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A simplification in the proof of the non-isomorphism between $H^1(\delta)$ and $H^1(\delta^2)$

by

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Abstract. The proof that $H^1(\delta)$ and $H^1(\delta^2)$ are not isomorphic is simplified. This is done by giving a new and simple proof to a martingale inequality of J. Bourgain.

In this note we present a simplification in the proof that $H^1(\delta)$ and $H^1(\delta^2)$ are not isomorphic. The analytic backbone of this result is a martingale inequality of J. Bourgain. Here we will strengthen this inequality and simplify its proof.

We let $\mathcal{D}$ denote the collection of dyadic intervals in the interval $[0,1]$. For $I \in \mathcal{D}$ we let h_I be the L^∞-normalized Haar function supported on the interval I. The Haar function h_I takes the value $+1$ on the left half of the interval I, and the value -1 on the right half of I. For a function h with Haar expansion $h = \sum b_I h_I$, $b_I \in \mathbb{R}$, we say that h belongs to the *dyadic BMO space* if

$$\sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \sum_{J \subseteq I} b_J^2 |J| \right)^{1/2} < \infty.$$

We write $\mathrm{BMO}(\delta)$ for the dyadic BMO space and we denote the above supremum by $\|h\|_{\mathrm{BMO}(\delta)}$. For $f = \sum a_J h_J$ we have the dyadic square function given as

$$S(f)(x) = \Big(\sum a_J^2 \mathbf{1}_J(x) \Big)^{1/2}.$$

If g is a positive integrable function then the martingale inequality of J. Bourgain relates the above expressions as follows:

$$\int g S(h) \ge \delta \int f h \, dx - \delta^{-1} \|S(f) - g\|_{L^1}^{1/2} \|S(f)\|_{L^1}^{1/2} \|h\|_{\mathrm{BMO}(\delta)},$$

where $\delta > 0$ is a universal constant. In [B] J. Bourgain uses this inequality to obtain estimates from below for $\int g S(h)$ under the hypotheses that $\int f h = 1$

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and that the error term $\|S(f)-g\|_{L^1}^{1/2}$ is small. It follows from the proof in [B] that improvements in the estimates for the error term translate into better estimates for the main theorem in [B]. It is the purpose of this note to improve this error term. We also obtain simple numerical constants from a simple straightforward proof.

Theorem 1. *Let $h \in \mathrm{BMO}(\delta)$, $g \in L^1([0,1])$, $g \geq 0$, and let f be a function with $S(f) \in L^1([0,1])$. Then*

$$\int g S(h)\,dx \geq \frac{1}{2}\int fh\,dx - 2\|S(f)-g\|_{L^1}\|h\|_{\mathrm{BMO}}.$$

Comment. The proof we give is just the standard proof of H^1-BMO duality as in [F-S], pp. 148–149. Clearly it is only the contrast to the original—quite delicate—argument of J. Bourgain [B] that justifies its presentation below.

Proof. We let $f=\sum a_I h_I$ and $h=\sum b_I h_I$ be the Haar expansions of f and h. Then we write

$$S(f,m)(x)=\Big(\sum_{|I|\leq 2^{-m}} a_I^2 \mathbf{1}_I(x)\Big)^{1/2}, \qquad h^{\#}(x)=\sup_{I\ni x}\bigg(\frac{1}{|I|}\sum_{J\subseteq I} b_J^2|J|\bigg)^{1/2}.$$

Now we define the following stopping time:

$$m(x)=\inf\{m : S(h,m)(x) < 2h^{\#}(x)\}.$$

We will use the following estimate which will be proved below:

$$|\{x\in I : 2^{-m(x)} < |I|\}| \leq |I|/2.$$

It follows from biorthogonality of the Haar functions, Fubini's theorem and the Cauchy–Schwarz inequality that

$$\begin{aligned}\int f(x)h(x)\,dx &\leq \int \sum_{I\in\mathcal{D}} |a_I b_I| \mathbf{1}_I(x)\,dx \leq 2\int \sum_{\{I:|I|<2^{-m(x)}\}} |a_I b_I|\mathbf{1}_I(x)\,dx\\ &\leq 2\int S(f)(x)S(h,m(x))(x)\,dx.\end{aligned}$$

We now add and subtract the function g, and we finish the proof using the defining property of the stopping time $m(x)$:

$$\begin{aligned}\int f(x)h(x)\,dx &\leq 2\int (S(f)(x)-g(x))S(h,m(x))(x)\,dx + 2\int g(x)S(h,m(x))\,dx\\ &\leq 4\int (S(f)(x)-g(x))h^{\#}(x)\,dx + 2\int g(x)S(h)(x)\,dx\\ &\leq 4\|S(f)-g\|_1\|h\|_{\mathrm{BMO}(\delta)} + 2\int g(x)S(h)(x)\,dx.\end{aligned}$$

We have used the equality $\|h^{\#}\|_\infty = \|h\|_{\mathrm{BMO}(\delta)}$ to obtain the last line. ■

It remains to prove the estimate $|\{x \in I : 2^{-m(x)} < |I|\}| \le |I|/2$. We fix $I \in \mathcal{D}$ and write $A = \{x \in I : 2^{-m(x)} < |I|\}$. Then we choose $m \in \mathbb{N}$ such that

$$|I| = 2^{-m}.$$

Note that for $x \in A$ we have the following pointwise estimate:

$$S^2(h, m(x) - 1)(x) \ge 4h^{\#2}(x) \ge 4\frac{1}{|\tilde{I}|}\int_{\tilde{I}} S^2(h, m-1)(t)\,dt,$$

where $\tilde{I}$ is the dyadic interval satisfying $I \subseteq \tilde{I}$, $|\tilde{I}| = 2|I|$. We also have

$$S^2(h, m-1)(x) \ge S^2(h, m(x) - 1)(x) \quad \text{for } x \in A.$$

Hence

$$\begin{aligned}\frac{1}{|\tilde{I}|}\int_{\tilde{I}} S^2(h, m-1)(x)\,dx &\ge \frac{1}{|\tilde{I}|}\int_A S^2(h, m(x)-1)(x)\,dx\\ &\ge 4\frac{|A|}{|\tilde{I}|}\cdot\frac{1}{|\tilde{I}|}\int_{\tilde{I}} S^2(h, m-1)(x)\,dx.\end{aligned}$$

Cancelling $|\tilde{I}|^{-1}\int_{\tilde{I}} S^2(h, m-1)(x)\,dx$ from both sides of the above estimate gives $|A| \le |I|/2$, as claimed.

Remarks. 1. Note that we actually proved more than we claimed. In fact we showed that the integral

$$\int g(x)\min\{S(h)(x), 2h^{\#}(x)\}\,dx$$

dominates the expression

$$\frac{1}{2}\int fh\,dx - 2\|S(f) - g\|_{L^1}\|h\|_{\mathrm{BMO}(\delta)}.$$

We should remark that this improvement of Bourgain's martingale inequality has further consequences. It allows us to break the proof of the non-isomorphism theorem in [B] into two independent pieces, in such a way that the only place where one uses the notion of "order-inversion" is in Lemma 5 of [B]. In this way the content of the present paper helps to clarify somewhat the role played by the concept of "order-inversion" in the proof of the non-isomorphism between H^1 spaces.

2. The above proof uses only well known and well understood tools developed to prove H^1-BMO duality. Therefore it is clear that the validity of Theorem 1 is not limited to the case of dyadic martingales. Analogous versions can be obtained, e.g., for the case of H^1 spaces consisting of harmonic functions in the upper half space $\mathbb{R}^{n+1}_+$ which are defined as follows. For an integrable function $f : \mathbb{R}^n \to \mathbb{R}$ we denote by $F : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ its harmonic

extension to the upper half space $\mathbb{R}^{n+1}_+ = \{(y,t) : y \in \mathbb{R}^n,\ t > 0\}$. Then the square function is

$$S(f)(x) = \Bigg(\int\limits_{\Gamma(x)} |\nabla F(y,t)|^2 t^{1-n}\, dt\, dx \Bigg)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$. For $h : \mathbb{R}^n \to \mathbb{R}$ locally integrable, we let

$$\|h\|_{\mathrm{BMO}(\mathbb{R}^n)} = \Bigg(\sup_Q \int\limits_Q \bigg| h(x) - \int\limits_Q h(y) \frac{dy}{|Q|} \bigg|^2 \frac{dx}{|Q|} \Bigg)^{1/2},$$

where the supremum is taken over all cubes in $\mathbb{R}^n$. Finally, we let $g \in L^1(\mathbb{R}^n)$ be a non-negative integrable function. With essentially the same proof as above we can show that

$$\int\limits_{\mathbb{R}^n} g(x) S(h)(x)\, dx \ge \delta \int\limits_{\mathbb{R}^n} f(x) h(x)\, dx - \delta^{-1} \|S(f) - g\|_{L^1(\mathbb{R}^n)} \|h\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

where $\delta > 0$ is a universal constant. (See [F-S], pp. 148–149.)

References

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