gamma)$ *such that* $|\lambda| \le 2$ *we have*

$$\|(\lambda-A)^{-1}\| \le \frac{C_1}{|\lambda-1|^{1+\alpha}}$$

with some $C_1 > 0$.

Proof of Lemma. Let $\Omega := \{\lambda \mid \lambda \in \operatorname{Ext}(\gamma) \text{ and } |\lambda| \le 2\}$. Then $u(\lambda) := |\lambda-1|^{1+\alpha}\|(\lambda-A)^{-1}\|$ is subharmonic in Ω (provided $\sigma(A) \subset \operatorname{Int}(\gamma)$). In fact, $f(\lambda) := (\lambda-1)^{1+\alpha}(\lambda-A)^{-1}$ is analytic outside the unit disc and so u must be subharmonic in Ω as u is just the norm of some analytic continuation of f into Ω.

All we need to do is to conclude that there exists γ of the given form such that u is bounded along it because u is bounded along $|\lambda| = 2$ and therefore the result follows from the maximum principle.

Let $\pi \ge \varphi \ge 0$ and

$$|\mu| \le \frac{1}{2}\cdot\frac{|e^{i\varphi}-1|^{1+\alpha}}{C}.$$

Then

$$\begin{aligned} \|e^{i\varphi}-\mu-A)^{-1}\| &\le \frac{C}{|e^{i\varphi}-1|^{1+\alpha}}\cdot\frac{1}{1-|\mu|\frac{C}{|e^{i\varphi}-1|^{1+\alpha}}} \\ &\le \frac{2C}{|e^{i\varphi}-1|^{1+\alpha}} \le \frac{C_1}{\varphi^{1+\alpha}} \end{aligned}$$

for small $|\varphi|$. The claim follows.

For each $n \ge 1$ we shall define an integration path γ_n to consist of three parts:

$$\gamma_n = \gamma_{n,A} \cup \gamma_{n,B} \cup \gamma_{n,C}.$$

Let $\beta > 0$ be small enough so that

$$(1+\beta/n)|\gamma(\varphi)| \le 1-\delta/2 \quad \text{for } \varphi_0 \le |\varphi| \le \pi.$$

Then put

$$\gamma_n(\varphi) := (1+\beta/n)\gamma(\varphi).$$

For c small enough, independent of n, we let $\gamma_{n,A}$ be the part of γ_n where $|\varphi| \le \varphi_n := (c/n)^{1/(1+\alpha)}$, while $\gamma_{n,B}$ corresponds to $\varphi_n \le |\varphi| \le \varphi_0$ and $\gamma_{n,C}$ to $|\varphi| \ge \varphi_0$.

Then we divide the integration into three parts according to the division of γ_n:

$$\|A^n(A-1)^k\| = \left\| \frac{1}{2\pi} \int_{\gamma_n} \lambda^n \lambda_1^k (\lambda - A)^{-1}\, d\lambda \right\| =: \|I_{n,A} + I_{n,B} + I_{n,C}\|.$$

Here

$$\|I_{n,C}\| \le (1-\delta/2)^n \frac{1}{2\pi} \int_{\gamma_{n,C}} \|(\lambda-1)^k(\lambda - A)^{-1}\|\,|d\lambda| = M_k(1-\delta/2)^n.$$

Likewise

$$\|I_{n,B}\| \le \text{const}(1+\beta/n)^n\, \frac{1}{\pi} \int_{\varphi_n}^{\varphi_0} e^{-n\varphi^{1+\alpha}} \varphi^{k-1-\alpha}\, d\varphi$$

as $|\lambda - 1|$ behaves like φ on that interval. The change of variable $\tau = n\varphi^{1+\alpha}$ yields a bound of the form $\text{const}\cdot(1/n)^{(k-\alpha)/(1+\alpha)}$.

Finally, we use the Kreiss condition. Choosing c small enough we have $|\lambda| - 1 \ge (1+\beta/n)e^{-c\varphi_n^{1+\alpha}} - 1 \ge \text{const}/n$, which gives

$$\|(\lambda - A)^{-1}\| \le \frac{C}{|\lambda| - 1} \le \text{const}\cdot n.$$

This yields

$$\|I_{n,A}\| \le \text{const}\cdot n \int_0^{\varphi_n} \varphi^k\, d\varphi \le \text{const}\cdot n(d/n)^{(k-\alpha)/(1+\alpha)}.$$

Proof of Theorem 11. Assumption (1.37) implies that A is power bounded for positive integers. Write $A = 1 + L$. Assume then that $\lambda < -1$ is such that (1.40) holds. Put $\alpha := -1/(\lambda - 1)$ so that (1.40) reads

$$\|(1+\alpha L)^{-n}\| \le C_1 \quad \text{for } n \ge 0.$$

Since $0 < \alpha < 1$ we conclude from the power boundedness of $1 + L$ that $1 + \alpha L$ is also power bounded, and thus it is bounded for both positive and negative integers.

From (1.37) we conclude that the spectrum of $1+\alpha L$ is in a cone pointing to the inside of the unit disc at 1. However, as the operator is also bounded on the negative integers the spectrum cannot intersect the inside of the disc and we conclude that $\sigma(1+\alpha L)=\{1\}$. An old result of Gelfand [G] then implies that $1+\alpha L=1$.

Proof of Theorem 14. Since

$$T(1)<\gamma=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}\frac{1}{|e^{i\theta}-1|}\,d\theta$$

we conclude from

$$\|(e^{i\theta}-A)^{-1}\|\geq\sup_{\lambda\in\sigma(A)}\frac{1}{|e^{i\theta}-\lambda|}$$

that $\sigma(A)$ is strictly inside the unit disc and in particular $M(1)$ is finite.

Without loss of generality we can assume that $f(\theta):=\|(e^{i\theta}-A)^{-1}\|$ attains its maximum $M(1)$ at $\theta=0$.

But then from $(1-A)^{-1}=(e^{i\theta}-A)^{-1}(1-(e^{i\theta}-1)(1-A)^{-1})$ we have

$$f(\theta)\geq\frac{M(1)}{1+M(1)|e^{i\theta}-1|}.$$

Now we estimate as follows (here $M=M(1)$):

$$\begin{aligned}T(1)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}f(\theta)\,d\theta\\&\geq\frac{1}{\pi}\int_{0}^{\pi/3}\log\frac{1}{|e^{i\theta}-1|}\,d\theta-\frac{1}{\pi}\int_{0}^{\pi/3}\log\left(1+\frac{1}{M|e^{i\theta}-1|}\right)d\theta\\&\geq\gamma-\frac{1}{3}\log(1+1/M)-\frac{1}{3M}\log(M+1),\end{aligned}$$

which implies the claim.

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