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On enveloping semigroups of nilpotent group actions generated by unipotent affine transformations of the torus

by

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Abstract. Let G be a group generated by a set of affine unipotent transformations $T : X \to X$ of the form $T(x) = Ax + \alpha$, where A is a lower triangular unipotent matrix, α is a constant vector, and X is a finite-dimensional torus. We show that the enveloping semigroup $E(X, G)$ of the dynamical system (X, G) is a nilpotent group and find upper and lower bounds on its nilpotency class. Also, we obtain a description of $E(X, G)$ as a quotient space.

1. Introduction. By a *dynamical system* we mean a pair (X, Γ), where X is a compact metric space with a metric d, and Γ is a group of self-homeomorphisms of X. When Γ is an infinite cyclic Abelian group generated by an invertible map T we usually denote the system (X, Γ) by (X, T).

A system (X, Γ) is called *distal* if for any distinct points $x, y \in X$,

$$\inf_{\gamma \in \Gamma} d(\gamma(x), \gamma(y)) > 0.$$

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus (which we identify with the unit circle). For a positive integer q, let $X_q = \mathbb{T}^q$ be the q-dimensional torus.

In this paper we consider *affine transformations* $T : X_q \to X_q$ given by

$$T(x) = Ax + \alpha, \tag{1.1}$$

where $x = (x_1, \ldots, x_q)$ is in X_q, $\alpha = (\alpha_1, \ldots, \alpha_q)$ is a fixed element of X_q, the plus sign denotes addition in the group X_q, and A is a unipotent $q \times q$ matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ a_{21} & 1 & 0 & \ldots & 0 \\ a_{31} & a_{32} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{q1} & a_{q2} & \ldots & a_{q\,q-1} & 1 \end{pmatrix}$$

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with integral entries. For the sake of simplicity we use Ax in place of $(Ax^T)^T$ in (1.1), with the understanding that vectors are to be treated either as row or column vectors, whichever is deemed necessary. This convention is used throughout the paper.

Recall that if $a, b \in \Gamma$, then the element $[a, b] = aba^{-1}b^{-1}$ of Γ is called the *commutator* of a and b. Let $\Gamma^{(1)} = [\Gamma, \Gamma]$ be the subgroup of Γ generated by the set of all commutators. We define inductively the subgroups

$$\Gamma^{(n+1)} = [\Gamma^{(n)}, \Gamma]$$

for all $n \in \mathbb{N}$. It is known that $\Gamma^{(n+1)}$ is a normal subgroup of $\Gamma^{(n)}$.

A group Γ is *nilpotent* if the descending series of subgroups

$$\Gamma = \Gamma^{(0)} \trianglerighteq \Gamma^{(1)} \trianglerighteq \Gamma^{(2)} \trianglerighteq \cdots,$$

called the *lower central series* of Γ, terminates at the identity group.

The *enveloping semigroup* of a dynamical system (X, Γ), denoted by $E(X, \Gamma)$, is defined to be the closure of Γ in X^X equipped with the product topology. It is known that $E(X, \Gamma)$ is a compact *left topological semigroup*, i.e., a semigroup in which the multiplication $(x, y) \mapsto xy$ is continuous in the left variable (we follow the terminology of [R]), and that the system (X, Γ) is distal if and only if $E(X, \Gamma)$ is a group (see for example [E]).

Motivated by the result of Namioka (see [N]) about the enveloping semigroup of the system (X_2, T) with $T(x, y) = (x + \alpha, y + x)$, where α is irrational, we computed in [P] the enveloping semigroup of (X_q, T), $q \in \mathbb{N}$, where T is of the form (1.1), and showed it is a nilpotent group.

A far reaching generalization of Namioka's result in a different direction is obtained in [G1] where the author considers a family of nil-systems of class 2 (which are distal) and shows that under certain natural assumptions such systems are characterized by the property that their enveloping semigroup is a 2-step nilpotent group. More examples of direct calculations of enveloping semigroups can be found in the survey [G2], which also contains an overview of recent developments in the general theory of enveloping semigroups.

In this paper we consider a class of dynamical systems (X_q, G), $q \in \mathbb{N}$, where the group G is generated by a set of affine unipotent transformations $T : X_q \to X_q$ of the form (1.1). We assume that the action of this group is *effective*, that is, for any distinct $g_1, g_2 \in G$ there exists an $x \in X_q$ such that $g_1(x) \neq g_2(x)$. Under this assumption the cardinality of G is that of the set of real numbers, since any element of G is of the form (1.1).

Having defined the system (X_q, G), we want to determine its enveloping semigroup $E(X_q, G)$. It is known and easy to see that the system (X_q, G) is distal. Indeed, given $x \in X_q$ and g in G, the action of g on x behaves at each coordinate of $g(x)$ like a rotation by an angle determined by the previous coordinates. Therefore, by the Ellis theorem alluded to above, $E(X_q, G)$ is a group.

We show that $E(X_q, G)$ is nilpotent and find the bounds on its nilpotency class. In particular, the nilpotency class never exceeds q. Also, we obtain a description of $E(X_q, G)$ as a quotient space.

The results obtained in the current work are more general (a completely new method is employed) than those of [P] (where a single transformation of the form (1.1) was considered). However, greater generality comes at the price of the lack of precise information about the exact nilpotency class of a given enveloping semigroup. If the group G is generated by a single transformation T, then we know exactly what the nilpotency class of $E(X_q, T)$ is, while in the case of G having more than one generator our method yields only a natural upper bound on the nilpotency class of $E(X_q, G)$. A lower bound is obtained provided we know the nilpotency class of the enveloping semigroup $E(X_q, T)$, where T is a selected generator of G.

The structure of this paper is as follows. Section 2 contains definitions, constructions and some of the properties of the topological spaces utilized in the next section.

In Section 3 we work with the group G generated by a set of transformations defined in (1.1). We show that the enveloping semigroup $E(X_q, G)$ of the dynamical system (X_q, G) is a nilpotent group. Moreover, we find lower and upper bounds on the nilpotency class of $E(X_q, G)$. As an immediate consequence we find that the dimension of the underlying torus serves as a universal upper bound. Also, we identify $E(X_q, G)$ with a quotient space of some naturally arising group of upper triangular matrices.

Appendix A contains mostly technical computations resulting in obtaining a compact and convenient formula for T^m, where T is given by (1.1), and m is an integer.

In Appendix B we collect some useful facts about compact left topological semigroups and compact left topological groups.

More information about nets, also known as generalized sequences (or Moore–Smith sequences), utilised in Sections 2 and 3, can be found in [K].

2. Building blocks. This section introduces some of the notions and constructions of the topological spaces needed in the next section, as well as some of their properties.

Definition 2.1. Let $\mathrm{End}(\mathbb{T})$ be the space of endomorphisms of the torus $\mathbb{T}$, i.e., of (not necessarily continuous) maps $f : \mathbb{T} \to \mathbb{T}$ satisfying

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{T}.$$

It is easy to see that $\mathrm{End}(\mathbb{T})$ is closed under addition, subtraction, and composition of maps. Moreover, $\mathrm{End}(\mathbb{T})$ under composition (denoted by fg, for $f, g \in \mathrm{End}(\mathbb{T})$) is a compact left topological subsemigroup of $\mathbb{T}^{\mathbb{T}}$ equipped with the product topology.

The following simple lemma is at the core of analogous results that the reader will encounter later.

Lemma 2.2. *Let $p \in \mathrm{End}(\mathbb{T})$ be of the form $p(x) = mx$ for all $x \in \mathbb{T}$, where m is an integer. Let $\{f_\gamma\}$ be a net in $\mathrm{End}(\mathbb{T})$ that converges to some $f \in \mathrm{End}(\mathbb{T})$. Then the net $\{pf_\gamma\}$ converges to pf.*

Proof. This follows from the fact that p is continuous. ∎

Definition 2.3. Let $\mathcal{F}$ be the space of $q \times q$ matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ f_{21} & 0 & 0 & \dots & 0 \\ f_{31} & f_{32} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{q1} & f_{q2} & \dots & f_{q\,q-1} & 0 \end{pmatrix},$$

where the entries f_{ij} for $i, j \in \{1, \dots, q\}$ are in $\mathrm{End}(\mathbb{T})$.

Note that $\mathcal{F}$ can be thought of as a subset of $(\mathrm{End}(\mathbb{T}))^{q\times q}$, and it is compact in the induced topology. Moreover, under matrix multiplication (defined the standard way, where the product of two elements of $\mathrm{End}(\mathbb{T})$ is their composition) $\mathcal{F}$ becomes a left topological semigroup. Also, the elements of $\mathcal{F}$ can be treated as lower triangular nilpotent matrices whose nilpotency class is bounded above by q.

Lemma 2.4. *Let $F^{(1)}$ be an element of $\mathcal{F}$ whose entries $f^{(1)}_{ij} \in \mathrm{End}(\mathbb{T})$ located below the main diagonal are of the form $f^{(1)}_{ij}(x) = m_{ij}x$ for all $x \in \mathbb{T}$, where m_{ij} are integers. Let $\{F_\gamma\}$ be a net in $\mathcal{F}$ that converges to some $F \in \mathcal{F}$. Then the net $\{F^{(1)}F_\gamma\}$ converges to $F^{(1)}F$.*

Proof. Note that the convergence of the net $\{F_\gamma\}$ is equivalent to the convergence of the net at each entry of the matrix F_γ. Each entry (below the main diagonal) of the composition $F^{(1)}F_\gamma$ consists of finite sums of compositions of elements in $\mathrm{End}(\mathbb{T})$, where the left component of each composition is $f^{(1)}_{ij}$, for some i, j, and the right component is an entry of F_γ. Applying Lemma 2.2 to these compositions yields the desired result. ∎

Let Id denote the $q \times q$ matrix

$$\begin{pmatrix} \mathrm{id} & 0 & \dots & 0 \\ 0 & \mathrm{id} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix}$$

with entries in End($\mathbb{T}$), where id denotes the identity, and let $\mathcal{F}' = \mathcal{F} \cup \{\mathrm{Id}\}$. Then $\mathcal{F}'$ is a compact left topological semigroup with identity. (Note that, since Id is an isolated point of $\mathcal{F}'$, the topology of $\mathcal{F}'$ consists of open sets of $\mathcal{F}$ and unions of such sets with $\{\mathrm{Id}\}$.)

DEFINITION 2.5. Let $\mathcal{B}$ be the space of $q \times q$ matrices of the form

$$\begin{pmatrix} \mathrm{Id} & F_1 & F_2 & \dots & F_{q-1} \\ 0 & \mathrm{Id} & F_1 & \dots & F_{q-2} \\ 0 & 0 & \mathrm{Id} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & F_1 \\ 0 & 0 & \dots & 0 & \mathrm{Id} \end{pmatrix},$$

where 0 and F_i for $i \in \{1, \dots, q-1\}$ are elements of $\mathcal{F}$.

Clearly, all the entries of such matrices belong to $\mathcal{F}'$, so we can think of $\mathcal{B}$ as a subspace of $(\mathcal{F}')^{q \times q}$. Note that $\mathcal{B}$ with the induced topology is compact. Moreover, since the operations of multiplying from the left and from the right by Id and by the zero matrix 0 are continuous on $\mathcal{F}'$, it is easy to see that $\mathcal{B}$ is a compact left topological semigroup with identity. In fact, more is true.

PROPOSITION 2.6. *$\mathcal{B}$ is a left topological group.*

Proof. Let $B \in \mathcal{B}$ be an idempotent (existence of which is guaranteed by Proposition B.1), that is,

$$B^2 = B. \tag{2.1}$$

By Proposition B.3 it is enough to show that B is the identity on $\mathcal{B}$. To this end note that if

$$B = \begin{pmatrix} \mathrm{Id} & F_1 & F_2 & \dots & F_{q-1} \\ 0 & \mathrm{Id} & F_1 & \dots & F_{q-2} \\ 0 & 0 & \mathrm{Id} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & F_1 \\ 0 & 0 & \dots & 0 & \mathrm{Id} \end{pmatrix},$$

then B^2 is of the form

$$\begin{pmatrix} \mathrm{Id} & F'_1 & F'_2 & \dots & F'_{q-1} \\ 0 & \mathrm{Id} & F'_1 & \dots & F'_{q-2} \\ 0 & 0 & \mathrm{Id} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & F'_1 \\ 0 & 0 & \dots & 0 & \mathrm{Id} \end{pmatrix},$$

where

$$F_i' = \sum_{k+l=i} F_k F_l = 2F_i + \sum_{k=1}^{i-1} F_k F_{i-k}$$

for $i \in \{1, \ldots, q-1\}$. If $i = 1$, then (2.1) implies $F_1 = 2F_1$, so that $F_1 = 0$. It follows by an easy induction that $F_i = 0$ for every $i \in \{1, \ldots, q-1\}$, hence B is the identity matrix of $\mathcal{B}$. ■

REMARK 2.7. Note that $\mathcal{B}$, having the structure of the set of upper triangular matrices, is a $(q-1)$-step nilpotent group.

The proof of the following lemma, almost identical to that of Lemma 2.4, is omitted.

LEMMA 2.8. *Let $B^{(1)}$ be an element of $\mathcal{B}$ whose entries $F_i^{(1)} \in \mathcal{F}$, where $i = 1, \ldots, q-1$, satisfy the hypothesis of Lemma 2.4. Let $\{B_\gamma\}$ be a net in $\mathcal{B}$ that converges to some $B \in \mathcal{B}$. Then the net $\{B^{(1)}B_\gamma\}$ converges to $B^{(1)}B$.*

DEFINITION 2.9. Let $B \in \mathcal{B}$ be a matrix of the form displayed in Definition 2.5, and let $v = (v_{q-1}, \ldots, v_0) \in (X_q)^q$ be a column vector. Define the element Bv of $(X_q)^q$ to have the ith coordinate equal to

$$\sum_{k=0}^{i} F_k v_{i-k}$$

for $i \in \{0, \ldots, q-1\}$, where $F_0 = \mathrm{Id}$, and the "product" $F_k v_{i-k}$ of a vector v_{i-k} in X_q (recall that $X_q = \mathbb{T}^q$) and a matrix F_k, whose entries are in $\mathrm{End}(\mathbb{T})$, is obtained in such a way that if v' is an entry of v_{i-k} and f' an entry of F_k, then $f'v'$ is defined to be $f'(v')$, the latter being an element of $\mathbb{T}$.

DEFINITION 2.10. Let $\mathcal{M}$ be the space of matrices of the form

$$\begin{pmatrix} B & v \\ 0 & 1 \end{pmatrix},$$

where $B \in \mathcal{B}$ and $v \in (X_q)^q$.

By identifying $\mathcal{M}$ with the product $\mathcal{B} \times (X_q)^q$, we can see that the space $\mathcal{M}$ is compact.

DEFINITION 2.11. Given

$$M_1 = \begin{pmatrix} B_1 & v^{(1)} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} B_2 & v^{(2)} \\ 0 & 1 \end{pmatrix}$$

in $\mathcal{M}$, define their product $M = M_1 M_2$ to be

$$M = \begin{pmatrix} B_1 B_2 & B_1 v^{(2)} + v^{(1)} \\ 0 & 1 \end{pmatrix}.$$

Endowed with this multiplication, the set $\mathcal{M}$ becomes a semigroup, and we can see that $\mathcal{M}$ is in fact a compact left topological semigroup with identity.

PROPOSITION 2.12. *$\mathcal{M}$ is a left topological group.*

Proof. Simple calculations show that the only idempotent of $\mathcal{M}$ is the identity, hence Proposition B.3 applies. ∎

LEMMA 2.13. *Let $M^{(1)}$ be an element of $\mathcal{M}$ of the form*

$$M^{(1)} = \begin{pmatrix} B^{(1)} & v^{(1)} \\ 0 & 1 \end{pmatrix},$$

where the matrix $B^{(1)}$ satisfies the hypothesis of Lemma 2.8. Let $\{M_\gamma\}$ be a net in $\mathcal{M}$ that converges to some $M \in \mathcal{M}$. Then the net $\{M^{(1)}M_\gamma\}$ converges to $M^{(1)}M$.

Proof. The idea of the proof is the same as that of Lemma 2.4. We leave the details to the reader. ∎

Now we want to study the structure of $\mathcal{M}$ more closely in order to show that it is a q-step nilpotent group. Let

$$\mathcal{M} = \mathcal{M}^{(0)} \trianglerighteq \mathcal{M}^{(1)} \trianglerighteq \mathcal{M}^{(2)} \trianglerighteq \cdots, \qquad \mathcal{B} = \mathcal{B}^{(0)} \trianglerighteq \mathcal{B}^{(1)} \trianglerighteq \mathcal{B}^{(2)} \trianglerighteq \cdots$$

be the lower central series of $\mathcal{M}$ and $\mathcal{B}$, respectively. By Remark 2.7, the group $\mathcal{B}$ is $(q-1)$-step nilpotent, so $\mathcal{B}^{(k)} = \{\mathrm{Id}_{\mathcal{B}}\}$ for $k \geq q-1$. Moreover, if $B \in \mathcal{B}^{(k)}$, where $k \geq 1$, then the entries of the first k diagonals above the main diagonal are all zeros.

LEMMA 2.14. *Let k be a non-negative integer. Suppose that the matrix*

$$\begin{pmatrix} B & v \\ 0 & 1 \end{pmatrix}$$

belongs to $\mathcal{M}^{(k)}$. Then $B \in \mathcal{B}^{(k)}$ and $v_0 = v_1 = \cdots = v_{k-1} = 0$.

Proof. The claim is clear for $k = 0$. Suppose that it holds for some $k \geq 0$. Let

$$M_1 = \begin{pmatrix} B_1 & v^{(1)} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} B_2 & v^{(2)} \\ 0 & 1 \end{pmatrix}$$

be in $\mathcal{M}$ and $\mathcal{M}^{(k)}$, respectively. The inductive hypothesis implies that $v_0^{(2)} = \cdots = v_{k-1}^{(2)} = 0$, and that $B_2 \in \mathcal{B}^{(k)}$. Computation of the commutator $M' = [M_1, M_2]$ yields

$$M' = \begin{pmatrix} [B_1, B_2] & -[B_1, B_2]v^{(2)} - B_1 B_2 B_1^{-1} v^{(1)} + B_1 v^{(2)} + v^{(1)} \\ 0 & 1 \end{pmatrix}.$$

We can easily see that the upper left entry of M' belongs to $\mathcal{B}^{(k+1)}$, hence so does the upper left entry of every element of $\mathcal{M}^{(k+1)}$.

Consider two vectors

$$v' = v^{(1)} - B_1 B_2 B_1^{-1} v^{(1)} \quad \text{and} \quad v'' = B_1 v^{(2)} - [B_1, B_2] v^{(2)}.$$

Using the fact that $v_0^{(2)} = \cdots = v_{k-1}^{(2)} = 0$, we see that, for any $B \in \mathcal{B}$, the kth coordinate of the vector Bv is zero. In particular, the jth coordinate of v'' is zero for $j = 1, \ldots, k$.

To deal with v' notice that, since $B_2 \in \mathcal{B}^{(k)}$, the matrix $B_1 B_2 B_1^{-1}$ is also in $\mathcal{B}^{(k)}$. Any matrix $B \in \mathcal{B}^{(k)}$ contains a smaller $(k+1) \times (k+1)$ identity matrix located in the bottom right corner. This implies that the last $k+1$ coordinates of $B_1 B_2 B_1^{-1} v^{(1)}$ are the same as those of $v^{(1)}$. In particular, the jth coordinate of v' is zero for $j = 1, \ldots, k$.

Hence the vector $v' + v''$ has the desired property. This finishes the proof of the inductive step. ■

As an immediate corollary we see that the group $\mathcal{M}$ is q-step nilpotent.

Definition 2.15. For $M \in \mathcal{M}$, where

$$M = \begin{pmatrix} B & v \\ 0 & 1 \end{pmatrix}$$

with B as in Definition 2.5, and $v = (v_{q-1}, \ldots, v_0) \in (X_q)^q$, let $L_M : X_q \to X_q$ be given by the formula

$$L_M(x) = \sum_{j=0}^{q-1} F_j x + \sum_{j=0}^{q-1} v_j \quad \text{for } x \in X_q.$$

Definition 2.16. Define a map φ from $\mathcal{M}$ to $X_q^{X_q}$ via the rule

$$\varphi(M) = L_M.$$

Proposition 2.17. *The map φ defined above is continuous.*

Proof. Let $M^{(r)}$ be a net in $\mathcal{M}$ convergent to some $M \in \mathcal{M}$. Suppose

$$M^{(r)} = \begin{pmatrix} B_r & v^{(r)} \\ 0 & 1 \end{pmatrix}$$

with

$$B_r = \begin{pmatrix} \mathrm{Id} & F_1^{(r)} & F_2^{(r)} & \ldots & F_{q-1}^{(r)} \\ 0 & \mathrm{Id} & F_1^{(r)} & \ldots & F_{q-2}^{(r)} \\ 0 & 0 & \mathrm{Id} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & F_1^{(r)} \\ 0 & 0 & \ldots & 0 & \mathrm{Id} \end{pmatrix}, \quad v^{(r)} = (v_{q-1}^{(r)}, \ldots, v_0^{(r)}) \in (X_q)^q,$$

and

$$M = \begin{pmatrix} B & v \\ 0 & 1 \end{pmatrix}$$

with

$$(2.2) \qquad B = \begin{pmatrix} \mathrm{Id} & F_1 & F_2 & \dots & F_{q-1} \\ 0 & \mathrm{Id} & F_1 & \dots & F_{q-2} \\ 0 & 0 & \mathrm{Id} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & F_1 \\ 0 & 0 & \dots & 0 & \mathrm{Id} \end{pmatrix}, \qquad v = (v_{q-1}, \dots, v_0) \in (X_q)^q.$$

This implies that the nets $\{F_j^{(r)}\}$ and $\{v_j^{(r)}\}$ converge to F_j and v_j, respectively, where $j = 0, 1, \dots, q-1$. Hence, for all $x \in X_q$ and $j = 0, 1, \dots, q-1$, the nets $\{F_j^{(r)} x\}$ converge to $F_j x$. Therefore, for all $x \in X_q$,

$$L_{M^{(r)}}(x) = \sum_{j=0}^{q-1} F_j^{(r)} x + \sum_{j=0}^{q-1} v_j^{(r)} \to \sum_{j=0}^{q-1} F_j x + \sum_{j=0}^{q-1} v_j = L_M(x),$$

which shows that $\varphi(M^{(r)}) = L_{M^{(r)}} \to L_M = \varphi(M)$, meaning that φ is continuous. ■

Definition 2.18. Let $\mathbb{M}$ be the subset of $\mathcal{M}$ consisting of all matrices

$$M = \begin{pmatrix} B & v \\ 0 & 1 \end{pmatrix},$$

where

$$B = \begin{pmatrix} \mathrm{Id} & F_1 & F_2 & \dots & F_{q-1} \\ 0 & \mathrm{Id} & F_1 & \dots & F_{q-2} \\ 0 & 0 & \mathrm{Id} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & F_1 \\ 0 & 0 & \dots & 0 & \mathrm{Id} \end{pmatrix}, \qquad v = (v_{q-1}, \dots, v_0),$$

such that

(1) the entries of the first $i-1$ diagonals below the main diagonal of F_i are all 0, for $i = 1, \dots, q-1$,
(2) the first i entries of the vector v_i are 0, for $i = 0, \dots, q-1$.

Proposition 2.19. *The set* $\mathbb{M}$ *with the matrix multiplication inherited from* $\mathcal{M}$ *is a left continuous group.*

Proof. It is clear that $\mathbb{M}$ is a compact subset of $\mathcal{M}$, and that $\mathbb{M}$ contains the identity. It is enough to show that $\mathbb{M}$ is a semigroup, since then it is automatically a compact left continuous subsemigroup of $\mathcal{M}$, and hence a compact left continuous group, by Proposition B.3. To this end assume that $M_1, M_2 \in \mathbb{M}$ and $M \in \mathcal{M}$ are as in Definition 2.11, where

$$B_j = \begin{pmatrix} \mathrm{Id} & F_1^{(j)} & F_2^{(j)} & \dots & F_{q-1}^{(j)} \\ 0 & \mathrm{Id} & F_1^{(j)} & \dots & F_{q-2}^{(j)} \\ 0 & 0 & \mathrm{Id} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & F_1^{(j)} \\ 0 & 0 & \dots & 0 & \mathrm{Id} \end{pmatrix}, \quad v^{(j)} = (v_{q-1}^{(j)}, \dots, v_0^{(j)}),$$

for $j = 1, 2$. Let

$$B = B_1 B_2 \quad \text{and} \quad v = B_1 v^{(2)} + v^{(1)}.$$

Note that, for $i = 1, \dots, q-1$, the entries of the ith diagonal above the main diagonal of B are all equal to

$$F_i = \sum_{k+l=i} F_k^{(1)} F_l^{(2)}.$$

The matrices $F_k^{(1)}$ and $F_l^{(2)}$ belong to $\mathcal{F}$ and by assumption the $k-1$ diagonals below the main diagonal of $F_k^{(1)}$, and the $l-1$ diagonals below the main diagonal of $F_l^{(2)}$, consist of zeros. Hence, for each of the products $F_k^{(1)} F_l^{(2)}$, the

$$(k-1) + (l-1) + 1 = (k+l) - 1 = i - 1$$

diagonals below the main diagonal consist of zeros. Therefore F_i has the required property (1) of Definition 2.18.

Turning to the vector v, we can see that it is enough to check that the vector $B_1 v^{(2)}$ satisfies property (2) of Definition 2.18. By Definition 2.9, for $i = 0, \dots, q-1$, the ith coordinate of $B_1 v^{(2)}$ is of the form

$$\sum_{k=0}^{i} F_k^{(1)} v_{i-k}^{(2)} = v_i^{(2)} + \sum_{k=1}^{i} F_k^{(1)} v_{i-k}^{(2)},$$

where $F_0^{(1)} = \mathrm{Id}$, so it is enough to deal with the sum on the right hand side above. If $i = 0$ we are done. Suppose that $i \geq 1$.

Recall that by our assumption the first $i-k$ coordinates of the vector $v_{i-k}^{(2)}$ are zeros. The first possibly non-zero row of the matrix $F_k^{(1)}$ has number at least $k+1$ (counting from top to bottom). It is easy to verify that the ith row of $F_k^{(1)}$ can have at most $i-k$ non-zero entries (when counting from left to right). This implies that the ith coordinate of the vector $F_k^{(1)} v_{i-k}^{(2)}$

(when counting from top to bottom) is zero, for $k = 1, \dots, i$. It follows that all coordinates above it are also zero, which proves that v has the required property.

Therefore, M belongs to $\mathbb{M}$, and not just $\mathcal{M}$, which shows that $\mathbb{M}$ is a semigroup. ∎

PROPOSITION 2.20. *The map φ is a continuous homomorphism from $\mathbb{M}$ to the semigroup $X_q^{X_q}$.*

Proof. The continuity follows from Proposition 2.17. Let M_1, M_2, and $M = M_1 M_2$ be as in the preceding proof. To show that φ is a homomorphism, it is enough to check that

$$L_M(x) = (L_{M_1} L_{M_2})(x) \quad \text{for all } x \in X_q.$$

Note that

$$\begin{aligned} L_M(x) &= \sum_{j=0}^{q-1} \Big(\sum_{k+l=j} F_k^{(1)} F_l^{(2)} \Big) x + \sum_{j=0}^{q-1} \sum_{k+l=j} F_k^{(1)} v_l^{(2)} + \sum_{j=0}^{q-1} v_j^{(1)} \\ &= \sum_{k+l \le q-1} F_k^{(1)} F_l^{(2)} x + \sum_{k+l \le q-1} F_k^{(1)} v_l^{(2)} + \sum_{j=0}^{q-1} v_j^{(1)}, \end{aligned}$$

and

$$(L_{M_1} L_{M_2})(x) = \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} F_k^{(1)} F_l^{(2)} x + \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} F_k^{(1)} v_l^{(2)} + \sum_{j=0}^{q-1} v_j^{(1)}.$$

Write

$$\begin{aligned} \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} F_k^{(1)} F_l^{(2)} &= \sum_{k+l \le q-1} F_k^{(1)} F_l^{(2)} + \sum_{k+l \ge q} F_k^{(1)} F_l^{(2)}, \\ \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} F_k^{(1)} v_l^{(2)} &= \sum_{k+l \le q-1} F_k^{(1)} v_l^{(2)} + \sum_{k+l \ge q} F_k^{(1)} v_l^{(2)}, \end{aligned}$$

and notice that, as M_1, M_2 are in $\mathbb{M}$, the second sum on the right hand side of each of the above two identities must be zero. Indeed, if $k + l \ge q$, then at least $q - 1$ diagonals below the main diagonal of $F_k^{(1)} F_l^{(2)}$ have to be zero, but there are only $q - 1$ such diagonals, so that $F_k^{(1)} F_l^{(2)} = 0$; moreover, at least q entries of the vector $F_k^{(1)} v_l^{(2)}$ have to be zero, but there are only q such entries to begin with, hence $F_k^{(1)} v_l^{(2)} = 0$.

This shows that, after applying the above simplifications, the formula for $(L_{M_1} L_{M_2})(x)$ is the same as the one for $L_M(x)$, for all $x \in X_q$. ∎

From now on we will always consider φ as a map from $\mathbb{M}$ to the semigroup $X_q^{X_q}$.

3. The enveloping semigroup of (X_q, G). In this section we consider transformations T_i of the form (1.1) acting on the space X_q, for some fixed $q \geq 2$, where $i \in I$, and I (as indicated in the Introduction) is an index set whose cardinality does not exceed that of the set of real numbers.

For given $i \in I$, let N_i be the nilpotent strictly lower triangular $q \times q$ matrix with integral entries associated with T_i.

The following formula, proved in Appendix A (see (A.7)), holds for all integers n:

$$T_i^n x = \sum_{j=0}^{d_i} \binom{n}{j} N_i^j x + \sum_{j=0}^{d_i} \binom{n}{j+1} N_i^j \alpha_i, \tag{3.1}$$

where $d_i + 1$ is the nilpotency class of N_i. Since the nilpotency class of N_i is bounded above by q for all $i \in I$, we may assume, without loss of generality, that

$$T_i^n x = \sum_{j=0}^{q-1} \binom{n}{j} N_i^j x + \sum_{j=0}^{q-1} \binom{n}{j+1} N_i^j \alpha_i. \tag{3.2}$$

Definition 3.1. Define G to be the group generated by the transformations T_i, i.e.,

$$G = \langle T_i : i \in I \rangle.$$

Consider the dynamical system (X_q, G). As explained in the Introduction, this system is distal, hence its enveloping semigroup $E(X_q, G)$ is a group. Our main goal is to prove that this group is nilpotent. The objects introduced in the previous section will play a prominent role in the proof.

Let $f \in \mathrm{End}(\mathbb{T})$ and let N be a nilpotent strictly lower triangular $q \times q$ matrix with integral entries of the form

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ n_{21} & 0 & 0 & \ldots & 0 \\ n_{31} & n_{32} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ n_{q1} & n_{q2} & \ldots & n_{q\,q-1} & 0 \end{pmatrix}.$$

Define

$$f \cdot N = N \cdot f = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ n_{21}f & 0 & 0 & \ldots & 0 \\ n_{31}f & n_{32}f & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ n_{q1}f & n_{q2}f & \ldots & n_{q\,q-1}f & 0 \end{pmatrix}.$$

Suppressing the multiplication sign, we can see that $fN \in \mathcal{F}$. Moreover, if $f_1N^{(1)}$ and $f_2N^{(2)}$ are two such elements, then it is easy to verify that

$$(f_1N^{(1)})(f_2N^{(2)}) = (f_1f_2)(N^{(1)}N^{(2)}),$$

where f_1f_2 denotes composition of maps in $\mathrm{End}(\mathbb{T})$, and $N^{(1)}N^{(2)}$ is a product of matrices.

Let T_i^n be as in (3.2). Define the following elements of $\mathcal{F}$:

$$B_j^{(i,n)} = \binom{n}{j} N_i^j \quad \text{and} \quad C_j^{(i,n)} = \binom{n}{j+1} N_i^j,$$

where we treat $\binom{n}{k}$ as an element of $\mathrm{End}(\mathbb{T})$ (via the rule $x \mapsto \binom{n}{k}x$). Also, define

$$v_j^{(i,n)} = C_j^{(i,n)} \alpha_i.$$

Then (3.2) becomes

$$T_i^n x = \sum_{j=0}^{q-1} B_j^{(i,n)} x + \sum_{j=0}^{q-1} v_j^{(i,n)}. \tag{3.3}$$

Notice that, for every pair (i,n), the matrix $B_j^{(i,n)}$ and the vector $v_j^{(i,n)}$ satisfy, respectively, properties (1) and (2) of Definition 2.18. Therefore the following definition makes sense.

DEFINITION 3.2. Define

$$B^{(i,n)} = \begin{pmatrix} \mathrm{Id} & B_1^{(i,n)} & B_2^{(i,n)} & \dots & B_{q-1}^{(i,n)} \\ 0 & \mathrm{Id} & B_1^{(i,n)} & \dots & B_{q-2}^{(i,n)} \\ 0 & 0 & \mathrm{Id} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & B_1^{(i,n)} \\ 0 & 0 & \dots & 0 & \mathrm{Id} \end{pmatrix}, \qquad v^{(i,n)} = (v_{q-1}^{(i,n)}, \dots, v_0^{(i,n)}).$$

Finally, define an element $M^{(i,n)}$ of $\mathbb{M}$ as follows:

$$M^{(i,n)} = \begin{pmatrix} B^{(i,n)} & v^{(i,n)} \\ 0 & 1 \end{pmatrix}.$$

REMARK 3.3. We emphasize that, for a given pair (i,n), the matrix $M^{(i,n)}$ satisfies the hypothesis of Lemma 2.13.

LEMMA 3.4. *Let n and m be integers. Then*

$$M^{(i,n)}M^{(i,m)} = M^{(i,n+m)}.$$

Proof. Let

$$M = M^{(i,n)}M^{(i,m)}.$$

Note that

$$M = \begin{pmatrix} B^{(i,n)}B^{(i,m)} & B^{(i,n)}v^{(i,m)} + v^{(i,n)} \\ 0 & 1 \end{pmatrix}.$$

Let

$$B = B^{(i,n)}B^{(i,m)}.$$

Then B is of the form (2.2), where

$$\begin{aligned} F_j &= \sum_{k+l=j} B_k^{(i,n)} B_l^{(i,m)} = \sum_{k+l=j} \binom{n}{k} N_i^k \binom{m}{l} N_i^l \\ &= \Big(\sum_{k+l=j} \binom{n}{k}\binom{m}{l}\Big) N_i^j = \binom{n+m}{j} N_i^j = B_j^{(i,n+m)} \end{aligned}$$

for $j = 0, 1, \dots, q-1$. Hence

$$B = B^{(i,n+m)}.$$

Letting

$$v = B^{(i,n)}v^{(i,m)} + v^{(i,n)},$$

we obtain

$$\begin{aligned} v_j &= v_j^{(i,n)} + \sum_{k=0}^{j} B_k^{(i,n)} v_{j-k}^{(i,m)} = C_j^{(i,n)}\alpha_i + \sum_{k=0}^{j} B_k^{(i,n)} C_{j-k}^{(i,m)}\alpha_i \\ &= \binom{n}{j+1} N_i^j \alpha_i + \sum_{k=0}^{j} \binom{n}{k} N_i^k \binom{m}{j-k+1} N_i^{j-k}\alpha_i \\ &= \Big(\sum_{k=0}^{j+1} \binom{n}{k}\binom{m}{j-k+1}\Big) N_i^j \alpha_i = \binom{n+m}{j+1} N_i^j \alpha_i \\ &= C_j^{(i,n+m)}\alpha_i = v_j^{(i,n+m)}. \end{aligned}$$

Therefore $M = M^{(i,n+m)}$. ∎

Corollary 3.5. *For any integer n we have*

$$(M^{(i,1)})^n = M^{(i,n)}.$$

Definition 3.6. Let $\mathbb{G}$ be the subgroup of $\mathbb{M}$ generated by the matrices $M^{(i,1)}$.

Remark 3.7. In view of Remark 3.3, every matrix M that belongs to $\mathbb{G}$ satisfies the hypothesis of Lemma 2.13.

Remark 3.8. Note that the definitions of φ and $M^{(i,n)}$ imply that

$$\varphi(M^{(i,n)}) = T_i^n.$$

Letting $\varphi_1 = \varphi|_{\mathbb{G}}$ we get the following result.

PROPOSITION 3.9. *The map φ_1 is a homomorphism from $\mathbb{G}$ onto G.*

Proof. By Proposition 2.20, it is enough to show that $\varphi_1 : \mathbb{G} \to G$ is onto. Let $M \in \mathbb{G}$. By the definition of $\mathbb{G}$ and by Corollary 3.5 we can write

$$M = M^{(i_{k_1}, n_1)} \ldots M^{(i_{k_l}, n_l)}$$

for some $i_{k_1}, \ldots, i_{k_l} \in I$, and some integers $n_1, \ldots, n_l$. By Remark 3.8 it is clear that $\varphi_1(M) \in G$, and that for any $g \in G$ we can find $M \in \mathbb{G}$ with $\varphi_1(M) = g$. ∎

Define $\mathbb{G}'$ to be the closure of $\mathbb{G}$ in $\mathbb{M}$. Remark 3.7 implies that we may apply Proposition B.2 to conclude that $\mathbb{G}'$ is a compact left topological subsemigroup of $\mathbb{M}$. Therefore, by Proposition B.3, it is a left topological group and as a subgroup of $\mathbb{M}$ it is a nilpotent group, with nilpotency class at most q.

Now we are going to describe the relationship between the group $\mathbb{G}'$ and the enveloping semigroup $E(X_q, G)$ of the dynamical system (X_q, G). Recall, that $E(X_q, G)$ is the closure of G in $X_q^{X_q}$.

Let $\varphi_2 = \varphi|_{\mathbb{G}'}$.

LEMMA 3.10. *The range of $\varphi_2 : \mathbb{G}' \to X_q^{X_q}$ is equal to $E(X_q, G)$.*

Proof. Notice that $\varphi_2|_{\mathbb{G}} = \varphi_1$, so that $\varphi_2(\mathbb{G}) = G$. Hence, by continuity of φ_2, we get

$$\varphi_2(\mathbb{G}') = \varphi_2(\overline{\mathbb{G}}) \subset \overline{G} = E(X_q, G).$$

Let $\tau \in E(X_q, G)$. Then there exists a net $\{g_\gamma\}$ in G that converges to τ, so we can find a net $\{M_\gamma\}$ in $\mathbb{G}$ with

$$\varphi_2(M_\gamma) = g_\gamma$$

for all γ. By the compactness of $\mathbb{G}'$, we can pass to a subnet $\{M_{\gamma'}\}$ that converges to some M in $\mathbb{G}'$. Thus, by continuity of φ_2, we get

$$\varphi_2(M) = \lim_{\gamma'} \varphi(M_{\gamma'}) = \lim_{\gamma'} g_{\gamma'} = \tau.$$

Therefore φ_2 maps $\mathbb{G}'$ onto $E(X_q, G)$. ∎

Consequently, the enveloping semigroup $E(X_q, G)$ is a homomorphic image of a nilpotent group $\mathbb{G}'$, with nilpotency class at most q. This proves the following theorem.

THEOREM 3.11. *The group $E(X_q, G)$ is k-step nilpotent, where $k \le q$.*

Fix a generator T_0 of G such that the matrix N associated with T_0 via (1.1) is $(d+1)$-step nilpotent, but not d-step nilpotent. Clearly, the enveloping semigroup $E(X_q, T_0)$ is a subgroup of $E(X_q, G)$. In [P] we show that even though $E(X_q, T_0)$ need not be exactly $(d+1)$-step nilpotent it is always at least d-step nilpotent. We also show that if $T_0 = A_0 x + \alpha_0$ as in (1.1), then if at least one of the non-zero entries of the vector $N_0^d\alpha_0$, where $N_0 = A_0 - \mathrm{Id}$, is irrational, then the group $E(X_q, T_0)$ is $(d+1)$-step nilpotent; otherwise it is d-step nilpotent. Therefore, we can strengthen Theorem 3.11 to get

Theorem 3.12. *The group $E(X_q, G)$ is k-step nilpotent for some k with $d \le k \le q$.*

Remark 3.13. Let d_G be the nilpotency class of the group G, and let d_E be the supremum of the nilpotency classes of the groups $E(X_q, T)$, where $T \in G$ (this supremum never exceeds q). It is clear that the nilpotency class of $E(X_q, G)$ is bounded from below by the number $\max\{d_G, d_E\}$, which will most likely provide a better lower estimate than that obtained in Theorem 3.12.

Example 3.14. Let N_1 and N_2 be $q \times q$ matrices defined in such a way that each entry of N_i, for $i = 1, 2$, is 0 except for the entries at the first diagonal below the main diagonal. For the purpose of this example we shall call this diagonal the *distinguished diagonal*. Let the entries of N_1 at the distinguished diagonal be $1, 0, 1, \ldots$; the last term of this sequence is either 0 or 1, depending on the dimension q. Finally, let the entries of N_2 at the distinguished diagonal be $0, 1, 0, \ldots$, so that the entries of the matrix $N = N_1 + N_2$ are 0 everywhere except at the distinguished diagonal, where they are all equal to 1.

It is easy to see that $N_1^2 = N_2^2 = 0$, and that $N^q = 0$ but $N^{q-1} \neq 0$, meaning that the matrix N is q-step nilpotent. Define

$$T_1(x) = x + N_1 x \quad \text{and} \quad T_2(x) = x + N_2 x,$$

and let G be the group generated by T_1 and T_2. According to Theorem 3.12 the lower bound on the nilpotency class of $E(X_q, G)$ is 1.

Consider the transformation

$$\begin{aligned} T(x) = T_1T_2(x) &= (\mathrm{Id} + N_1)(\mathrm{Id} + N_2)x \\ &= (\mathrm{Id} + N_1 + N_2 + N_1N_2)x = x + (N + N_1N_2)x. \end{aligned}$$

Notice that the first diagonal below the main diagonal of N_1N_2 consists only of zeros, which implies that the nilpotency class of N_1N_2 is strictly lower than that of N. Therefore, since the entries of N_1N_2 are nonnegative, the nilpotency class of $N + N_1N_2$ is equal to that of N.

It follows from [P] that $E(X_q, T)$ is a $(q-1)$-step nilpotent group, since the matrix $N + N_1N_2$ is q-step nilpotent. Therefore, the nilpotency class of $E(X_q, G)$ is bounded from below by $q - 1$. This constitutes a significant improvement over Theorem 3.12, as indicated in Remark 3.13.

In the next theorem we provide a description of the enveloping semigroup $E(X_q, G)$ as a quotient space.

THEOREM 3.15. *Let $\mathbb{G}'$ and φ_2 be as above. Let $\mathbb{K}$ be the kernel of φ_2. Then $E(X_q, G)$ is topologically isomorphic to $\mathbb{G}'/\mathbb{K}$.*

Proof. Obviously, $\mathbb{G}'/\mathbb{K}$ is a compact space. Proposition B.4 implies that it is a left topological group. Since $\mathbb{G}'$ is compact and $E(X_q, G)$ is Hausdorff, by Proposition B.5 the continuous onto homomorphism φ_2 is open. Therefore, by Proposition B.6, the map

$$\Phi : \mathbb{G}'/\mathbb{K} \to E(X_q, G)$$

given by

$$\Phi(M\mathbb{K}) = \varphi_2(M) = L_M \quad \text{for } M \in \mathbb{G}'$$

is a topological isomorphism. ∎

As a closing remark we note that when the group G is generated by a single transformation T it can be shown that the kernel of φ_2 is trivial. It follows that $E(X_q, T)$ is topologically isomorphic to $\mathbb{G}'$, which provides a new insight into the structure of these enveloping semigroups.

Appendix A: Supplementary computations. Let $T : X_q \to X_q$ be as in (1.1), and let $N = A - \mathrm{Id}$. Then N is a $(d+1)$-step nilpotent matrix for some integer $d < q$, i.e., $N^d \neq 0$ and $N^{d+1} = 0$.

Recall that in [P] the following formula for the powers T^n with $n > d$ was derived:

$$T^n(x) = \sum_{i=0}^{d} \binom{n}{i} N^i x + \sum_{i=0}^{d} \binom{n}{i+1} N^i \alpha. \tag{A.1}$$

If we define the symbol $\binom{n}{i}$ to be 0 for $i > n$, then it is easy to verify that the formula (A.1) holds true for all positive integers n.

We are going to show that (A.1) can be extended to all integers $n \in \mathbb{Z}$. Clearly T^0 is the identity map, so it remains to establish (A.1) for negative integers n.

Let

$$A' = \sum_{i=0}^{d} (-1)^i N^i = \mathrm{Id} + \sum_{i=1}^{d} (-1)^i N^i,$$

and let $\alpha' = -A'\alpha$. We claim that

$$T^{-1}x = A'x + \alpha'. \tag{A.2}$$

To verify this note that

$$T(A'x + \alpha') = \Big[(\mathrm{Id} + N)\Big(\mathrm{Id} + \sum_{i=1}^{d} (-1)^i N^i\Big)x\Big] + [(\mathrm{Id} + N)\alpha' + \alpha]. \tag{A.3}$$

The product of the matrices in the first bracketed expression in (A.3) simplifies to

$$\mathrm{Id} + N + \sum_{i=1}^{d}(-1)^i N^i + \sum_{i=1}^{d}(-1)^i N^{i+1} = \mathrm{Id} + (-1)^d N^{d+1} = \mathrm{Id},$$

since N is $(d+1)$-step nilpotent, while the second bracketed expression in (A.3) becomes

$$-(\mathrm{Id}+N)\Big(\mathrm{Id} + \sum_{i=1}^{d}(-1)^i N^i\Big)\alpha + \alpha = 0.$$

Thus (A.2) holds.

When we apply the formula (A.1) to the transformation T^{-1}, we obtain

$$\text{(A.4)}\quad T^{-n}x = \sum_{k=0}^{d}\binom{n}{k}\Big(\sum_{i=1}^{d}(-1)^i N^i\Big)^k x + \sum_{k=0}^{d}\binom{n}{k+1}\Big(\sum_{i=1}^{d}(-1)^i N^i\Big)^k \alpha',$$

where the sum involving constant terms splits further into two sums:

$$-\sum_{k=0}^{d}\binom{n}{k+1}\Big(\sum_{i=1}^{d}(-1)^i N^i\Big)^k \alpha - \sum_{k=0}^{d}\binom{n}{k+1}\Big(\sum_{i=1}^{d}(-1)^i N^i\Big)^{k+1} \alpha.$$

To simplify this formula, we need the following lemmas.

Lemma A.1. *Let N be a $(d+1)$-step nilpotent matrix. Then*

$$\Big(\sum_{i=1}^{d}(-1)^i N^i\Big)^k = \sum_{i=k}^{d}\binom{i-1}{k-1}(-1)^i N^i.$$

Proof. We proceed by induction on k. The formula is obviously valid for $k=1$. Suppose it is true for some $k \geq 1$. Let

$$L = \Big(\sum_{i=1}^{d}(-1)^i N^i\Big)^{k+1}.$$

By the inductive hypothesis and the nilpotency of N we get

$$\begin{aligned} L &= \Big(\sum_{i=k}^{d}\binom{i-1}{k-1}(-1)^i N^i\Big)\Big(\sum_{j=1}^{d}(-1)^j N^j\Big) \\ &= \sum_{l=k+1}^{d}\sum_{i=k}^{l-1}\binom{i-1}{k-1}(-1)^l N^l = \sum_{l=k+1}^{d}\binom{l-1}{k}(-1)^l N^l. \ \blacksquare \end{aligned}$$

Lemma A.2. *For integers r, s, m, p, with $r, s \geq 0$, we have*

$$\sum_{k}\binom{r}{m+k}\binom{s}{p+k} = \binom{r+s}{r-m+p}.$$

Proof. See identity (5.23) in [GKP]. ∎

COROLLARY A.3.

$$\sum_{k=1}^{i}\binom{n}{k}\binom{i-1}{k-1}=\binom{n+i-1}{i}.$$

Proof. Use Lemma A.2 with $r=i-1$, $s=n$, $m=-1$, and $p=0$. ∎

LEMMA A.4. *Let N be a $(d+1)$-step nilpotent matrix. Then*

$$\sum_{k=0}^{d}\binom{n}{k}\Big(\sum_{i=1}^{d}(-1)^iN^i\Big)^k=\sum_{k=0}^{d}\binom{n+k-1}{k}(-1)^kN^k.$$

Proof. Clearly

$$\sum_{k=0}^{d}\binom{n}{k}\Big(\sum_{i=1}^{d}(-1)^iN^i\Big)^k=\mathrm{Id}+\sum_{k=1}^{d}\binom{n}{k}\Big(\sum_{i=1}^{d}(-1)^iN^i\Big)^k,$$

hence, after applying Lemma A.1, changing the order of summation, and applying Corollary A.3, we get

$$\begin{aligned}\sum_{k=1}^{d}\binom{n}{k}\Big(\sum_{i=1}^{d}(-1)^iN^i\Big)^k&=\sum_{k=1}^{d}\sum_{i=k}^{d}\binom{n}{k}\binom{i-1}{k-1}(-1)^iN^i\\&=\sum_{i=1}^{d}\sum_{k=1}^{i}\binom{n}{k}\binom{i-1}{k-1}(-1)^iN^i\\&=\sum_{i=1}^{d}\binom{n+i-1}{i}(-1)^iN^i.\end{aligned}$$

This yields the desired formula. ∎

In a similar fashion one can show that

$$\sum_{k=0}^{d}\binom{n}{k+1}\Big(\sum_{i=1}^{d}(-1)^iN^i\Big)^{k+1}=\sum_{k=1}^{d}\binom{n+k-1}{k}(-1)^kN^k, \tag{A.5}$$

and, modulo another application of Lemma A.2 (this time with $r=i-1$, $s=n$, $m=-1$, and $p=1$),

$$\sum_{k=0}^{d}\binom{n}{k+1}\Big(\sum_{i=1}^{d}(-1)^iN^i\Big)^{k}=n\cdot\mathrm{Id}+\sum_{k=1}^{d}\binom{n+k-1}{k+1}(-1)^kN^k. \tag{A.6}$$

Combination of (A.5), (A.6), and Lemma A.4 leads to a more manageable form of (A.4), namely

$$T^{-n}x=\sum_{k=0}^{d}\binom{n+k-1}{k}(-1)^kN^kx+\textit{constant term},$$

where the *constant term* is equal to

$$-\sum_{k=1}^{d}\binom{n+k-1}{k}(-1)^k N^k\alpha - n\alpha - \sum_{k=1}^{d}\binom{n+k-1}{k+1}(-1)^k N^k\alpha.$$

The formula for $T^{-n}x$ simplifies to

$$T^{-n}x = \sum_{k=0}^{d}\binom{n+k-1}{k}(-1)^k N^k x + \sum_{k=0}^{d}\binom{n+k-1}{k+1}(-1)^{k+1} N^k\alpha.$$

After applying the identity $\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}$, this yields

$$T^{-n}x = \sum_{k=0}^{d}\binom{-n}{k} N^k x + \sum_{k=0}^{d}\binom{-n}{k+1} N^k\alpha, \tag{A.7}$$

where $n \geq 1$, which extends (A.1), as claimed.

Appendix B: Some facts about left topological groups and semigroups. In this appendix we collect some results about left topological semigroups and groups which were used in the main body of the paper.

For the proofs of Propositions B.1–B.3 see Appendix B in [V]. For the proofs of Propositions B.4–B.6 see Chapter 1.5 in [AT].

Note that what we call a left topological semigroup/group is called a right topological semigroup/group in both [AT] and [V], and vice versa. However, the standard device of reversing the order of multiplication allows one to transfer properties between left and right topological semigroups. Indeed, assuming that S is a semigroup, and considering the two multiplications $\mu_1(p,q) = pq$ and $\mu_2(p,q) = qp$ on S, one can easily verify that (S,μ_1) is a left (right) topological semigroup if and only if (S,μ_2) is a right (left) topological semigroup.

Proposition B.1. *A compact left topological semigroup contains an idempotent.*

Proposition B.2. *Let E be a subsemigroup of a left topological semigroup S. Suppose that, for every $p \in E$, the map $q \mapsto pq$ from S to itself is continuous. Then the closure of E in S is a left topological subsemigroup of S. (Here S need not be compact.)*

Proposition B.3. *A compact left topological semigroup whose unique idempotent is the identity is necessarily a compact left topological group, and any of its compact subsemigroups is a compact subgroup.*

Proposition B.4. *Suppose that G is a left topological group, and that H is a closed normal subgroup of G. Then G/H with the quotient topology and multiplication is a left topological group.*

Proposition B.5. *Let $f : G \to H$ be a continuous onto homomorphism of left topological groups. If G is compact and H is Hausdorff, then f is open.*

Proposition B.6. *Let G and H be left topological groups, and let f be an open continuous homomorphism of G onto H. Then the kernel N of f is a closed normal subgroup of G, and the fibers $f^{-1}(h)$ with $h \in H$ coincide with the cosets of N in G. The mapping $\Phi : G/N \to H$ which assigns to a coset gN the element $f(g) \in H$ is a topological isomorphism.*

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